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Houcine BEN DALI

b-ÉNUMÉRATION DE CARTES ET POLYNÔMES DE JACK

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Abstract

We are interested in connections between *symmetric functions* and the enumeration of *maps*, which are graphs drawn on surfaces, not necessarily orientable. We consider generating series of some families of maps with colored vertices, including *bipartite maps* and *constellations*. In these generating series, some properties of the combinatorial structure of the map are controlled, and each map is counted with a weight correlated to its *non-orientability*. We focus on two families of conjectures connecting these series to *Jack polynomials*, a one parameter deformation of Schur symmetric functions.

The *Matching-Jack conjecture*, introduced by Goulden and Jackson in 1996, suggests that the expansion of a mutliparametric Jack series in the power-sum symmetric functions has non-negative integer coefficients. Moreover, these coefficients count bipartite maps with controlled degrees of all vertices and faces. Using techniques of differential operators recently introduced by Chapuy and Dołęga, we prove the Matching-Jack conjecture for a particular specialization of the generating series. We use this result and a new connection with the *Farahat-Higman algebra* to prove the "integrality part" in the conjecture.

In another direction, we establish a *combinatorial formula* for the power-sum expansion of Jack polynomials using *layered maps*, a family of decorated bipartite maps introduced in this thesis. We deduce this formula from a more general one that we provide for *Jack characters*. Actually, this result generalizes a formula conjectured by Stanley and proved by Féray in 2010 for the characters of the symmetric group. We combine this formula with an approach based on a family of operators introduced by Nazarov and Sklyanin in order to prove a *conjecture of Lassalle* from 2008 about the positivity and the integrality of Jack characters in *Stanley's coordinates*.

Finally, we use the map expansion of Jack characters in order to prove that the generating series of bipartite maps with controlled vertex and face degrees satisfies a family of differential equations that completely characterizes it. Similar differential equations are also provided for the series of constellations.

Key words: Combinatorial maps, non-orientability, symmetric functions, Jack polynomials, differential operators.

Résumé

On s'intéresse à des liens entre les *fonctions symétriques* et l'énumération des *cartes*, qui sont des graphes dessinés sur des surfaces, pas nécessairement orientables. On considère des séries génératrices de certaines familles de cartes avec des sommets colorés, notamment des *cartes bipartites* et des *constellations*. Dans ces séries génératrices, certaines propriétés de la structure combinatoire de la carte sont contrôlées, et chaque carte est comptée avec un poids corrélé à sa *non-orientabilité*. On se concentre sur deux familles de conjectures reliant ces séries aux *polynômes de Jack*, une déformation à un paramètre des *fonctions de Schur*.

La *conjecture Matching-Jack*, introduite par Goulden et Jackson en 1996, suggère que le développement d'une certaine série de Jack à plusieurs alphabets de variables dans la base des sommes de puissances a des coefficients entiers et positifs. De plus, ces coefficients compteraient des cartes biparties avec contrôle des degrés de tous les sommets et de toutes les faces. En utilisant des techniques d'opérateurs différentiels récemment introduites par Chapuy et Dołęga, on prouve la conjecture Matching-Jack pour une spécialisation particulière de la série génératrice. On utilise ensuite ce résultat et un nouveau lien avec *l'algèbre de Farahat–Higman* pour prouver la "partie intégralité" de la conjecture.

Dans une autre direction, on établit une formule combinatoire pour le développement en sommes de puissances des polynômes de Jack, en utilisant *les cartes à niveaux*, une famille de cartes biparties décorées introduite dans cette thèse. En fait, cette formule découle d'une formule plus générale qu'on prouve pour les *caractères de Jack*. Ce résultat généralise une formule conjecturée par Stanley et prouvée par Féray en 2010 pour les caractères du groupe symétrique. En combinant cette formule avec une approche basée sur une famille d'opérateurs introduite par Nazarov et Sklyanin, on prouve *une conjecture de Lassalle* de 2008 sur la positivité et l'intégralité des caractères de Jack dans les coordonnées de Stanley.

Finalement, nous utilisons la formule combinatoire obtenue pour les caractères de Jack afin de prouver que la série génératrice des cartes biparties avec contrôle des degrés de sommets et des faces satisfait une famille d'*équations différentielles* qui la caractérise. Ce résultat s'étend également aux séries des constellations.

Mots clés: Cartes combinatoires, non-orientabilité, fonctions symétriques, polynômes de Jack, opérateurs différentiels.

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Contents

Introduction (en français)

Cartes

Une *carte* est un graphe dessiné sur une surface (orientable ou non). L'*énumération des cartes* a été initiée par Tutte [Tut62b, Tut62a, Tut63] dans le cas planaire (cartes dessinées sur la sphère), et les premiers résultats concernant les cartes sur d'autres surfaces orientables ont été obtenus par Lehman et Walsh [LW72a, LW72b].

Au cours des dernières décennies, l'étude des cartes est devenue un domaine très actif avec de fortes connexions à la combinatoire analytique, à la physique mathématique et aux probabilités [BC86, LZ04, Eyn16, CS04], impliquant diverses méthodes telles que les séries génératrices, les techniques d'intégrales de matrices et les méthodes bijectives : voir par exemple [BDG04, La 09, Cha11, Eyn16, AL20]. Il convient de mentionner que les cartes sur des surfaces orientables sont bien plus étudiées que les cartes sur des surfaces générales (orientables ou non).

Nous étudions ici les propriétés énumératives des cartes à travers leurs *séries génératrices*, qui sont essentiellement des sommes de cartes comptées avec des poids qui tiennent compte de certaines propriétés de leur structure. Nous cherchons à comprendre ces séries génératrices du point de vue de la *combinatoire algébrique* en les reliant à certaines familles de *fonctions symétriques*, et plus précisément les *polynômes de Jack*.

Fonctions symétriques et polynômes de Jack

Les *polynômes de Jack* $J_\lambda^{(\alpha)}$ sont des fonctions symétriques indexées par une partition entière λ et un paramètre de déformation α. Ils interpolent, à un facteur de normalisation près, entre les *fonctions de Schur* pour $\alpha = 1$ et les *fonctions zonales* pour $\alpha = 2$. Ils ont été introduits par Jack [Jac71] en tant qu'outil important en statistique, et il s'est avéré plus tard qu'ils apparaissent assez naturellement dans de nombreux contextes : ils jouent un rôle crucial dans l'étude de divers modèles de mécanique statistique et de probabilité tels que les β-ensembles et les généralisations des intégrales de Selberg [OO97, Kad97, Joh98, DE02, Meh04, For10]. De plus, ils sont fortement liés au modèle de Calogero-Sutherland de la mécanique quantique [LV95] et aux partitions aléatoires [BO05, DF16, BGG17, Mol15, DŚ19]. Enfin, on a découvert qu'ils possèdent une riche structure combinatoire [Sta89, Mac95, GJ96a, KS97, CD22, Mol23].

Knop et Sahi ont donné dans [KS97] une interprétation combinatoire pour les coefficients du polynôme de Jack $J_\lambda^{(\alpha)}$ dans la base des *fonctions symétriques monomiales* en termes de *tableaux* de forme λ. Plus récemment, d'autres formules ont été obtenues par Haglund et Wilson dans [HW20] pour le développement des polynômes de Jack dans les fonctions de Schur et dans les *fonctions sommes de puissances*. Ces formules sont données en termes de *graphes d'inversion de tableaux* comptés avec certains poids de *produits d'équerres* α-*déformés*.

Remarquablement, il a été découvert dans divers domaines [GJ96a, CE06, Las08b, Las09, AGT10] que lorsque la paramétrisation originale des polynômes de Jack par α est remplacée par sa version décalée $b := \alpha - 1$ (avec différentes formulations dans les références mentionnées), d'autres propriétés énumératives fascinantes des polynômes de Jack émergent, notamment liées aux cartes.

Deux familles de conjectures

L'interaction entre les fonctions symétriques et l'énumération des cartes s'est révélée enrichissante tant du point de vue des cartes que des fonctions symétriques. On s'intéresse ici à des problèmes qui suggèrent que le développement des polynômes de Jack dans la base des sommes de puissances est liée à l'énumération des cartes. Ces problèmes consistent principalement en deux familles de conjectures :

- Dans une direction, nous cherchons à trouver une série génératrice de cartes biparties avec contrôle des profils qui a un développement utilisant les polynômes de Jack. Cela est suggéré par les conjectures de Goulden-Jackson [GJ96a] (la conjecture Matching-Jack et la b-conjecture).

- Dans une autre direction, nous aimerions trouver une formule combinatoire pour les polynômes de Jack en termes de cartes. Cela est lié aux conjectures combinatoires de Hanlon [Han88] et de Dołęga–Féray–Śniady [DFŚ14], ainsi qu'à une conjecture de positivité de Lassalle [Las08a].

Ces problèmes visent en fait à généraliser des résultats connus pour les fonctions de Schur et les fonctions zonales, qui correspondent respectivement aux polynômes de Jack pour $\alpha = 1$ et $\alpha = 2$. En effet, la théorie des représentations peut être utilisée pour relier d'une part les fonctions de Schur aux séries génératrices des cartes orientables [JV90, Fér10, FŚ11a], et d'autre part les fonctions zonales aux séries génératrices des cartes non orientables [GJ96b, FŚ11b].

Le cas des polynômes de Jack pour un α général semble alors plus difficile car les outils de la théorie des représentations n'existent pas pour tout α, ce qui nécessite le développement de nouvelles techniques.

On verra tout au long de cette thèse que ces deux familles de conjectures sont étroitement liées. En particulier, un objet combinatoire principal commun à ces conjectures est donné par des *séries génératrices de non-orientabilité*. Dans ces séries, les cartes sont comptées avec un poids corrélé à leur "non-orientabilité". Cette notion de poids de non-orientabilité a été introduite par Goulden et Jackson [GJ96a] et appliquée dans de nombreux travaux [La 09, DFŚ14, CD22]. Elle jouera un rôle central dans cette thèse.

Avant d'expliquer ces conjectures, on introduit les cartes biparties.

Cartes biparties

Une carte bipartie est une carte dont les sommets sont colorés en deux couleurs (noir et blanc), et telle que tout arête lie deux sommets de couleurs différentes.

À une carte bipartie, on associe trois partitions d'entiers, formées respectivement par les degrés des sommets blancs, des sommets noirs et des faces. Ce triplet de partitions s'appelle le *profil* de la carte.

La plupart des résultats obtenus dans cette thèse pour les cartes biparties s'étendent bien aux cas des constellations, une généralisation à plusieurs couleurs des cartes biparties. Les constellations sont en bijection avec *les revêtements de la sphère* au-dessus d'un nombre arbitraire de points de ramification [LZ04, CD22] et sont directement liées aux *nombres de Hurwitz* [BS00, CD22].

Conjectures de Goulden–Jackson

Dans cette première famille de conjectures, on étudie des séries génératrices de cartes biparties avec contrôle de profil. Ce sont des séries génératrices dans lesquelles un poids à trois *alphabets de variables* est utilisé. Chacun de ces alphabets encode une des partitions du profil. Il est bien connu que ces séries génératrices peuvent êtres exprimées en utilisant les fonctions de Schur dans le cas orientable [JV90], et les fonctions zonales dans le cas non-orientable [GJ96b].

En utilisant les polynômes de Jack, Goulden et Jackson ont introduit une série $\tau^{(\alpha)}$ qui donne la série des cartes orientables quand $\alpha = 1$ et la série des cartes non-orientables quand $\alpha = 2$. Ils ont conjecturé que cette série a une interprétation combinatoire pour tout α, et qu'elle compte des cartes biparties considérées avec *des poids de non-orientabilité*. Ceci correspond à la *conjecture Matching-Jack*. La *b-conjecture* est une variante de cette conjecture pour les séries de cartes connexes $\log\left(\tau^{(\alpha)}\right)$.

En fait, ces deux conjectures sont équivalentes à dire que les coefficients du développement des séries $\tau^{(\alpha)}$ et $\log\left(\tau^{(\alpha)}\right)$ dans les bases de sommes de puissance, notées respectivement $c^{\pi}_{\mu,\nu}(\alpha)$ et $h^{\pi}_{\mu,\nu}(\alpha)$, sont des **polynômes** dans le paramètre $b := \alpha - 1$ avec des coefficients **entiers** et **positifs**. Bien qu'elles puissent être formulées d'une manière indépendante des cartes, l'interprétation combinatoire derrière ces conjectures a été essentielle dans plusieurs résultats dans cette direction [La 09, DF16, KV16, Doł17, DF17, KPV18, CD22]. On présente brièvement certains de ces résultats.

Dołęga et Féray ont démontré dans [DF16] la polynomialité dans la conjecture Matching-Jack, et ils en ont déduit la polynomialité dans la b-conjecture dans [DF17].

Dans [CD22], Chapuy et Dołęga ont démontré la b-conjecture pour une spécialisation de la fonction $\log\left(\tau^{(\alpha)}\right)$. Cette spécialisation consiste à considérer des séries génératrices de cartes biparties dans lesquelles on contrôle les degrés des sommets blancs, les degrés des faces et le nombre des sommets noirs. D'une manière équivalente, ils prouvent que certaines *sommes marginales* des coefficients $h^{\pi}_{\mu,\nu}$ s'interprètent en termes de cartes considérées avec des poids de non-orientabilité.

Malgré ces résultat partiels, la conjecture Matching-Jack et la b-conjecture sont toujours ouvertes et aucune implication n'est connue entre les deux.

Caractères de Jack et conjecture de Lassalle

Un autre problème important liant les polynômes de Jack aux cartes consiste à établir une interprétation combinatoire du développement en sommes de puissances d'un polynôme de Jack en termes de séries de non-orientabilité de cartes. Un objet important dans cette étude est donné par les *caractères de Jack*. Nous commençons par expliquer l'origine de ce problème.

Une approche très efficace de la théorie des représentations asymptotiques du groupe symétrique, initiée par Kerov et Olshanski [KO94], traite les caractères irréductibles normalisés $\chi^\lambda(\mu)$ comme des fonctions sur les diagrammes de Young λ avec une normalisation bien choisie, qu'on note ici $\theta_\mu^{(1)}(\lambda)$. Kerov et Olshanski ont démontré que ces caractères normalisés ont des propriétés de symétrie intéressantes en tant que polynômes dans plusieurs descriptions de λ (tailles des parts, contenus, coordonnées de Frobenius...).

Stanley [Sta04] a étudié les caractères irréductibles normalisés du groupe symétrique en utilisant cette approche duale, et il a observé que si on les exprime dans une nouvelle description de λ qu'il a appelée les *coordonnées multirectangulaires*, alors elles ont des propriétés de positivité et d'intégralité remarquables. Il démontre cette propriété pour des partitions λ rectangulaires en donnant une formule explicite des caractères, et il conjecture une formule plus générale pour tout λ. Cette formule permet d'écrire le caractère $\theta_\mu^{(1)}(\lambda)$ comme somme signée de paires de permutations stabilisant respectivement les colonnes et les lignes d'un tableau de forme λ.

Ce résultat a été démontré par Féray dans [Fér10] en utilisant la théorie des représentations du groupe symétrique et a été un outil clé dans des avancées importantes dans l'étude asymptotique des caractères du groupe symétrique [FŚ11a]. Féray et Śniady ont obtenu dans [FŚ11b] une formule combinatoire analogue en termes de matchings pour les *caractères zonaux* $\theta_\mu^{(2)}(\lambda)$, qui sont directement liés au développement en somme de puissance des fonctions zonales.

Les formules de [Fér10] et [FŚ11b] se réécrivent en termes de séries génératrices de cartes biparties avec une décoration supplémentaire, qu'on appelle ici *les cartes à niveaux*. Dans ces séries, on contrôle les degrés des faces ainsi que certains paramètres liés à cette "structure en niveaux". Ces formules combinatoires permettent en particulier d'obtenir des *développements topologiques* pour les fonctions de Schur et les fonctions zonales, qui ressemblent aux développements à genre fixé connus pour *les matrices aléatoires* [Meh04, EKR15]. Il devient alors naturel de se demander si les polynômes de Jack pour n'importe quel α admettent un tel développement topologique. Cependant, et comme pour les conjectures de Goulden-Jackson, le cas général nécessite de nouvelles méthodes.

L'approche de Kerov et Olshanski pour étudier les caractères du groupe symétrique a été étendue au cas des polynômes de Jack par Lassalle [Las08a, Las09], où l'objet principal d'étude est *le caractère de Jack* $\theta_\mu^{(\alpha)}$, et qui est obtenu comme les coefficients d'un polynôme de Jack dans la base des sommes de puissance.

Basé sur les formules combinatoires de [Fér10, FŚ11b], Lassalle a généralisé la conjecture de Stanley ; une fois que le paramètre α est remplacé par le paramètre $b := \alpha - 1$ les caractères $\theta_\mu^{(\alpha)}(\lambda)$ sont des polynômes dans le paramètre b et les coordonnées de Stanley, avec des coefficients **entiers** et **positifs**. Une version combinatoire de cette conjecture a été formulée dans [DFŚ14] et suggère que les caractères de Jack peuvent s'écrire comme des séries génératrices de cartes non-orientables considérées avec des poids de non-orientabilité, dans le même esprit que dans les conjectures de Goulden–Jackson.

En plus des cas $\alpha = 1$ et $\alpha = 2$, cette conjecture a été prouvée lorsque le diagramme de la partition λ est un rectangle [DFŚ14, Ben22]. Malgré ces cas particuliers, la conjecture de Lassalle est restée non prouvée pendant les 15 dernières années. La preuve de cette conjecture est l'un des principaux résultats de cette thèse.

L'outil principal : les opérateurs différentiels

Le principal outil utilisé dans ce travail pour relier les séries génératrices des cartes aux polynômes de Jack sont les opérateurs différentiels, et plus précisément les opérateurs $\mathcal{B}_n^{(\alpha)}$ introduits par Chapuy et Dołęga dans [CD22]. L'intérêt de ces opérateurs vient du fait que d'une part, ils peuvent être utilisés pour encoder des opérations combinatoires sur les cartes, et d'autre part, leur action sur les polynômes de Jack est donnée par des formules simples.

Ces deux "facettes" des opérateurs $\mathcal{B}_n^{(\alpha)}$ sont reflétées par deux types de formules ; ils peuvent être écrits en utilisant des "variables catalytiques", dans le même esprit que celles utilisées dans une *équation de Tutte*. Cette écriture permet de donner une interprétation de ces opérateurs en termes de cartes. De plus, ils sont obtenus en utilisant des commutateurs itérés à partir de l'*opérateur de Laplace–Beltrami*, un opérateur qui agit diagonalement sur les polynômes de Jack.

Organisation de la thèse et résultats principaux

Chapitre 1 :

On donnera les définitions précises des objets combinatoires et algébriques nécessaires pour énoncer les différentes conjectures auxquelles on s'intéresse dans ce travail. On expliquera ensuite le lien entre l'approche d'opérateurs différentiels de Chapuy-Dołęga et la construction des cartes. Finalement, on donnera une formulation précise des résultats principaux de cette thèse.

Chapitre 2 :

Le résultat principal du Chapitre 2 est basé sur [Ben22]. Cependant, la preuve donnée ici est différente.

On établit la conjecture Matching-Jack pour une spécialisation de la fonction $\tau^{(\alpha)}$. Dans la preuve, on définit une famille de poids de non-orientabilité sur les cartes non connexes qui permet de donner une interprétation combinatoire pour certaines somme marginales des coefficients $c^{\pi}_{\mu,\nu}$. Ce résultat est un analogue du résultat de Chapuy–Dołęga [CD22] pour la b-conjecture.

Dans la preuve, on établit une construction de la fonction $\tau^{(\alpha)}$ spécialisée avec les opérateurs $\mathcal{B}_n^{(\alpha)}$, en utilisant une équation différentielle fournie dans [CD22].

Chapitre 3 :

Le Chapitre 3 est basé sur le travail de [Ben23a].

On utilise le résultat du Chapitre 2 pour prouver l'intégralité dans la conjecture Matching-Jack. En effet, on déduit l'intégralité des coefficients $c^{\pi}_{\mu,\nu}$ de l'intégralité des somme marginales précédemment obtenue en utilisant un lien entre certains coefficients $c^{\pi}_{\mu,\nu}$ et *l'algèbre de Farahat–Higman*.

Cette algèbre a été introduite dans [FH59] afin d'étudier les coefficients de structure des classes de conjugaison dans le centre de l'algèbre du groupe symétrique. L'algèbre de Farahat-Higman est connue pour être isomorphe à l'algèbre des fonctions symétriques ; voir [GJ94, CGS04]. Elle est également liée à l'algèbre des permutations partielles introduite par Ivanov et Kerov dans [IK99]. On s'intéresse ici à une *version graduée* de l'algèbre de Farahat–Higman.

Chapitre 4 :

Le Chapitre 4 est basé sur un travail en commun avec Maciej Dołęga [Ben23a].

On établit une interprétation combinatoire pour les caractères de Jack en termes de cartes à niveaux comptées avec des poids de non-orientabilité. Ce résultat est une interpolation entre la formule de Stanley–Féray pour les caractères du groupe symétrique [Fér10] et la formule de Féray–Śniady pour les caractères zonaux [FŚ11b]. Comme nous l'avons déjà mentionné, les idées utilisées dans les démonstrations des cas particuliers $\alpha \in \{1, 2\}$ ne s'appliquent pas au cas général. L'outil clé dans notre preuve est l'approche de calcul différentiel développée par Chapuy et Dołęga dans [CD22]. Nous combinons cette approche avec une *caractérisation algébrique* due à Féray pour les caractères de Jack comme des fonctions *symétriques décalées* satisfaisant certaines conditions d'annulation dans l'esprit de [KS96]. Dans cette preuve, nous obtenons plusieurs relations de commutation entre les opérateurs différentiels de cartes, en utilisant des méthodes inspirées par la théorie des algèbres de Lie. Nous montrons que ces relations algébriques reflètent les propriétés combinatoires et algébriques souhaitées dans le théorème de caractérisation.

La deuxième contribution principale du chapitre consiste à prouver la conjecture de Lassalle sur les caractères de Jack. La "partie positivité" de la conjecture est obtenue grâce à la formule combinatoire établie dans la première partie. Afin d'obtenir la "partie intégralité", nous utilisons une autre famille d'opérateurs liée au *système intégrable de Nazarov–Sklyanin* [NS13]. Cette étape fait intervenir une autre famille d'objets combinatoires récemment introduite par Moll ; *les chemins de Łukasiewicz à rubans* (voir [Mol23, CDM23]).

Chapitre 5 :

Le Chapitre 5 est basé sur [Ben24].

Un problème classique dans l'énumération des cartes consiste à établir des *équations différentielles* pour les séries génératrices qui peuvent être utilisées pour obtenir des formules de récurrence pour le nombre de cartes satisfaisant des propriétés données.

Nous établissons une équation différentielle pour la série génératrice des cartes biparties avec contrôle du profil complet, ainsi que pour leur série α-déformée. Ce résultat est nouveau même dans le cas orientable pour lequel nous donnons une preuve combinatoire. L'approche utilisée ici est différente de celle donnée par les équations de type Tutte [Tut62b, BC86, CD22], dans lesquelles nous ne pouvons pas suivre les trois alphabets du profil. Cependant, contrairement aux équations de Tutte, les équations que nous obtenons ici sont signées.

Chapitre 6 :

Dans le chapitre 6, on présentera quelques problèmes ouverts motivés par les résultats de cette thèse. On y discutera quelques problèmes combinatoires liées à la b-conjecture ainsi que d'autres conjectures sur les caractères de Jack. On présentera également une généralisation des conjectures de Goulden–Jackson et de Lassalle au cas des *polynômes de Macdonald* (une généralisation à deux paramètres des polynômes de Jack).

Chapter 1

Introduction (in English)

Roughly, a map is a graph drawn on a surface (orientable or not), considered up to a homeomorphism of the surface. The enumeration of maps has been initiated by Tutte [Tut62b, Tut62a, Tut63] in the planar case (maps drawn on the sphere), and the first results about maps on other orientable surfaces have been obtained by Lehman and Walsh [LW72a, LW72b].

In the last decades, the study of maps has become a well developed area with strong connections to analytic combinatorics, mathematical physics and probability [BC86, LZ04, Eyn16, CS04], involving various methods such as generating series, matrix integral techniques and bijective methods, see *e.g.* [BDG04, La 09, Cha11, Eyn16, AL20]. It is worth mentioning that maps on orientable surfaces are widely more studied than maps on general surfaces (orientable or not).

We study here the enumerative properties of maps through their **generating series**, which are roughly sums of maps counted with weights which keep track of some of their properties (*e.g.* vertex degrees). We seek to understand these generating series from the point of view of **algebraic combinatorics** by connecting them to some families of **symmetric functions**, namely **Jack polynomials**.

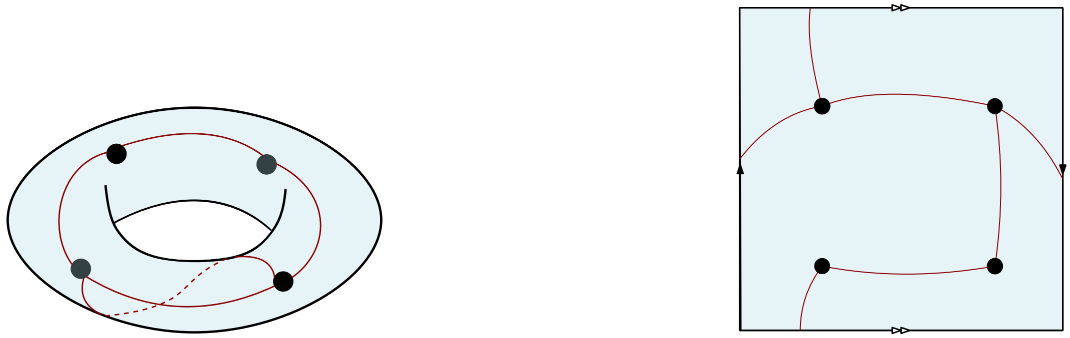
Structure of the Introduction

In Section 1.1, we give some definitions related to maps. In Section 1.2, we introduce symmetric functions. We then present in Section 1.3 the main families of problems connecting maps to Jack polynomials. In Section 1.4, we introduce the main tools used in this thesis. Finally, we present the main results in Section 1.5.

1.1 Maps

1.1.1 Definitions

In this section, we give two definitions of maps. The first one is topological while the second one is more combinatorial. We refer to [LZ04] for more details about these definitions.



(a) (b)

Figure 1.1: Two examples of connected maps. On the left, an orientable map on the torus. On the right, a non-orientable map on the Klein bottle. The square represents here the Klein bottle; the left-hand side of the square should be glued to the right-hand one (with a twist) and the top side should be glued to the bottom one (without a twist), as indicated by the arrows.

Maps as graphs on surfaces

A *connected map* is a connected graph embedded into a surface such that all the connected components of the complement of the graph are simply connected (see [LZ04, Definition 1.3.6]). These connected components are called the *faces* of the map. We consider maps up to homeomorphisms of the surface. A connected map is *orientable* if the underlying surface is orientable.

More generally, a *map*[1] is an unordered collection (possibly empty) of connected maps. A map is orientable if each one of its connected components is orientable, otherwise we say that the map is *non-orientable*. Moreover, a face of the map is a face of one of its connected components.

By convention, we require that there are no isolated vertices in a map. As a consequence, the empty map is the only map with 0 edges.
We give in Fig. 1.1 examples of orientable and non-orientable maps.

Combinatorial definition

In this thesis, we will think of a map as a discrete object. Indeed, maps considered up to homeomorphism, can be defined as graphs with some additional structure which we now explain.

In the orientable case, a map is a graph endowed with a cyclic order on the edges around each vertex, see Fig. 1.2 for an example. This description of maps allows one to give an encoding in terms of permutations; see [Cor75, LZ04] and Section 1.1.4.

In the general case, a map can be represented by a *ribbon graph* (also called a band diagram); see [GJ96b]. A ribbon graph is a graph with cyclic order on edges around each vertex, and such that each edge is represented by a ribbon which can be twisted at most once. In Fig. 1.3a, we give a ribbon graph representation of the non-orientable map from Fig. 1.1b.

[1] This is not the standard definition of a map; in most of the literature, maps are necessarily connected.

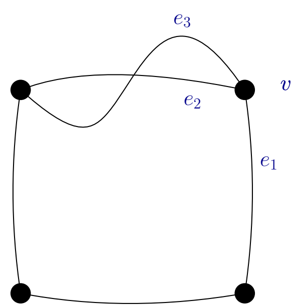


Figure 1.2: Two different orientable maps with the same underlying graph; the cyclic order around the vertex v is different on the two maps. The map on the left corresponds to the map on the torus given in Fig. 1.1 while the map on the right is planar (is embedded on the sphere).

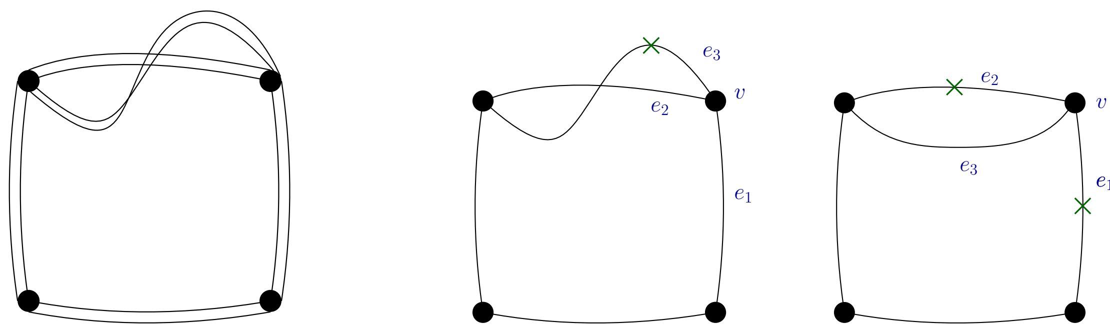


Figure 1.3: Different representations of the same non-orientable map. On the left a non-orientable map represented as a ribbon graph. In the middle and on the right two representations of the map as a simple graph with twisted edges. The representation on the right is obtained from the one in the middle by changing the side of the surface from which we represent the vertex v (see also Fig. 1.4).

To simplify figures, we usually represent a ribbon graph as a graph and we mark the twisted edges with a cross; see Fig. 1.3b.

However, this representation is not unique; there are different ways to represent the same map as a ribbon graph. Indeed, such a representation is obtained by choosing, for each vertex of the map one of the two sides of the surface from which we represent it, see Fig. 1.4. As a consequence, there are exactly $2^{|\mathcal{V}(M)|}$ ways to represent a map M as a graph, $|\mathcal{V}(M)|$ being its number of vertices. In Fig. 1.3c we give a second graph representation of the map of Fig. 1.3b obtained by changing the side from which we represent one vertex.

Actually, this representation of maps as ribbon graphs contains all the information about its combinatorial and geometric structure. For example, faces are obtained as follows; the twists and the cyclic ordering around vertices define a canonical way to travel along edges, and each one of the (non-oriented) cycles formed represents a face, see Fig. 1.5.

Given a connected ribbon graph M, there is exactly one surface on which it can be embedded while respecting the condition that faces are simply connected; we refer to [LZ04, Section 1.3] for more details. Moreover, the genus g of this surface is given by the Euler

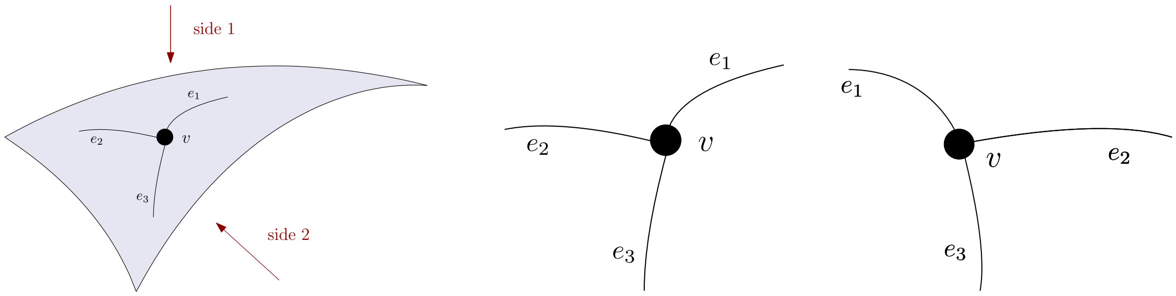


Figure 1.4: The two possible representations of the neighborhood of a vertex drawn on a surface. In the middle the representation corresponding to Side 1 and on the right the representation corresponding to Side 2.

Figure 1.5: Traveling along the edges of a map to obtain faces. On the left an orientable map with three faces and on the right a non-orientable map with one face.

formula

$$2g - 2 = |M| - |\mathcal{V}(M)| - |\mathcal{F}(M)|, \tag{1.1}$$

where $|M|$, $|\mathcal{V}(M)|$ and $|\mathcal{F}(M)|$ are respectively the number of edges, vertices and faces of the map.

Finally, we have the following alternative characterization of orientability: a map is orientable if each one of its faces can be endowed with an orientation such that for every edge e of the map the two edge-sides of e are oriented in opposite ways. In Figure 1.6 we have an edge e whose sides are incident to two faces F_1 and F_2 (not necessarily distinct), and that are oriented in opposite ways. In this case we say that the orientation of the faces is *consistent*.

In this thesis, maps will always be represented as ribbon graphs. We conclude this subsection with the definition of a corner. A pair of edge-sides which appear consecutively while travelling along a face F is called a *corner* of F. An *oriented corner* is a corner endowed with an order on its pair of edge-sides. A corner of a vertex v is a corner whose edge-sides are incident to v.

1.1.2 Bipartite maps, rooting and labelling

We are particularly interested here in *bipartite* maps, *i.e.* maps whose vertices are colored in two colors, white and black, and such that each edge connects two vertices of different colors. Unless stated otherwise, all maps considered are bipartite.

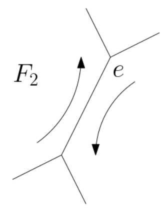
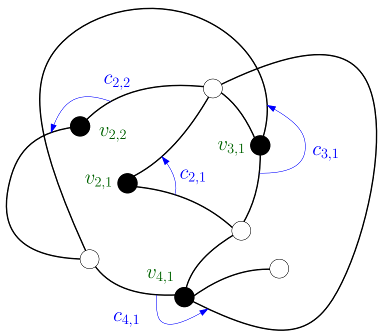
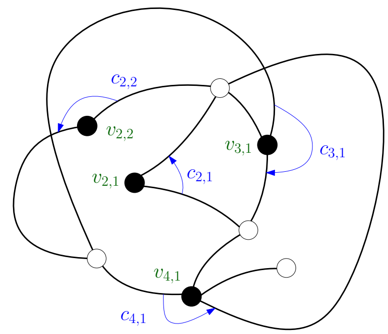


Figure 1.6: Consistent orientation from either side of an edge e.



(a) An oriented vertex-labelled map. The orientation of all black vertices are consistent.



(b) A non-oriented vertex-labelled map. The underlying map is orientable, however the orientation of $v_{3,1}$ is not consistent with the orientations of $v_{2,1}, v_{2,2}$ and $v_{4,1}$.

Figure 1.7: Two examples of vertex-labelled bipartite maps. The root of vertex $v_{d,i}$ is denoted $c_{d,i}$.

In order to eliminate symmetries and enumerate maps with trivial automorphism groups, we consider maps with marked corners or labelled edges.

Definition 1.1.1. *We say that a connected map is rooted if it has a distinguished black oriented corner c. The corner c will be called the root corner and the edge following the root corner is called the root edge. We say that a bipartite map is vertex-labelled if:*

1. *for each $d \geq 1$, black vertices of same degree d are labelled $v_{d,1}, v_{d,2}, \ldots$.*
2. *each black vertex has a marked oriented corner. This corner is called the vertex root.*

A vertex-labelled bipartite map is ***oriented*** *if the map is orientable and the orientation endowed by the vertex roots are consistent, i.e. any two black corners incident to the same face have roots oriented in the same direction.*

In Fig. 1.7, we give two examples of vertex-labelled maps. The map of Fig. 1.7b is a vertex-labelled map which is orientable but not oriented. These definitions will be useful to describe the encoding of maps with matchings in Section 1.1.4.

Notice that a map is orientable if and only if there exists a vertex-labelling for which the map is oriented. The orientations of the vertices in such labelling can be obtained by fixing an orientation for each one of the connected underlying surfaces. Equivalently, we can start by fixing the orientation of one face in each connected component, and then choose the orientations of the other faces so that they are consistent as in Fig. 1.6.

1.1.3 Types and profile

Let M be a map. We call the *size* of M and we denote $|M|$ its number of edges. We define the *degree* of a vertex or a face, as the number of edges incident to it. Note that in a bipartite map, all faces have even degree.

In order to define the profile of a map, we need to introduce *integer partitions*. A partition $\lambda = [\lambda_1, \lambda_2, \dots \lambda_\ell]$ of a non-negative integer n is a finite non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

We associate to a map M three integer partitions of $|M|$ defined as follows;

- its *face-type*, denoted $\lambda^\diamond(M)$, is the partition obtained by reordering the face degrees divided by 2.
- its *black-type*, denoted $\lambda^\bullet(M)$, is the partition obtained by reordering the degrees of the black vertices.
- its *white-type*, denoted $\lambda^\circ(M)$, is the partition obtained by reordering the degrees of the white vertices.

The *profile* of M, is then the tuple[2] of partitions $(\lambda^\bullet(M), \lambda^\diamond(M), \lambda^\circ(M))$. We finally denote the set of white and black vertices of a map M respectively by $\mathcal{V}_\circ(M)$ and $\mathcal{V}_\bullet(M)$, respectively.

Example 1.1.2. The maps of Fig. 1.7 are both of profile $([4,3,2,2],[7,2,2],[4,3,3,1])$.

Remark 1.1.3. Note that if a bipartite M has only white vertices of degree 2, then we can forget white vertices and think of M as a (non bipartite) map. As a consequence, maps can be thought of as bipartite maps for which $\lambda^\circ = [2, 2, \dots, 2]$. Moreover, it has been observed in [Cor75, LZ04] that bipartite maps are more natural objects to study from an algebraic point of view than simple maps.

1.1.4 Maps encoding with permutations and matchings

In this section, we explain how maps can be encoded with *permutations* in the orientable case, and with *matchings* in the general one.

Orientable case

We denote by $\mathfrak{S}_n$ the symmetric group of size n. If σ is a permutation in $\mathfrak{S}_n$, then we define its *cycle type* as the partition of n given by the cycle lengths. The cycle type of σ will be denoted $\mathrm{ct}(\sigma)$.

Example 1.1.4. For any partition $\pi \vdash n$, the permutation

$$\sigma_\pi := (1, 2, \dots, \pi_1)(\pi_1 + 1, \dots, \pi_2) \dots (\pi_{\ell(\pi)-1} + 1, \dots, n)$$

is of cycle type π (we use here the cycle notation of permutations). For example, when $\pi = [3, 3, 1]$, we have

$$\sigma_\pi = (1, 2, 3)(4, 5, 6)(7).$$

[2] The convention adapted here for the order on the profile's partitions is different from the one used in other references [CD22, Ben22].

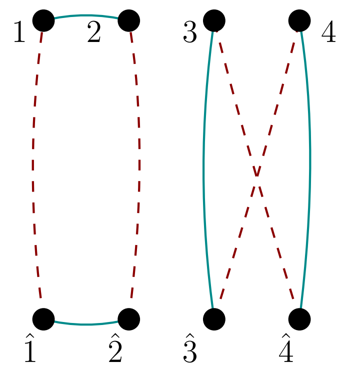


Figure 1.8: The two matchings of Example 1.1.6; δ_1 is represented with dashed edges and δ_2 in plain edges.

We have the following correspondence between orientable maps and permutations. We refer to [LZ04, Sections 1.3 and 1.5] for a proof. This correspondence will be explained in more generality in the proof of Proposition 1.1.10.

Proposition 1.1.5. *Fix three partitions π, μ, ν of the same size n and a permutation σ of cycle type π. There is a bijection between vertex-labelled oriented maps of profile (π, μ, ν) and the set*

$$\{\sigma_1, \sigma_2 \in \mathfrak{S}_n \text{ such that } \sigma_1 \cdot \sigma_2 = \sigma \text{ and } \operatorname{ct}(\sigma_1) = \mu, \operatorname{ct}(\sigma_2) = \nu\}. \tag{1.2}$$

General case

For $n \geq 1$, we consider the set

$$\mathcal{N}_n := \{1, \widehat{1}, ..., n, \widehat{n}\}.$$

We call *matching* on $\mathcal{N}_n$ a set partition of $\mathcal{N}_n$ into pairs. A matching δ on $\mathcal{N}_n$ is *bipartite* if each one of its pairs is of the form $\{i, \widehat{j}\}$. Given two matchings δ_1 and δ_2, we define the graph $G(\delta_1, \delta_2)$ as the graph with vertex set $\mathcal{N}_n$ and edge set $\delta_1 \cup \delta_2$. We use the convention that if $\{i, j\}$ is a pair in δ_1 and δ_2, then $G(\delta_1, \delta_2)$ has a double edge between i and j. It is then clear that $G(\delta_1, \delta_2)$ is a 2-regular graph, and all its connected components are cycles of even size.

We then define $\Lambda(\delta_1, \delta_2)$ as the partition of n obtained by reordering the half-sizes of the connected components of the graph formed by δ_1 and δ_2.

Example 1.1.6. We consider the two matchings $\delta_1 = \Big\{\{1, \widehat{1}\}, \{2, \widehat{2}\}, \{3, \widehat{4}\}, \{4, \widehat{3}\}\Big\}$ and $\delta_2 = \Big\{\{1, 2\}, \{\widehat{1}, \widehat{2}\}, \{3, \widehat{3}\}, \{4, \widehat{4}\}\Big\}$ on $\mathcal{N}_4$; see Fig. 1.8. Then δ_1 is bipartite and δ_2 is not. Moreover, one has $\Lambda(\delta_1, \delta_2) = [2, 2]$.

Remark 1.1.7. Note that the symmetric group $\mathfrak{S}_n$ is in natural bijection with the set of bipartite matchings of $\mathcal{N}_n$ via the map $\sigma \longmapsto \delta_\sigma$ where δ_σ is the bipartite matching whose pairs are $(i, \widehat{\sigma(i)})_{1 \leq i \leq n}$. Moreover, if $\sigma_1, \sigma_2 \in \mathfrak{S}_n$, then $\Lambda(\delta_{\sigma_1}, \delta_{\sigma_2}) = \operatorname{ct}(\sigma_1 \cdot \sigma_2^{-1})$.

Fix two partitions μ, ν of size $n \geq 1$, and two matchings δ_1, δ_2 on $\mathcal{N}_n$. We introduce the set

$$\mathfrak{F}_{\mu,\nu}^{\delta_1,\delta_2} := \{\delta \text{ matching on } \mathcal{N}_n \text{ such that } \Lambda(\delta, \delta_1) = \mu \text{ and } \Lambda(\delta, \delta_2) = \nu\}.$$

If δ_1 and δ_2 are bipartite, we also define

$$\widetilde{\mathfrak{F}}^{\delta_1,\delta_2}_{\mu,\nu} := \{\delta \text{ bipartite on } \mathcal{N}_n \text{ such that } \Lambda(\delta,\delta_1)=\mu \text{ and } \Lambda(\delta,\delta_2)=\nu\}.$$

Remark 1.1.8. If (δ_1,δ_2) and (δ_1',δ_2') satisfy

$$\Lambda(\delta_1,\delta_2)=\Lambda(\delta_1',\delta_2'),$$

then there is a bijection between $\mathfrak{F}^{\delta_1,\delta_2}_{\mu,\nu}$ and $\mathfrak{F}^{\delta_1',\delta_2'}_{\mu,\nu}$. If we also assume that $\delta_1,\delta_2,\delta_1',\delta_2'$ are bipartite then there is also a bijection between $\widetilde{\mathfrak{F}}^{\delta_1,\delta_2}_{\mu,\nu}$ and $\widetilde{\mathfrak{F}}^{\delta_1',\delta_2'}_{\mu,\nu}$. We refer to [HSS92] for more details.

We introduce the reference matchings

$$\varepsilon_n := \{\{1,\hat{1}\},\{2,\hat{2}\},...,\{n,\hat{n}\}\},$$

and for any partition π

$$\delta_\pi := \{\{1,\hat{2}\},\{2,\hat{3}\}..\{\pi_1-1,\widehat{\pi_1}\},\{\pi_1,\hat{1}\},\{\pi_1+1,\widehat{\pi_1+2}\},...\}.$$

In other terms, $\delta_\pi=\delta_{\sigma_\pi}$ with the notation of Example 1.1.4 and Remark 1.1.7.

Example 1.1.9. When $\pi=[3,3,1]$, we have

$$\delta_\pi=\{\{1,\hat{2}\},\{2,\hat{3}\},\{3,\hat{1}\},\{4,\hat{5}\},\{5,\hat{6}\},\{6,\hat{4}\},\{7,\hat{7}\}\}$$

We then have the following correspondence between bipartite maps and matchings.

Proposition 1.1.10 ([GJ96b, Section 4]). *Fix three partition $\pi,\mu,\nu\vdash n\geq 1$, and two bipartite matchings δ_1 and δ_2 on $\mathcal{N}_n$ such that $\Lambda(\delta_1,\delta_2)=\pi$. Then there exists a bijection between the set of matchings of $\mathfrak{F}^{\delta_1,\delta_2}_{\mu,\nu}$ and vertex-labelled maps of profile (π,ν,μ). Moreover, the image of $\widetilde{\mathfrak{F}}^{\delta_1,\delta_2}_{\mu,\nu}$ by this bijection correspond to oriented vertex-labelled maps.*

Proof. To simplify notation, we will prove the proposition for $\delta_1=\varepsilon_n$ and $\delta_2=\delta_\pi$ (this is enough to conclude since $\mathfrak{F}^{\varepsilon_n,\delta_\pi}_{\mu,\nu}$ and $\mathfrak{F}^{\delta_1,\delta_2}_{\mu,\nu}$ have the same cardinality by Remark 1.1.8).

Starting from a vertex-labelled map M of profile (π,ν,μ), we associate to it a matching $\delta\in\mathfrak{F}^{\varepsilon_n,\delta_\pi}_{\mu,\nu}$ as follows. First, we number the edge sides by $\hat{1},1,\hat{2},2\dots,\hat{n},n$ as follows; we start from the corner $c_{\pi_1,1}$ (the root of $v_{\pi_1,1}$, the black vertex of maximal degree and labelled by 1) and turning around the vertex $v_{\pi_1,1}$. We then restart with the corner $c_{\pi_1,2}$ and so on. We give an example of this labelling in Fig. 1.9b. With this numbering, one can notice that the numbers of two sides of the same edge form a pair of the matching ε_n. Moreover, two sides forming a black corner are a pair of δ_π. We then define δ as the matching connecting two numbers forming a white corners of the map. Using this numbering, a white vertex of degree r in M corresponds to a cycle of size $2r$ in the graph $G(\varepsilon_n,\delta)$. As a consequence, we get that $\Lambda(\varepsilon_n,\delta)=\mu$. Similarly, the cycles of $G(\delta_\pi,\delta)$ correspond to the faces of M. Hence, $\Lambda(\delta_\pi,\delta)=\nu$.

Conversely, given a matching δ, one can always construct a map M as follows:

- We start from $\ell(\pi)$ black vertices, with degrees π_1,π_2,..., $\pi_{\ell(\pi)}$, and we number the edge sides as above; the pairs of ε_n correspond to edges and the pairs of δ_π correspond to black corners; see Fig. 1.9c.
- We then attach the edge sides and we add the white vertices, in such a way that the matching δ corresponds to the matching of white corners. In this operation, we may need to twist some of the edges so that the labelling conventions are respected on the black and the white vertices. In the example of Fig. 1.9, the edge numbered by $\{7, \hat{7}\}$ is the only twisted edge. We recall that in general, there is not a unique choice of the set of twisted edges; see Section 1.1.1.
- For each black vertex, we choose its root as the oriented corner followed by the edge side numbered by the minimal label with a hat. Finally, we number the vertices of same degree in an increasing order of the numbers of the edges incident to it.

When the matching δ is bipartite, it is easy to see that in the associated map the orientations induced by the black vertex roots are consistent in the sense of Definition 1.1.1, and the map is then oriented. □

Example 1.1.11. In Fig. 1.9, we illustrate the correspondence of Proposition 1.1.10 with $n = 11$, the profile

$$\pi = [4, 3, 2, 2], \quad \nu = [7, 2, 2], \quad \mu = [4, 3, 3, 1]$$

and the matchings $\delta_1 = \varepsilon_{11}, \delta_2 = \delta_\pi$ and

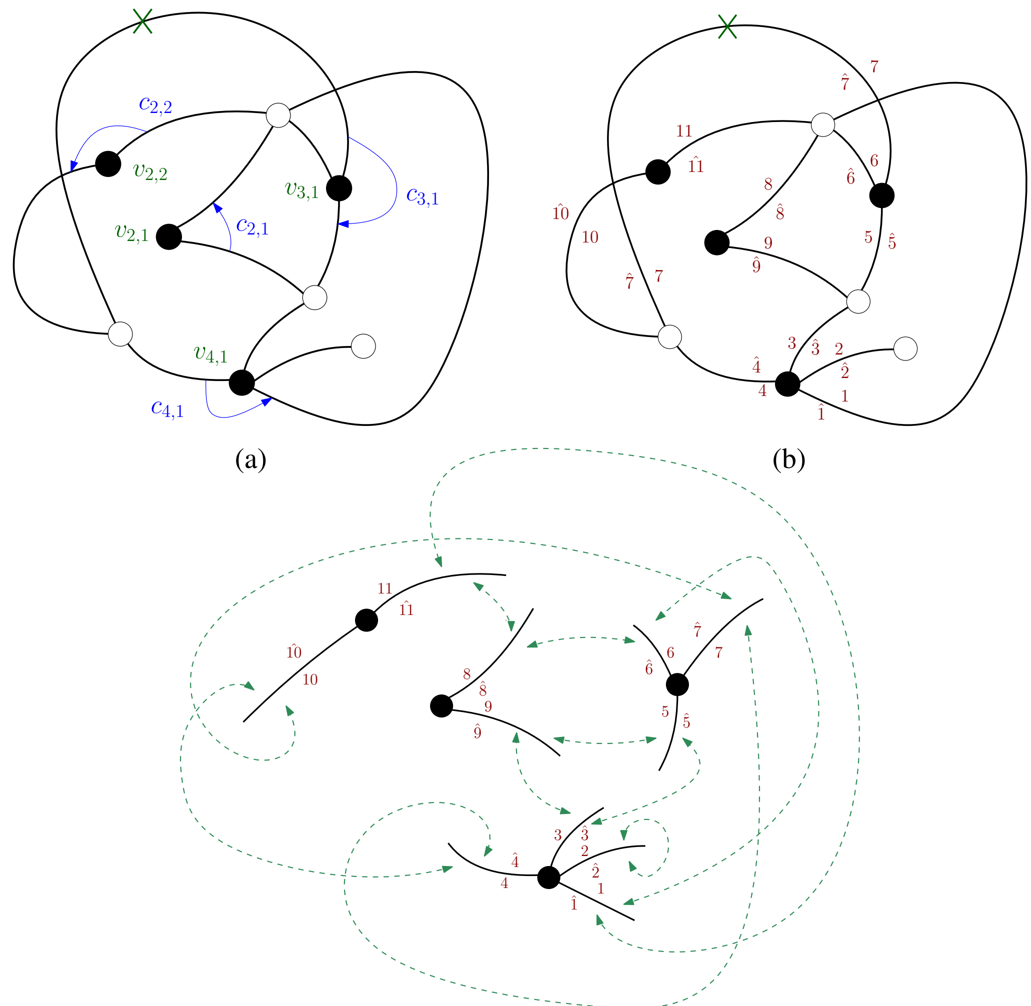
$$\delta = \big\{\{1, 6\}, \{2, \hat{2}\}, \{3, \hat{9}\}, \{4, \hat{10}\}, \{5, 9\}, \{7, \hat{4}\}, \{8, \hat{11}\}, \{10, \hat{7}\}, \{11, \hat{1}\}, \{\hat{3}, \hat{5}\}, \{\hat{6}, \hat{8}\}\big\}\,.$$

Usually, the bijection given in Proposition 1.1.10 is described with operations of gluing faces and not vertices; see *e.g.* [DFŚ14, Section 3.2]. Since we study here vertex-labelled maps we prefer this description (this is also related to the decomposition equations presented later in the thesis, in which we add vertices rather than faces; see *e.g.* Proposition 2.2.3). One can pass from one description to the other by duality operations.

Remark 1.1.12. Actually, the encoding of vertex-labelled bipartite maps with matchings explained in Proposition 1.1.10 induces the classical encoding of orientable bipartite maps with permutations stated in Proposition 1.1.5. Indeed, starting with an oriented vertex-labelled map of profile (π, ν, μ), the matching δ provided by the correspondence of Proposition 1.1.10 is bipartite and satisfies $\Lambda(\varepsilon, \delta) = \mu$ and $\Lambda(\delta_\pi, \delta) = \nu$. To such matching we associate the permutation σ defined by $\delta = \delta_\sigma$. The three permutations $(\sigma_\pi, \sigma, \sigma^{-1} \cdot \sigma_\pi)$ satisfy then $\mathrm{ct}(\sigma_\pi) = \pi$, $\mathrm{ct}(\sigma) = \mu$ and $\mathrm{ct}(\sigma^{-1} \cdot \sigma_\pi) = \nu$.

This bijection between maps and matchings provides two different points of view to study the same combinatorial structure. Indeed, some operations like edge deletion are more natural on maps (see Section 2.1), while their algebraic properties are easier to understand on matchings (see Section 3.5.1).

By Proposition 1.1.5, counting orientable maps corresponds to counting factorizations in the symmetric group as in Eq. (1.2). The latter is known to be related to the characters



(a) (b)

(c)

Figure 1.9: An example of the correspondence between vertex-labelled maps and matchings. Fig. 1.9a is a vertex-labelled map. Fig. 1.9b is the associated numbering of edge sides. Finally, Fig. 1.9c illustrates the map as a list of isolated black vertices equipped with a matching on their edge sides.

of the symmetric group by Frobenius's formula; see [LZ04, Theorem A.1.10]. Similarly, the cardinality of $\mathfrak{F}^{\delta_1,\delta_2}_{\mu,\nu}$ can be determined using the characters of the double coset algebra; [HSS92, Lemma 3.3]. Combining this with the correspondence between maps and matchings (Proposition 1.1.10) gives a formula for the generating series of orientable and general maps; see Section 1.3.1.

1.1.5 Constellations

A natural generalization of bipartite maps is given by k-*constellations*, which are a family of maps whose vertices are colored in $k+1$ colors. Bipartite maps correspond then to the case $k = 1$. In the orientable case, k-constellations are related to the factorizations of the identity in the symmetric group into $k+2$ permutations [BS00]. Recently, a model of constellations

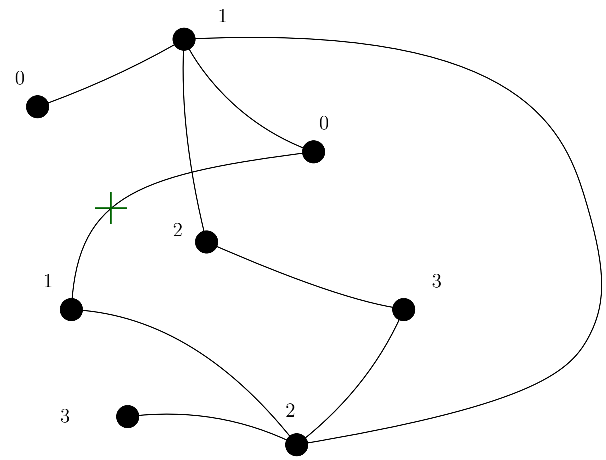


Figure 1.10: A non-orientable 3-constellation of size 3.

on non-orientable surfaces has been introduced in Chapuy and Dołęga in [CD22].

Definition 1.1.13 ([CD22, Definition 2.2]). *Let $k \geq 1$. A k-constellation is a map, connected or not, whose vertices are colored with colors $\{0, 1, ..., k\}$ such that*[3]*:*

1. *Each vertex of color 0 (respectively k) has only neighbors of color 1 (respectively $k-1$).*

2. *For $0 < i < k$, a vertex of color i has only neighbors of colors $i - 1$ and $i + 1$, and each corner of such vertex separates two vertices of colors $i - 1$ and $i + 1$.*

The size of a constellation is defined as the number of edges incident to a vertex of color 0.

An example of a 3-constellation of size 3 is illustrated in Figure 1.10. One can easily check that bipartite maps correspond to 1-constellations. As in the case of bipartite maps, to a k-constellation of size n we can associate a *profile*, defined as the $k + 2$-tuple of integer partitions $(\pi, \mu^0, ..., \mu^k)$ of size n, which contains respectively the information of the face degrees, as well as the degrees of the vertices of color i for $0 \leq i \leq k$.

The encoding of bipartite maps with matchings explained in Proposition 1.1.10 can be generalized to constellations; given two matchings δ_1 and δ_2, there is a correspondence between k-tuples of matchings and labelled k-constellations with 0-colored vertices of degrees $\Lambda(\delta_1, \delta_2)$. Furthermore, this correspondence preserves the notion of profile. We refer to [Ben22, Theorem 3.1] for a precise statement.

Remark 1.1.14. Constellations also appear in enumerative geometry. Namely, orientable constellations with control of all color types are in bijection with the ramified coverings of the sphere above an arbitrary number of points with the full ramification profiles; see [LZ04, Section 1.2]. We refer to [CD22, Section 2.2] for a similar result in the non-orientable case.

[3] We use here the convention of [CD22], what we call k-constellation is often called $k + 1$-constellation in the orientable case.

1.2 Symmetric functions and Jack polynomials

We are interested here in connections between generating series of maps and symmetric functions. In this section, we introduce the space of symmetric functions and we define Jack polynomials. We first start by giving some notation related to integer partitions.

1.2.1 Partitions

A *partition* $\lambda = [\lambda_1, ..., \lambda_k]$ is a weakly decreasing sequence of positive integers $\lambda_1 \geq ... \geq \lambda_k > 0$. We denote by $\mathbb{Y}$ the set of all integer partitions, including the empty partition. The integer k is called the *length* of λ and is denoted $\ell(\lambda)$. The size of λ is the integer $|\lambda| := \lambda_1 + \lambda_2 + ... + \lambda_k$. If n is the *size* of λ, we say that λ is a partition of n and we write $\lambda \vdash n$. The integers λ_1,...,λ_k are called the *parts* of λ. We denote by 2λ the partition given by $2\lambda := [2\lambda_1, ..., 2\lambda_k]$.

For $i \geq 1$, we denote $m_i(\lambda)$ the number of parts of size i in λ. We then set

$$z_\lambda := \prod_{i\geq 1} m_i(\lambda)! i^{m_i(\lambda)}. \tag{1.3}$$

One may notice that if σ is permutation with cycle type λ then the centraliser of σ in $\mathfrak{S}_{|\lambda|}$ has size z_λ.

We denote by $\leq$ the *dominance partial* ordering on partitions, defined by

$$\mu \leq \lambda \iff |\mu| = |\lambda|, \quad \text{and} \quad \mu_1 + ... + \mu_i \leq \lambda_1 + ... + \lambda_i \text{ for } i \geq 1. \tag{1.4}$$

For every partition λ and $i \geq 1$, we set $\lambda_i = 0$ if $i > \ell(\lambda)$.
We identify a partition λ with its *Young diagram*, defined by

$$\lambda := \{(i,j), 1 \leq j \leq \ell(\lambda), 1 \leq i \leq \lambda_j\}.$$

The conjugate partition of λ, denoted λ', is the partition associated to the Young diagram obtained by reflecting the diagram of λ with respect to the line $j = i$:

$$\lambda' := \{(i,j), 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}. \tag{1.5}$$

Fix a cell $\Box := (i,j) \in \lambda$. We define the *arm-length* of $\Box$ by

$$a_\lambda(\Box) := |\{(r,j) \in \lambda, r > i\}| = \lambda_j - i,$$

and its *leg-length* by

$$\ell_\lambda(\Box) := |\{(i,r) \in \lambda, r > j\}| = \lambda'_i - j.$$

See Fig. 1.11 for an example. The hook-length product of a Young diagram is defined by

$$H_\lambda := \prod_{\Box\in\lambda} (a_\lambda(\Box) + \ell_\lambda(\Box) + 1) \tag{1.6}$$

We now consider a formal parameter α. Stanley has introduced in [Sta89] two α-deformations of the hook-length product;

$$\mathrm{hook}_\lambda^{(\alpha)} := \prod_{\Box\in\lambda} (\alpha a_\lambda(\Box) + \ell_\lambda(\Box) + 1), \quad \mathrm{hook}_\lambda'^{(\alpha)} := \prod_{\Box\in\lambda} (\alpha(a_\lambda(\Box) + 1) + \ell_\lambda(\Box)).$$

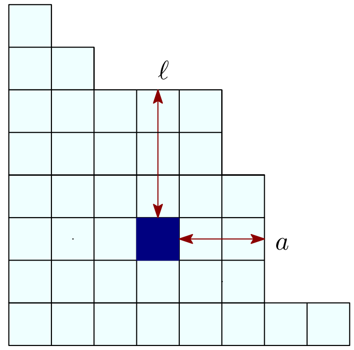


Figure 1.11: The Young diagram of the partition $\lambda = [8,6,6,6,5,5,2,1]$ using the French convention. The arrows illustrate the arm-length and the leg-length of the cell $\square = (4,3)$.

They both coincide with the classical hook-length product when $\alpha = 1$;

$$H_\lambda = \mathrm{hook}_\lambda^{(1)} = \mathrm{hook}_\lambda'^{(1)}.$$

Finally, we define the *α-content* of a cell $\square := (i,j)$ by

$$c_\alpha(\square) := \alpha(i-1) - (j-1). \tag{1.7}$$

1.2.2 The space of symmetric functions

We consider an infinite alphabet of variables $\boldsymbol{x} := (x_1, x_2, ..)$.

Definition 1.2.1. *We say that a polynomial f in k variables is symmetric if it is invariant under the action of the symmetric group $\mathfrak{S}_k$;*

$$f(x_1,\dots,x_k) = f(x_{\sigma(1)},\dots,x_{\sigma(k)}); \textit{ for any } \sigma \in \mathfrak{S}_k.$$

A symmetric function f is the projective limit of a sequence $(f_k)_{k\geq 1}$ of symmetric polynomials of bounded degrees, such that for every $k \geq 1$, the function f_k is a symmetric polynomial in k variables and

$$f_{k+1}(x_1,\dots,x_k,0) = f_k(x_1,\dots,x_k).$$

Moreover, f has degree n if the (f_k) have degree n for k large enough.

We denote by $\mathcal{S}$ the algebra of symmetric functions on $\mathbb{Q}$ and by $\mathcal{S}^{(n)}$ its subspace of functions of degree n, together with the zero function.

Example 1.2.2. The function $\sum_{1\leq i<j} x_i x_j$ is a symmetric function of degree 2. However, $\prod_{i\geq 1} x_i$ is not a symmetric function because it does not have finite degree.

For every partition λ of length k, we denote by m_λ the *monomial symmetric function* defined by

$$m_\lambda(\boldsymbol{x}) := \sum_{\beta=(\beta_1,\dots,\beta_k)} \sum_{1\leq i_1<\dots<i_k} x_{i_1}^{\beta_1} \dots x_{i_k}^{\beta_k},$$

where the sum is taken over all reorderings β of the partition λ. Moreover, let p_λ denote the *power-sum symmetric function*, defined as follows; if $n \geq 1$ then

$$p_n(\boldsymbol{x}) := \sum_{i \geq 1} x_i^n,$$

and if $\lambda = [\lambda_1, \ldots, \lambda_k]$ then

$$p_\lambda(\boldsymbol{x}) = p_{\lambda_1}(\boldsymbol{x}) \ldots p_{\lambda_k}(\boldsymbol{x}).$$

It is well known that $(m_\lambda)_{\lambda \vdash n}$ and $(p_\lambda)_{\lambda \vdash n}$ are both bases of $\mathcal{S}^{(n)}$ (see *e.g.* [Mac95, Section I. 2]).

Example 1.2.3. When $\lambda = [2, 2]$, we have

$$\begin{aligned} p_{[2,2]}(\boldsymbol{x}) &= \left(\sum_{i \geq 1} x_i^2\right)\left(\sum_{j \geq 1} x_j^2\right) \\ &= \sum_{i \geq 1} x_i^4 + \sum_{\substack{i,j \geq 1 \\ i \neq j}} x_i^2 x_j^2 \\ &= m_{[4]}(\boldsymbol{x}) + 2m_{[2,2]}(\boldsymbol{x}). \end{aligned}$$

We now consider a parameter α and the space $\mathcal{S}_\alpha := \mathbb{Q}(\alpha) \otimes \mathcal{S}$ of symmetric functions with rational coefficients in α.

Since power-sum functions are a basis of the symmetric functions algebra, $\mathcal{S}_\alpha$ can be identified with the polynomial algebra $\mathcal{P} := \mathrm{Span}_{\mathbb{Q}(\alpha)}\{p_\lambda\}_{\lambda \in \mathbb{Y}}$. If f is a symmetric function in the alphabet $\boldsymbol{x}$, it will be convenient where there is non ambiguity to denote with the same letter the function and the associated polynomial in the alphabet of power-sum functions $\boldsymbol{p} := (p_1, p_2, ..)$;

$$f(\boldsymbol{x}) \equiv f(\boldsymbol{p}). \tag{1.8}$$

We consider the scalar product $\langle ., . \rangle$ on $\mathcal{S}_\alpha$ defined by

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \alpha^{\ell(\lambda)} \delta_{\lambda,\mu}, \tag{1.9}$$

for any partitions λ, μ, where $\delta_{\lambda,\mu}$ denotes the Kronecker delta, and z_λ is the factor defined in Eq. (1.3). This scalar product is an α-deformation of the *Hall scalar product*, obtained by setting $\alpha = 1$.

If $f \in \mathcal{S}_\alpha$, then we denote $f^\perp$ the dual of the multiplication by f, with respect to this scalar product.

Lemma 1.2.4. *For any $n \geq 1$, we have*

$$p_n^\perp = \alpha \frac{n\partial}{\partial p_n}.$$

More generally, for any partition μ

$$p_\mu^\perp = \alpha^{\ell(\mu)} \frac{\mu_1 \partial}{\partial p_{\mu_1}} \cdots \frac{\mu_{\ell(\mu)} \partial}{\partial p_{\mu_{\ell(\mu)}}}$$

Proof. One can check that

$$\langle p_n p_\mu, p_\nu \rangle = \left\langle p_\mu, \alpha \frac{n\partial}{\partial p_n} p_\nu \right\rangle,$$

for any partitions μ and ν. Since power-sum functions are a basis of $\mathcal{S}_\alpha$, we obtain the first equation of the lemma. In order to get the second one, we use the fact that for any symmetric functions f and g we have

$$(f \cdot g)^\perp = g^\perp \cdot f^\perp. \qquad \square$$

1.2.3 Jack polynomials

Jack polynomials $J_\lambda^{(\alpha)}$ are symmetric functions indexed by an integer partition λ and a deformation parameter α. They interpolate, up to scaling factors, between Schur functions for $\alpha = 1$ and zonal polynomials for $\alpha = 2$. Originally, they were introduced by Jack in [Jac71] as an important tool in statistics, but it turned out that they appear quite naturally in many different contexts: they play a crucial role in studying various models of statistical mechanics and probability such as β-ensembles and generalizations of Selberg integrals [OO97, Kad97, Joh98, DE02, Meh04, For10]. Furthermore, they are strongly related to the Calogero–Sutherland model from quantum mechanics [LV95] and to random partitions [BO05, DF16, BGG17, Mol15, DŚ19]. Finally, they were found to have a rich combinatorial structure [Sta89, Mac95, GJ96a, KS97, CD22, Mol23].

Macdonald has established the following characterization theorem for Jack polynomials, which we take as a definition.

Theorem 1.2.5 ([Mac95, Chapter VI, Section 10])**.** *Jack polynomials $(J_\lambda^{(\alpha)})_{\lambda \in \mathbb{Y}}$ are the unique family of symmetric functions in $\mathcal{S}_\alpha$ indexed by partitions, satisfying the following properties:*

- *Orthogonality: $\langle J_\lambda^{(\alpha)}, J_\mu^{(\alpha)} \rangle_\alpha = 0$, for $\lambda \neq \mu$.*
- *Triangularity: there exist coefficients $a_\nu^\lambda \in \mathbb{Q}(\alpha)$, such that*

$$J_\lambda^{(\alpha)} = \sum_{\nu \leq \lambda} a_\nu^\lambda m_\nu,$$

  where $\leq$ is the dominance order (see Eq. (1.4)).

- *Normalization:*

$$[p_{1^n}] J_\lambda^{(\alpha)} = 1, \tag{1.10}$$

  where $[p_{1^n}] J_\lambda^{(\alpha)}$ denotes the coefficient of p_{1^n} in the power-sum expansion of $J_\lambda^{(\alpha)}$.

Moreover, Jack polynomials form a basis of $\mathcal{S}_\alpha$.

Actually, the "existence" part in this theorem is remarkable, while "the "uniqueness" is not difficult. This definition can be used in practice to generate Jack polynomials using a Gram-Schmidt orthogonalization process.

Schur functions are recovered by specializing Jack polynomials at $\alpha = 1$;

$$J_\lambda^{(1)} = H_\lambda s_\lambda, \tag{1.11}$$

where H_λ is the hook product defined in Eq. (1.6). Another special case of Jack polynomials is when $\alpha = 2$ for which we obtain zonal polynomials Z_λ;

$$J_\lambda^{(2)} = Z_\lambda. \tag{1.12}$$

Schur and zonal functions are two well studied families of symmetric functions related to the theory of random matrices and their zonal spherical functions in the complex and the real case respectively [HSS92, Mac95].

We denote by $j_\lambda^{(\alpha)}$ the squared-norm of $J_\lambda^{(\alpha)}$;

$$j_\lambda^{(\alpha)} := \langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)} \rangle_\alpha. \tag{1.13}$$

This norm has the following combinatorial formula (see [Sta89, Theorem 5.8]).

$$j_\lambda^{(\alpha)} := \mathrm{hook}_\lambda^{(\alpha)} \cdot \mathrm{hook}_\lambda^{\prime(\alpha)}. \tag{1.14}$$

In particular, we have

$$j_\lambda^{(1)} = H_\lambda^2 \quad \text{and} \quad j_\lambda^{(2)} = H_{2\lambda}. \tag{1.15}$$

If u is a variable, then we define the alphabet $\underline{u} := (u, u, \dots)$. Then $\mathbb{Q}(\alpha)[\![u]\!]$ is the space of formal power series in u with coefficients in $\mathbb{Q}(\alpha)$. We consider the following specialization from $\mathcal{S}_\alpha$ to $\mathbb{Q}(\alpha)[\![u]\!]$

$$f(\underline{u}) := f(\boldsymbol{p})\big|_{p_1 = p_2 = \dots = u}. \tag{1.16}$$

In particular we have

$$p_\mu(\underline{u}) = u^{\ell(\mu)},$$

for any partition μ. The following theorem, due to Macdonald, gives a combinatorial formula for Jack polynomials under this specialization.

Theorem 1.2.6 ([Mac95, Chapter VI, Eq. (10.25)])**.** *For every $\lambda \in \mathbb{Y}$, we have*

$$J_\lambda^{(\alpha)}(\underline{u}) = \prod_{\square \in \lambda} (u + c_\alpha(\square)).$$

Example 1.2.7. For $\lambda = [2, 2]$, one has[4]

$$J_{[2,2]}(\boldsymbol{p}) = p_1^4 + 2(\alpha - 1)p_2 p_1^2 - 4\alpha p_3 p_1 + (\alpha^2 + \alpha + 1)p_2^2 + (-\alpha^2 + \alpha)p_4.$$

Hence,

$$\begin{aligned} J_{[2,2]}(\underline{u}) &= u^4 + 2(\alpha - 1)u^3 + (\alpha^2 - 3\alpha + 1)u^2 + (-\alpha^2 + \alpha)u \\ &= u(u-1)(u+\alpha)(u+\alpha-1). \end{aligned}$$

Moreover, the diagram of $[2, 2]$ has 4 cells of α-contents 0, -1, α and $\alpha - 1$. Then,

$$\prod_{\square \in [2,2]} (u + c_\alpha(\square)) = u(u-1)(u+\alpha)(u+\alpha-1).$$

[4] Such a formula can be obtained using a computer program such as Maple or SageMath.

Knop and Sahi have given in [KS97] a combinatorial interpretation for the coefficients of the Jack polynomial $J_\lambda^{(\alpha)}$ in the monomial basis in terms of tableaux of shape λ. More recently, other formulas have been obtained by Haglund and Wilson in [HW20] for the expansion of Jack polynomials in Schur functions and power-sum functions. These formulas are given in terms of inversion graphs of tableaux counted with some α-deformed hook weights.

1.2.4 The Laplace–Beltrami Operator

Another remarkable property of Jack polynomials is that they are eigenfunctions of *the Laplace–Beltrami operator*; that is the operator on $\mathcal{S}_\alpha$ defined by

$$D^{(\alpha)} = \frac{1}{2}\left(\alpha \sum_{i,j\geq 1} p_{i+j}\frac{ij\partial^2}{\partial p_i \partial p_j} + \sum_{i,j\geq 1} p_i p_j \frac{(i+j)\partial}{\partial p_{i+j}} + (\alpha-1)\cdot \sum_{i\geq 1} p_i \frac{i(i-1)\partial}{\partial p_i}\right). \quad (1.17)$$

Remark 1.2.8. One can already notice that $D^{(\alpha)}$ is positive in the parameter $b := \alpha - 1$ but not in α.

This operator can be thought of as a defining operator for the Jack polynomials.

Proposition 1.2.9 ([Sta89]). *Jack polynomials are the unique family of symmetric functions such that for each $\lambda \in \mathbb{Y}$,*

- $D^{(\alpha)} J_\lambda^{(\alpha)} = \left(\sum_{\square\in\lambda} c_\alpha(\square)\right) J_\lambda^{(\alpha)}$;
- *there exist coefficients $a_\nu^\lambda \in \mathbb{Q}(\alpha)$ such that*

$$J_\lambda^{(\alpha)} = \mathrm{hook}_\lambda^{(\alpha)} m_\lambda + \sum_{\nu<\lambda} a_\nu^\lambda m_\nu.$$

Actually, the proof of Theorem 1.2.6 is based on this characterization, see also [Mac95] and [CD22, Definition-Proposition 5.1]. A third characterization of Jack polynomials will be given using shifted symmetric functions in Theorem 4.1.4.

Proposition 1.2.9 will not be directly used here, but it has been an important tool in the work of Chapuy and Dołęga [CD22] in which they introduce a new family of operators connecting Jack polynomials to generating series of maps. These operators will play a crucial role in this thesis; see Section 1.4.2 and Section 2.2.

1.3 Two families of conjectures

The interaction between symmetric functions and the enumeration of maps has showed itself to be enlightening both from the map and the symmetric function perspectives. We are interested here in problems suggesting that there are connections between the expansion of Jack polynomials in the power-sum basis and the enumeration of maps. These problems are mainly given by two families of conjectures;

- in one direction we want to find a generating series of bipartite maps with controlled profiles which has an expansion using Jack polynomials. This is suggested by Goulden–Jackson's conjectures; see Section 1.3.1.
- in an another direction, we would like to find a combinatorial formula for Jack polynomials in terms of maps. This is related to combinatorial conjectures of Hanlon [Han88] and Dołęga–Féray–Śniady [DFŚ14] and a positivity conjecture of Lassalle [Las08a]. This will be presented in Section 1.3.2.

We will see throughout this thesis that these two families of conjectures are closely related.

Actually, these problems aim to generalize known results for Schur and zonal functions, which correspond respectively to Jack polynomials for $\alpha = 1$ and $\alpha = 2$. Indeed, representation theory can be used to relate Schur functions to generating series of orientable maps in one hand [JV90, Fér10, FŚ11a], and zonal functions to generating series of non-orientable maps [GJ96b, FŚ11b]. The case of Jack polynomials for general α seems then more challenging because the tools of representation theory do not exist, which requires the development of new techniques.

1.3.1 Goulden–Jackson conjectures

Generating series with control of the profile

We now introduce the first type of generating series of maps which will be studied in this thesis. In these series, we control the three partitions of the profile $(\lambda^\bullet(M), \lambda^\diamond(M), \lambda^\circ(M))$.

To this purpose, we consider one variable t and three infinite alphabets of variables $\boldsymbol{p} := (p_1, p_2, \dots)$, $\boldsymbol{q} := (q_1, q_2, \dots)$ and $\boldsymbol{r} := (r_1, r_2, \dots)$. We associate to an edge the weight t, to a face of degree $2k$ the weight q_k and to a white vertex (resp. black vertex) of degree k the weight p_k (resp. r_k). Hence, a bipartite map M has the weight

$$t^{|M|} p_{\lambda^\bullet(M)} q_{\lambda^\diamond(M)} r_{\lambda^\circ(M)},$$

where if λ is a partition, we write $p_\lambda := p_{\lambda_1} p_{\lambda_2} \dots$, and use the same notation for $\boldsymbol{q}$ and $\boldsymbol{r}$. We can think of $\boldsymbol{p}$ as the "power-sum alphabet" in an underlying alphabet $\boldsymbol{x}$ as in Section 1.2.2. We do the same for the alphabets $\boldsymbol{q}$ and $\boldsymbol{r}$ which will be power-sum alphabets in two new underlying alphabets $\boldsymbol{y}$ and $\boldsymbol{z}$.

Counting maps with a prescribed profile is a hard combinatorial problem (no bijections or decomposition equations are known). However, combining the correspondence between maps and permutations given by Proposition 1.1.5, and a formula due to Frobenius for the characters of the symmetric group, it is possible to write the generating series of bipartite maps using a change of basis from Schur symmetric functions to power-sum symmetric functions. More precisely, we have the following formula (see [JV90])

$$\sum_{\lambda \in \mathbb{Y}} t^{|\lambda|} H_\lambda s_\lambda(\boldsymbol{p}) s_\lambda(\boldsymbol{q}) s_\lambda(\boldsymbol{r}) = \sum_M \frac{t^{|M|}}{z_{\lambda^\bullet(M)}} p_{\lambda^\bullet(M)} q_{\lambda^\diamond(M)} r_{\lambda^\circ(M)}, \tag{1.18}$$

where the sum in the right-hand side is taken over oriented vertex-labelled bipartite maps. We recall that, $s_\lambda(\boldsymbol{p})$, $s_\lambda(\boldsymbol{q})$ and $s_\lambda(\boldsymbol{r})$ denote Schur functions in three independent power-sum bases.

Remark 1.3.1. Note that the coefficient $z_{\lambda^\bullet(M)}$ in the denominator is related to the labelling convention we are considering here; z_λ corresponds to the number of ways of labelling the black vertices of an oriented map (see Definition 1.1.1) without taking account of possible symmetries (see also Lemma 1.3.4 below).

Remark 1.3.2. By Remark 1.1.3, the generating series of simple maps can be obtained from the series of Eq. (1.18) by taking the specialization $r_2 = 1$ and $r_i = 0$ for $i \neq 2$.

The series of Eq. (1.18) is actually related to the generating series of weighted Hurwitz numbers [GPH17], with strong links to the topological recursion [ACEH18, BDBKS20].

One can now deduce from Eq. (1.18) a formula for the generating series of connected bipartite maps by taking the logarithm. More precisely,

$$\log\left(\sum_{\lambda\in\mathbb{Y}} t^{|\lambda|} H_\lambda s_\lambda(\boldsymbol{p}) s_\lambda(\boldsymbol{q}) s_\lambda(\boldsymbol{r})\right) = \sum_M \frac{t^{|M|}}{|M|} p_{\lambda^\bullet(M)} q_{\lambda^\circ(M)} r_{\lambda^\diamond(M)}, \tag{1.19}$$

where the sum in the right-hand side is now taken over oriented rooted connected bipartite maps.

Remark 1.3.3. Usually, we consider generating series with a fixed genus of the surface [LW72a, BC86, BC91, CFF13, AL20]. Actually, when we work with generating series of connected maps with controlled profile, the genus is implicitly controlled by the Euler formula (Eq. (1.1)).

The equivalence between Eq. (1.18) and Eq. (1.19) can be checked using the following lemma.

Lemma 1.3.4. *Fix three partitions π,μ,ν of the same size $n \geq 1$. Let $\mathrm{VL}^\pi_{\mu,\nu}$ (resp. $\mathrm{R}^\pi_{\mu,\nu}$) denote the number of oriented vertex-labelled (resp. oriented rooted) connected maps of profile (π, μ, ν). Then*

$$\frac{\mathrm{VL}^\pi_{\mu,\nu}}{z_\pi} = \frac{\mathrm{R}^\pi_{\mu,\nu}}{n}.$$

Proof. We count in two different ways the number of connected oriented maps of profile (π, μ, ν) which are both rooted and vertex-labelled. We can start from a rooted map and then fix a vertex labelling; there are $z_\pi \cdot \mathrm{R}^\pi_{\mu,\nu}$ choices. But we can also start from a vertex-labelled map and then choose a root, there are $n \cdot \mathrm{VL}^\pi_{\mu,\nu}$ choices. This finishes the proof of the lemma. □

Remark 1.3.5. Note that, because of symmetries, the quotient $\frac{\mathrm{VL}^\pi_{\mu,\nu}}{z_\pi}$ does not correspond to the number of connected maps with profile (π, μ, ν) and without any labelling or rooting. For example, when $\pi = [1, 1]$ and $\mu = \nu = [2]$, one can check that $z_\pi = 2$ and $\mathrm{VL}^\pi_{\mu,\nu} = 1$, and in particular the quotient is not integer.

Remark 1.3.6. One may notice that a "minimal" notion of rooting for non-connected maps would be to root each one of the connected component instead of labelling vertices. However, the normalization factor in the associated generating series would depend on the sizes of the connected components of the map (instead of normalizing by $z_{\lambda^\bullet}$ as in Eq. (1.18)). But unlike the vertex degrees, the connected component sizes are not controlled by the profile. This explains the normalization by $z_{\lambda^\bullet}$ introduced by Goulden–Jackson in [GJ96a].

Using zonal polynomials, Goulden and Jackson have given in [GJ96b] an analogous expression of Eqs. (1.18) and (1.19) for the generating series of maps on non-orientable surfaces.

Theorem 1.3.7 ([GJ96b, Corollary 4.4])**.** *We have,*

$$\sum_{\lambda\in\mathbb{Y}} t^{|\lambda|}\frac{Z_\lambda(\boldsymbol{p})Z_\lambda(\boldsymbol{q})Z_\lambda(\boldsymbol{r})}{H_{2\lambda}} = \sum_{M}\frac{t^{|M|}}{2^{|\mathcal{V}_\bullet(M)|}z_{\lambda^\bullet(M)}}p_{\lambda^\bullet(M)}q_{\lambda^\circ(M)}r_{\lambda^\diamond(M)}, \tag{1.20}$$

where the sum is taken over vertex-labelled bipartite maps, oriented or not. Equivalently,

$$2\log\left(\sum_{\lambda\in\mathbb{Y}} t^{|\lambda|}\frac{Z_\lambda(\boldsymbol{p})Z_\lambda(\boldsymbol{q})Z_\lambda(\boldsymbol{r})}{H_{2\lambda}}\right) = \sum_{M}\frac{t^{|M|}}{|M|}p_{2\lambda^\diamond(M)}q_{\lambda^\bullet(M)}r_{\lambda^\circ(M)}, \tag{1.21}$$

where the sum runs over rooted connected bipartite maps, oriented or not.

This result is based on the encoding of maps with matchings and on the representation theory of the Gelfand pair $(\mathfrak{S}_{2n},\mathfrak{B}_n)$, where $\mathfrak{B}_n$ denotes the hyperoctahedral group, see also [HSS92].

The disadvantage of this representation theory approach is that it is quite rigid and it is hard to generalize to the case of generating series of maps with additional weights.

In Section 3.5, we give a new proof of Eqs. (1.18) to (1.21) which does not use representation theory.

The function $\tau^{(\alpha)}$

Using Jack polynomials, Goulden and Jackson have introduced in [GJ96a] an α-deformation of the generating series of bipartite maps:

$$\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) := \sum_{n\geq 0} t^n \sum_{\xi\vdash n}\frac{1}{j_\lambda^{(\alpha)}}J_\lambda^{(\alpha)}(\boldsymbol{p})J_\lambda^{(\alpha)}(\boldsymbol{q})J_\lambda^{(\alpha)}(\boldsymbol{r}) \in \mathbb{Q}(\alpha)[\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}][\![t]\!], \tag{1.22}$$

where $j_\lambda^{(\alpha)}$ is the square norm of the Jack polynomial $J_\lambda^{(\alpha)}$, see Eq. (1.13).

The function $\tau^{(\alpha)}$ has been introduced as an interpolation between the generating series of bipartite maps on orientable surfaces (Eq. (1.18)) and general surfaces (Eq. (1.20)), obtained by setting respectively the parameter α to 1 and 2; see Eqs. (1.11), (1.12) and (1.15). This function is related to the β-ensembles of random matrix theory, which can be obtained by some specializations of the variables $\boldsymbol{q}$ and $\boldsymbol{r}$, see [La 09, CD22]. Under some specializations, the function $\tau^{(\alpha)}$ satisfies a family of partial differential equations known in theoretical physics as the Virasoro constraints, see [KZ15, BCD23]. It has been also used in [CD22] to define a deformation of Hurwitz numbers (see Remark 1.3.15 below).

We now state the two conjectures of Goulden and Jackson relating Jack polynomials to maps.

The b-conjecture

We consider the series

$$\widehat{\tau}^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) = \alpha \log\left(\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\right) \tag{1.23}$$

For any partitions π,μ and ν of the same size, we define coefficients $h^{\pi}_{\mu,\nu}(\alpha)$ as the rational coefficients in α obtained by the expansion;

$$\widehat{\tau}^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) = \sum_{n\geq 1}\frac{t^n}{n}\sum_{\pi,\mu,\nu\vdash n} h^{\pi}_{\mu,\nu}(\alpha) p_\pi q_\mu r_\nu. \tag{1.24}$$

With this notation, Theorem 1.3.7 can be reformulated as follows.

$$h^{\pi}_{\mu,\nu}(1) = \left|\{\text{Oriented rooted connected bipartite maps of profile } (\pi,\mu,\nu)\}\right|, \tag{1.25}$$

$$h^{\pi}_{\mu,\nu}(2) = \left|\{\text{Rooted connected bipartite maps of profile } (\pi,\mu,\nu),\ \text{oriented or not}\}\right|. \tag{1.26}$$

We now state the b-conjecture of Goulden–Jackson.

Conjecture 1 (b-conjecture. [GJ96a, Conjecture 6.2])**.** *Fix $n\geq 1$. For any partitions π,μ,ν of size n, the coefficient $h^{\pi}_{\mu,\nu}$ is a polynomial in $b := \alpha - 1$ with non-negative integer coefficients.*

In order to give a combinatorial reformulation of this conjecture, Goulden and Jackson have introduced the following definition.

Definition 1.3.8 ([GJ96a])**.** *A statistic of non-orientability (*SON*) on bipartite maps is a statistic ϑ with non-negative integer values, such that $\vartheta(M) = 0$ if and only if M is orientable.*

Goulden and Jackson have conjectured that the coefficients $h^{\pi}_{\mu,\nu}$ still count maps for any b.

Conjecture 2 (b-conjecture; combinatorial reformulation. [GJ96a, Conjecture 6.3])**.** *There exists a statistic of non-orientability ϑ on rooted connected bipartite maps, such that for any partitions π,μ and ν*

$$h^{\pi}_{\mu,\nu}(\alpha) = \sum_{M} b^{\vartheta(M)},$$

where the sum is taken over all rooted connected bipartite maps of profile (π,μ,ν).

Given Eqs. (1.25) and (1.26), one can see that Conjecture 1 and Conjecture 2 are equivalent. In addition to the special cases $b = 0$ and $b = 1$ that follow from connections with representation theory, several partial results related to the b-conjecture have been established [La 09, DF16, Doł17, DF17, CD22]. We present here some of them.

It follows from the definitions that the coefficients $h^{\pi}_{\mu,\nu}$ are rational functions in b. The polynomiality is however not immediate and has been proved by Dołęga and Féray.

Theorem 1.3.9 ([DF17, Theorem 1.2])**.** *For all partitions $\pi,\mu,\nu\vdash n\geq 1$, the coefficient $h^{\pi}_{\mu,\nu}$ is a polynomial in b with rational coefficients.*

Another important recent progress in the direction of the b-conjecture is related to marginal sums which we now explain. Chapuy and Dołęga have considered in [CD22] the following specialization of the series $\tau^{(\alpha)}$;

$$\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\underline{u}) := \sum_{\xi\in\mathbb{Y}} t^{|\xi|}\frac{J_\xi^{(\alpha)}(\boldsymbol{p})J_\xi^{(\alpha)}(\boldsymbol{q})J_\xi^{(\alpha)}(\underline{u})}{j_\xi^{(\alpha)}} \in \mathbb{Q}(\alpha)[\boldsymbol{p},\boldsymbol{q},u][\![t]\!], \tag{1.27}$$

obtained from Eq. (1.22) by specializing the alphabet $\boldsymbol{r}$ as in Eq. (1.16). We consider the coefficients $\bar{h}^\pi_{\mu,k}$ defined for any partitions of the same size $\pi,\mu\vdash n\geq 1$, and any integer $1\leq k\leq n$, by the expansion

$$\widehat{\tau}^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\underline{u}) = \sum_{n\geq 1}\frac{t^n}{n}\sum_{\pi,\mu\vdash n}\sum_{1\leq k\leq n}\bar{h}^\pi_{\mu,k}(\alpha)p_\pi q_\mu u^k. \tag{1.28}$$

Equivalently, these coefficients can be written as marginal sums of the coefficients $h^\pi_{\mu,\nu}$:

$$\bar{h}^\pi_{\mu,k} = \sum_{\substack{\nu\vdash n\\ \ell(\nu)=k}} h^\pi_{\mu,\nu}.$$

From the point of view of maps, this specialization sends the weight $p_{\lambda^\bullet(M)}q_{\lambda^\diamond(M)}r_{\lambda^\circ(M)}$ onto $p_{\lambda^\bullet(M)}q_{\lambda^\diamond(M)}u^{|\mathcal{V}_\circ(M)|}$. In other terms, we control the face-type, the black-type and the number of white vertices. Chapuy and Dołęga have proved the b-conjecture under this specialization.

Theorem 1.3.10 ([CD22, Theorem 5.10])**.** *Fix $n,m\geq 1$. For any partitions $\pi,\mu\vdash n$, the marginal coefficient $\bar{h}^\pi_{\mu,m}$ is a polynomial in b with non-negative integer coefficients. Moreover, there exists an explicit* SON *ϑ on rooted connected bipartite maps, such that*

$$\bar{h}^\pi_{\mu,m} = \sum_M b^{\vartheta(M)},$$

where the sum is taken over rooted connected bipartite maps M such that $\lambda^\diamond(M)=\pi$, $\lambda^\bullet(M)=\mu$ and $|\mathcal{V}_\circ(M)|=m$.

In order to obtain this result, the authors develop a new approach based on differential calculus. More precisely, they introduce a new family of differential operators which are used to "construct" generating series of maps and which act nicely on Jack polynomials (see Section 1.4.2 for the definition of these operators).

The Matching-Jack conjecture

The Matching-Jack conjecture can be thought of as a "disconnected" version of the b-conjecture.

We start by defining the coefficients $c^\pi_{\mu,\nu}(\alpha)$ by the following expansion of $\tau^{(\alpha)}$ in the power-sum basis:

$$\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) = 1+\sum_{n\geq 1}t^n\sum_{\pi,\mu,\nu\vdash n}\frac{c^\pi_{\mu,\nu}(\alpha)}{z_\pi\alpha^{\ell(\pi)}}p_\pi q_\mu r_\nu. \tag{1.29}$$

With this notation, Eqs. (1.18) and (1.20) can be written as follows;

$$c^{\pi}_{\mu,\nu}(1) = |\{\text{Oriented vertex-labelled bipartite maps of profile } (\pi,\mu,\nu)\}|, \tag{1.30}$$

$$c^{\pi}_{\mu,\nu}(2) = |\{\text{vertex-labelled bipartite maps of profile } (\pi,\mu,\nu), \text{ oriented or not}\}|, \tag{1.31}$$

for any partitions π,μ,ν of the same size n. Moreover, using the encoding of maps with matchings (Proposition 1.1.10), these equations become

$$c^{\pi}_{\mu,\nu}(1) = c^{\pi}_{\nu,\mu}(1) = \left|\widetilde{\mathfrak{F}}^{\delta_1,\delta_2}_{\mu,\nu}\right|,$$

$$c^{\pi}_{\mu,\nu}(2) = c^{\pi}_{\nu,\mu}(2) = \left|\mathfrak{F}^{\delta_1,\delta_2}_{\mu,\nu}\right|,$$

where δ_1 and δ_2 are two matchings on $\mathcal{N}_n$ satisfying $\Lambda(\delta_1,\delta_2) = \pi$.

Remark 1.3.11. Note that unlike the coefficients $h^{\pi}_{\mu,\nu}$, the coefficients $c^{\pi}_{\mu,\nu}$ are not completely symmetric in the variables π, μ and ν. Combinatorially, this asymmetry is related to the notion of vertex-labelling used here.

The coefficients $c^{\pi}_{\mu,\nu}(\alpha)$ are the main objects of the Matching-Jack conjecture, formulated by Goulden and Jackson.

Conjecture 3 (Matching-Jack conjecture. [GJ96a, Conjecture 3.5])**.** *Fix $n \geq 1$. For any partitions $\pi,\mu,\nu \vdash n$, the coefficient $c^{\pi}_{\mu,\nu}$ is polynomial in the parameter $b := \alpha - 1$ with non-negative integer coefficients.*

This conjecture has the following combinatorial reformulation.

Conjecture 4 (Matching-Jack conjecture; combinatorial formulation. [GJ96a, Conjecture 4.2])**.** *Fix a partition $\pi \vdash n \geq 1$, and two bipartite matchings δ_1 and δ_2 on $\mathcal{N}_n$ such that $\Lambda(\delta_1,\delta_2) = \pi$. There exists a statistic $\mathrm{st}_{\delta_1,\delta_2}$ on the matchings of $\mathcal{N}_n$ with non-negative integer values, such that*

- $\mathrm{st}_{\delta_1,\delta_2}(\delta) = 0$ *if and only if δ is bipartite.*
- *for any partitions $\mu,\nu \vdash n$, we have*

$$c^{\pi}_{\mu,\nu} = \sum_{\delta \in \mathfrak{F}^{\delta_1,\delta_2}_{\mu,\nu}} b^{\mathrm{st}_{\delta_1,\delta_2}(\delta)}.$$

In order to reformulate this conjecture in terms of maps, we introduce the following definition, which is a variant of Definition 1.3.8.

Definition 1.3.12. *A strong statistic of non-orientability (*SSON*) on vertex-labelled bipartite maps is a statistic $\boldsymbol{\vartheta}$ with non-negative integer values, such that $\boldsymbol{\vartheta}(M) = 0$ if and only if M is oriented.*

The term "strong" used here is related to the fact that we are taking into account the orientation of the labelling and not only the orientability of the map. SSONs will always be denoted with a bold letter in order to differentiate them from SONs. Conjecture 3 has then the following equivalent formulation.

Figure 1.12: The four vertex-labelled maps of profile $([3],[3],[3])$. On the left the only orientable (and oriented) vertex-labelled map of profile $([3],[3],[3])$. On the right the only non-orientable map with the three different vertex-labelling given by the roots c_1, c_2 and c_3 (since the map has exactly one black vertex, fixing a vertex-labelling corresponds to marking an oriented corner on the black vertex).

Conjecture 5 (Matching-Jack conjecture; second combinatorial formulation)**.** *There exists a* SSON $\boldsymbol{\vartheta}$ *on vertex-labelled bipartite maps, such that for any partitions* π, μ *and* ν *of the same size*

$$c^{\pi}_{\mu,\nu}(\alpha) = \sum_{M} b^{\boldsymbol{\vartheta}(M)},$$

where the sum is taken over all vertex-labelled bipartite maps of profile (π,μ,ν).

Example 1.3.13. We give the first four terms of the expansion of $\tau^{(\alpha)}$ as a formal power series in t;

$$\begin{aligned}\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) =& 1 + t\cdot\frac{p_1q_1r_1}{\alpha} + t^2\left(b\cdot\frac{p_2q_2r_2}{2\alpha} + \frac{p_2q_2r_{1,1}}{2\alpha} + \frac{p_2q_{1,1}r_2}{2\alpha} + (1+b)\cdot\frac{p_{1,1}q_2r_2}{2\alpha^2}\right.\\ &\left.+(1+b)\cdot\frac{p_{1,1}q_{1,1}r_{1,1}}{2\alpha^2}\right) + t^3\left((2b^2+b+1)\cdot\frac{p_3q_3r_3}{3\alpha} + 3b\cdot\frac{p_3q_3r_{2,1}}{3\alpha}\right.\\ &+ 3b\cdot\frac{p_3q_{2,1}r_3}{3\alpha} + \frac{p_3q_3r_{1,1,1}}{3\alpha} + \frac{p_3q_{1,1,1}r_3}{3\alpha} + 3\cdot\frac{p_3q_{2,1}r_{2,1}}{3\alpha}\\ &+ 2(b^2+b)\cdot\frac{p_{2,1}q_3r_3}{2\alpha^2} + 2b\cdot\frac{p_{2,1}q_3r_{2,1}}{2\alpha^2} + 2b\cdot\frac{p_{2,1}q_{2,1}r_3}{2\alpha^2} + b\cdot\frac{p_{2,1}q_{2,1}r_{2,1}}{2\alpha^2}\\ &+ \frac{p_{2,1}q_{2,1}r_{1,1,1}}{2\alpha^2} + \frac{p_{2,1}q_{1,1,1}r_{2,1}}{2\alpha^2} + 2(b+1)^2\cdot\frac{p_{1,1,1}q_3r_3}{6\alpha^3}\\ &\left.+3(b+1)\cdot\frac{p_{1,1,1}q_{2,1}r_{2,1}}{6\alpha^3} + \frac{p_{1,1,1}q_{1,1,1}r_{1,1,1}}{6\alpha^3}\right) + o(t^3).\end{aligned}$$

In particular,

$$c^{[3]}_{[3],[3]} = 2b^2 + b + 1.$$

The four monomials of this coefficient correspond to the four vertex-labelled bipartite maps of profile $([3],[3],[3])$; see Fig. 1.12. A SSON answering Conjecture 5 should associate the value 0 to the map of Fig. 1.12a, the value 1 to one of the three maps of Fig. 1.12b and the value 2 to the two others.

Even though the Matching-Jack conjecture can be formulated independently from maps or matchings (see Conjecture 3), the combinatorial interpretation behind this conjecture has been essential in several breakthroughs in its direction [La 09, DF16, KV16, Doł17, KPV18]. The conjecture itself is still open. The following polynomiality result is due to Dołęga and Féray.

Theorem 1.3.14 ([DF16, Corollary 4.2])**.** *For every $n \geq 1$, and any $\pi, \mu, \nu \vdash n$, the coefficient $c^{\pi}_{\mu,\nu}$ is a polynomial in b with rational coefficients.*

One of the main results of this thesis consists in proving the Matching-Jack conjecture for marginal sums (an analog for Chapuy–Dołęga result on the b-conjecture); see Theorem 1.5.1.

Connection between the two conjectures. In addition to being both open, no implications are known between the Matching-Jack and the b-conjectures. Indeed, in the cases $b = 0$ and $b = 1$, the equivalence between the formulas for the connected and the disconnected series (Eqs. (1.18) and (1.19) and Theorem 1.3.7) can be easily checked using simple combinatorial arguments (see *e.g.* Lemma 1.3.4). This equivalence is no longer clear for general b because of the statistics of non-orientability and the different labelling used in the two cases (in the b-conjecture we consider rooted maps, while maps are vertex-labelled in the Matching-Jack conjecture). This explains for example the fact that Dołęga and Féray have used in [DF17] new ideas in order to deduce the polynomiality property in the b-conjecture (Theorem 1.3.9) from the polynomiality result for the Matching-Jack conjecture obtained in [DF16] (Theorem 1.3.14).

Generalization to constellations

We consider $k+2$ alphabets $\boldsymbol{p}$, $\boldsymbol{q}^{(0)}$, $\boldsymbol{q}^{(1)}$, $\dots \boldsymbol{q}^{(k)}$. We define a generalization with $k+2$ alphabets of the series $\tau^{(\alpha)}$;

$$\tau^{(\alpha)}_k(t, \boldsymbol{p}, \boldsymbol{q}^{(0)}, \dots, \boldsymbol{q}^{(k)}) := \sum_{n\geq 0} t^n \sum_{\theta \vdash n} \frac{1}{j^{(\alpha)}_\theta} J^{(\alpha)}_\theta(\boldsymbol{p}) J^{(\alpha)}_\theta(\boldsymbol{q}^{(0)}) \dots J^{(\alpha)}_\theta(\boldsymbol{q}^{(k)}).$$

Using the encoding of constellations with permutations in the orientable case and matchings in the general one, it is possible to show that for $\alpha = 1$ (resp. $\alpha = 2$), the series $\tau^{(\alpha)}_k$ corresponds to the generating series of orientable (resp. orientable or not) constellations with control of the degrees of faces and vertices of all colors $0 \leq i \leq k$. This corresponds to a k-generalization of Eq. (1.18) and Eq. (1.20). We refer to [JV90] and [Ben22] for detailed statements and proofs.

This leads us to a generalization of the Matching-Jack and b-conjectures.

Conjecture 6 (Generalized b-conjecture. [Ben22, Conjecture 3])**.** *We define the coefficients $h^{\pi}_{\mu^{(0)},\dots,\mu^{(k)}}(\alpha)$ by*

$$\alpha \log\left(\tau^{(\alpha)}_k(t, \boldsymbol{p}, \boldsymbol{q}^{(0)}, \dots, \boldsymbol{q}^{(k)})\right) = \sum_{n\geq 1} \frac{t^n}{n} \sum_{\pi, \mu^{(0)}, \dots, \mu^{(k)} \vdash n} h^{\pi}_{\mu^{(0)},\dots,\mu^{(k)}}(\alpha) p_\pi q^{(0)}_{\mu^{(0)}} \dots q^{(k)}_{\mu^{(k)}}. \tag{1.32}$$

Then $h^{\pi}_{\mu^{(0)},\dots,\mu^{(k)}}$ is polynomial in b with non-negative integer coefficients.

Conjecture 7 (Generalized Matching-Jack conjecture. [Ben22, Conjecture 2])**.** *We define the coefficients $c^{\pi}_{\mu^{(0)},\dots,\mu^{(k)}}(\alpha)$ by*

$$\tau^{(\alpha)}_k(t, \boldsymbol{p}, \boldsymbol{q}^{(0)}, \dots, \boldsymbol{q}^{(k)}) = 1 + \sum_{n\geq 1} t^n \sum_{\pi, \mu^{(0)}, \dots, \mu^{(k)} \vdash n} \frac{c^{\pi}_{\mu^{(0)},\dots,\mu^{(k)}}(\alpha)}{z_\pi \alpha^{\ell(\pi)}} p_\pi q^{(0)}_{\mu^{(0)}} \dots q^{(k)}_{\mu^{(k)}}. \tag{1.33}$$

Then $c^{\pi}_{\mu^{(0)},\dots,\mu^{(k)}}$ is polynomial in b with non-negative integer coefficients.

As in the case $k = 1$, these conjectures have combinatorial reformulations; the coefficients $h^{\pi}_{\mu^{(0)},\dots,\mu^{(k)}}$ and $c^{\pi}_{\mu^{(0)},\dots,\mu^{(k)}}$ should count constellations with non-orientability weights.

It turns out that the Matchings-Jack conjecture for $k = 1$ implies the conjecture for any $k \geq 1$, since these coefficients $c^{\pi}_{\mu,\nu}$ satisfy a multiplicativity property (see Proposition 3.3.1). We are not aware of such property for the coefficients $h^{\pi}_{\mu,\nu}$, so the positivity for these generalized coefficients is a priory more general than the positivity in the case $k = 1$.

Remark 1.3.15 (Link to b-deformed Hurwitz numbers). In both orientable and non-orientable cases, generating series of constellations have been a useful tool to understand *Hurwitz numbers* which we now define.

Fix $k, n \geq 1$ and $\ell \geq 2$, and let $\mu^{(1)}, \dots, \mu^{(k)} \vdash n$. The classical *Hurwitz number* $H^{\ell}_{\mu^{(1)},\dots,\mu^{(k)}}$ is defined as $\frac{1}{n}$ times the number of following decompositions in $\mathfrak{S}_n$;

$$\mathbb{1}_{\mathfrak{S}_n} = \tau_1 \dots \tau_\ell \sigma_1 \dots \sigma_k,$$

where $(\tau_i)_{1\leq i\leq \ell}$ are transpositions, for every $1 \leq i \leq k$ the permutation σ_i has cycle type $\mu^{(i)}$, and such that $\tau_1, \dots, \tau_\ell, \sigma_1, \dots, \sigma_k$ act transitively on $\mathfrak{S}_n$. The cases $k = 1$ and $k = 2$ are the most studied cases and are known respectively as simple and double Hurwitz numbers.

Using a specialization of the function $\tau^{(\alpha)}_k$, Chapuy and Dołęga have introduced in [CD22] a b-deformation of Hurwitz numbers as follows;

$$H^{\ell}_{\mu^{(1)},\dots,\mu^{(k)}}(\alpha) := h^{\mu^{(1)}}_{\mu^{(2)},\dots,\mu^{(k)},\underbrace{[2,1^{n-2}],\dots,[2,1^{n-2}]}_{\ell \text{ times}}}.$$

The case $k = 2$ of these numbers has been studied in [CD22] and a *monotone* version has been treated in [BCD23, Ruz23, CDO24]. Moreover, the classical Hurwitz numbers can be recovered by setting $\alpha = 1$.

1.3.2 Lassalle's conjecture and dual problems

Another important problem involving Jack polynomials and maps, consists in establishing a combinatorial interpretation of the power-sum expansion of one single Jack polynomial. This question has been first raised by Hanlon in [Han88].

Young's formula and Hanlon's conjecture

The expansion of Schur functions in the power-sum basis is given by the characters of the symmetric group (see [Mac95, Chapter 1]);

$$s_\lambda(\boldsymbol{p}) = \sum_{\mu \vdash |\lambda|} \frac{\chi^\lambda(\mu)}{z_\mu} p_\mu.$$

Here, $\chi^\lambda(\mu)$ is the irreducible character of $\mathfrak{S}_n$ associated to λ evaluated at a permutation of cycle type μ.

Using Young symmetrizers, the expansion of a Schur function s_λ in the power-sum basis can be written as a sum of pairs of permutations; see [Han88, Eq. (1.1)] (see also [FŚ11a, Proposition 5] for a full proof of this result);

$$H_\lambda s_\lambda = J_\lambda^{(\alpha=1)}(\boldsymbol{p}) = \sum_{\substack{\sigma_\circ \in \mathrm{CS}(T_\lambda),\\ \sigma_\bullet \in \mathrm{RS}(T_\lambda)}} (-1)^{\ell(\mathrm{ct}(\sigma_\circ))} p_{\mathrm{ct}(\sigma_\circ \sigma_\bullet)}, \tag{1.34}$$

where T_λ is a fixed bijective filling of the Young diagram λ by the numbers $1, \dots, n$ and the subgroups $\mathrm{RS}(T_\lambda), \mathrm{CS}(T_\lambda) < \mathfrak{S}_n$ are the row and column stabilizers of T_λ. We also recall that $\mathrm{ct}(\sigma)$ is the cycle type of σ.

Inspired by Young's formula (1.34), Hanlon asked in [Han88] if there exists a function $\mathrm{stat}\colon\ \mathrm{RS}(T_\lambda) \times \mathrm{CS}(T_\lambda) \to \mathbb{Z}_{\geq 0}$ such that

$$J_\lambda^{(\alpha)}(\boldsymbol{p}) = \sum_{\substack{\sigma_\circ \in \mathrm{CS}(T_\lambda),\\ \sigma_\bullet \in \mathrm{RS}(T_\lambda)}} \alpha^{\mathrm{stat}(\sigma_\circ, \sigma_\bullet)} (-1)^{|\sigma_\circ|} p_{\mathrm{ct}(\sigma_\circ \sigma_\bullet)}. \tag{1.35}$$

Using the correspondence between orientable maps and permutations given in Proposition 1.1.5, it is actually possible to reformulate Eq. (1.34) and Hanlon's conjecture using orientable maps endowed with a decoration of their edges by the cells of λ, which satisfies some particular conditions.

Since then, substantial progress has been made in understanding the structure of Jack polynomials and it became clearer that a natural one-parameter deformation of (1.34) shall involve non-orientable maps.

More precisely, following Goulden–Jackson's conjectures discussed in the previous section, it is natural to replace the right-hand side of Eq. (1.35) as a sum over matchings or non-oriented bipartite maps, counted with a non-orientability weight $b^{\vartheta(M)}$. Such a formula, has been conjectured by Dołęga–Féray–Śniady in [DFŚ14]. This question is supported by the work of Féray–Śniady [FŚ11b] in which they have established a combinatorial formula for zonal polynomials (Jack polynomials for $\alpha = 2$) in terms of matchings. In the following, we will reformulate these results in terms of maps. To this purpose, we need to introduce a family of decorated bipartite maps; *layered maps*.

Layered maps

The following definition has been introduced in [BD23].

Definition 1.3.16. *Fix $k \geq 0$. We say that a bipartite map M is k-layered if its vertices are partitioned into k sets (which may be empty), called the layers of the map;*

$$\mathcal{V}_\circ(M) = \bigcup_{1 \leq i \leq k} \mathcal{V}_\circ^{(i)}(M), \quad \textit{and} \quad \mathcal{V}_\bullet(M) = \bigcup_{1 \leq i \leq k} \mathcal{V}_\bullet^{(i)}(M),$$

which satisfy the following condition: if v is a white vertex in a layer i, then all its neighbors are in layers $j \leq i$, and it has at least one neighbor in the layer i.

For $1 \leq i \leq k$, we define the partition $\nu_\bullet^{(i)}(M)$ obtained by ordering the degrees of the black vertices in the layer i. A k-layered map is vertex-labelled if:

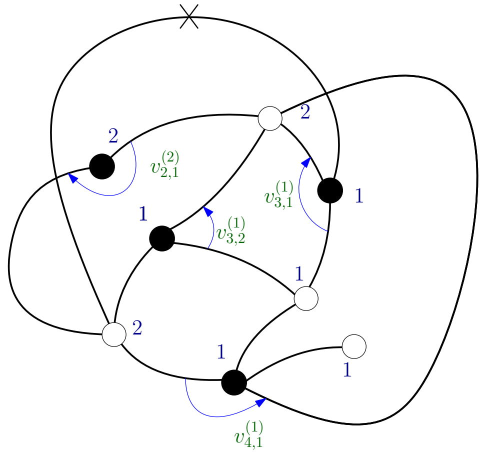


Figure 1.13: A vertex-labelled 2-layered (non-orientable) map. Here, the integer next to a vertex indicates its layer, and $v_{j,m}^{(i)}$ denotes the black vertex of degree j numbered by m in the layer i. Blue arrows illustrate vertex roots.

- *in each layer* $1 \leq i \leq k$*, the black vertices having the same degree* $j \geq 1$ *are numbered by* $1, 2, \dots,$
- *each black vertex has a distinguished oriented corner.*

Note that a vertex-labelled 1-layered map is simply a vertex-labelled map. An example of a vertex-labelled 2-layered map is given in Fig. 1.13. Note that a k-layered map can be seen as $(k+1)$-layered map with an empty layer $k+1$. We call a *layered map* a k-layered map for some $k \geq 1$.

Remark 1.3.17. This definition of layered maps is closely related to maps equipped with a (non-bijective) decoration of their edges with the cells of a Young diagram introduced in [DFŚ14, Section 1.6]. However, we prefer to present this definition as above since it will play a slightly different role in the combinatorial model used in Chapter 4.

Combinatorial formulas for Schur and zonal functions

Using representation theory tools, Féray has proved a second combinatorial formula for Schur functions [Fér10] conjectured by Stanley [Sta06] (another proof has been given in [FŚ11a]). We formulate here this result using layered maps.

Theorem 1.3.18 ([Fér10, Theorem 1])**.** *For any partition* λ*,*

$$J_\lambda^{(\alpha=1)}(\boldsymbol{p}) = \sum_M \frac{(-1)^{|\lambda|}}{z_{\lambda^\bullet(M)}} p_{\lambda^\circ(M)} \prod_{1 \leq i \leq \ell(\lambda)} (-\lambda_i)^{|\mathcal{V}_\circ^{(i)}(M)|}, \tag{1.36}$$

the sum runs over vertex-labelled $\ell(\lambda)$*-layered oriented maps of size* $|\lambda|$.

This formula turned out to be more useful for asymptotic purposes (see [FŚ11a]) than the one given in (1.34).

Féray and Śniady have also obtained a similar formula for zonal polynomials (Jack polynomials for $\alpha = 2$) in terms of non-orientable maps.

Theorem 1.3.19 ([FŚ11b, Theorem 1.2])**.** *For any partition λ,*

$$J_\lambda^{(\alpha=2)}(\boldsymbol{p}) = \sum_M \frac{(-1)^{|\lambda|}}{2^{|\mathcal{V}_\bullet(M)|} z_{|\mathcal{V}_\bullet(M)|}} p_{\lambda^\diamond(M)} \prod_{1 \le i \le \ell(\lambda)} (-2\lambda_i)^{|\mathcal{V}_\circ^{(i)}(M)|} \tag{1.37}$$

where the sum is taken over all vertex-labelled $\ell(\lambda)$-layered maps of size $|\lambda|$, oriented or not.

The combinatorial formulas given above of Jack polynomials for $\alpha \in \{1, 2\}$ resemble the *topological expansions* known from random matrices: such formulas involve sums over objects with a notion of genus, and for which the "leading term" is given by the objects of genus zero; see *e.g.* [Meh04, EKR15].

However, the representation theory tools have been substantial to obtain the formulas for the cases $\alpha \in \{1, 2\}$. The question of establishing such topological expansions for Jack polynomials for general α becomes then more compelling, and as for Goulden–Jackson's conjectures, the general case requires developing new methods.

Normalized characters and Stanley–Féray formulas

A very successful approach to the asymptotic representation theory of the symmetric group, initiated by Kerov and Olshanski [KO94], treats normalized irreducible characters $\chi^\lambda(\mu)$ as functions on Young diagrams λ. More precisely, they considered for a fixed μ the function

$$\theta_\mu^{(1)}(\lambda) := \frac{|\lambda|!}{(|\lambda| - |\mu|)! z_\mu} \cdot \frac{\chi^\lambda(\mu \cup 1^{|\lambda| - |\mu|})}{\chi^\lambda(1^{|\lambda|})}, \text{ for } |\lambda| \ge |\mu|.$$

They proved that these normalized characters satisfy nice symmetry properties as polynomials in several descriptions of λ (part sizes, contents, Frobenius coordinates...). Within this approach, a formula equivalent to (1.36) became the key ingredient in achieving a breakthrough in asymptotic representation theory of the symmetric groups [FŚ11a].

Using this dual approach, Stanley [Sta04] studied the normalized irreducible characters of the symmetric group $\theta_\mu^{(1)}$, and he observed that if one expresses them in other variables than $\lambda_1, \lambda_2, \dots$, which he called *multirectangular coordinates*, then they have very special positivity and integrality properties.

Definition 1.3.20 ([Sta04])**.** *Let $k \ge 1$ and let $s_1 \ge s_2 \ge \cdots \ge s_k \ge 1$ and $r_1, \dots r_k$ be two sequences of non negative integers. We say that $(s_1, s_2, \dots, s_k)$ and $(r_1, \dots, r_k)$ are multirectangular coordinates (or also Stanley's coordinates) for a partition λ and we denote $\lambda = \boldsymbol{s}^{\boldsymbol{r}}$, if λ is the union of k rectangles of sizes $s_i \times r_i$, or equivalently $\lambda = [s_1^{r_1} \dots s_k^{r_k}]$, see Fig. 1.14 for an example.*

Since we do not require that the sequence $\boldsymbol{s}$ be strictly decreasing, the multirectangular coordinates are not unique in general. If λ is a partition of multirectangular coordinates $(s_1, \dots, s_k)$ and $(r_1, \dots, r_k)$, we write, for any partition μ,

$$\tilde{\theta}_\mu^{(1)}(\boldsymbol{s}, \boldsymbol{r}) := \theta_\mu^{(1)}(\lambda), \tag{1.38}$$

where $\boldsymbol{r} = (r_1, \dots, r_k, 0 \dots)$ and $\boldsymbol{s} = (s_1, \dots, s_k, 0, \dots)$.

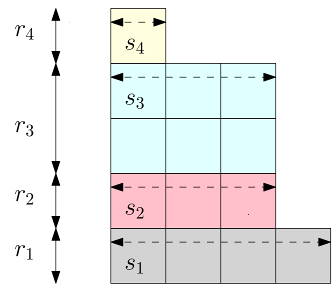


Figure 1.14: The Young diagram of the partition $[4, 3, 3, 3, 1]$ as the union of 4 rectangles, with $\boldsymbol{s} = (4, 3, 3, 1)$ and $\boldsymbol{r} = (1, 1, 2, 1)$ as multirectangular coordinates.

Stanley found in [Sta04] an explicit formula for $\theta_\mu^{(1)}(\lambda)$ when λ is a rectangle, and he conjectured a formula for general $\boldsymbol{r}, \boldsymbol{s}$, which implies that the normalized irreducible character $(-1)^{|\mu|} z_\mu \tilde{\theta}_\mu^{(1)}(\boldsymbol{s}, \boldsymbol{r})$ is a polynomial in the variables $-s_1, -s_2, \ldots, r_1, r_2, \ldots$ with **non-negative integer coefficients**, and the aforementioned consequences in asymptotic representation theory follow. This formula, nowadays known as Féray–Stanley formula, is a generalized version of (1.36) which corresponds to $|\lambda| = |\mu|$ and $r_i = 1$ for any $i \geq 1$ (see [Fér10] or [FŚ11a]).

Jack characters, Lassalle's conjecture and related problems

The approach of Kerov and Olshanski to study the characters of the symmetric group was extended to the Jack case by Lassalle [Las08a, Las09], where the primal object of study is *the Jack character* $\theta_\mu^{(\alpha)}$. It is the function on Young diagrams defined by:

$$\theta_\mu^{(\alpha)}(\lambda) := \begin{cases} 0 & \text{if } |\lambda| < |\mu|, \\ \binom{|\lambda| - |\mu| + m_1(\mu)}{m_1(\mu)} [p_{\mu, 1^{|\lambda| - |\mu|}}] J_\lambda^{(\alpha)} & \text{if } |\lambda| \geq |\mu|, \end{cases} \tag{1.39}$$

where $m_1(\mu)$ is the number of parts equal to 1 in the partition μ. Equivalently, we have;

$$\exp\left(\frac{\partial}{\partial p_1}\right) J_\lambda^{(\alpha)}(\boldsymbol{p}) = \sum_{\mu \in \mathbb{Y}} \theta_\mu^{(\alpha)}(\lambda) p_\mu. \tag{1.40}$$

Moreover, we define $\tilde{\theta}_\mu^{(\alpha)}(\boldsymbol{s}, \boldsymbol{r})$ as the Jack character expressed in the multirectangular coordinated as in Eq. (1.38).

Lassalle has stated in [Las08b] the following conjecture which extends the positivity and the integrality properties of the Stanley–Féray formula.

Conjecture 8. *The normalized Jack characters expressed in the Stanley's coordinates* $(-1)^{|\mu|} z_\mu \tilde{\theta}_\mu^{(\alpha)}(\boldsymbol{s}, \boldsymbol{r})$ *are polynomials in the variables* $b, -s_1, -s_2, \ldots, r_1, r_2, \ldots$ *with non-negative integer coefficients, where* $b := \alpha - 1$.

Lassalle speculated that this reparametrization reflects a true combinatorial meaning of Jack characters, but he was unable to find its combinatorial interpretation, even at a conjectural level. In [DFŚ14], the authors have given a combinatorial reformulation of this conjecture using layered maps counted with non-orientability weights.

Besides the case $\alpha = 1$, the case $\alpha = 2$ of this conjecture can be deduced from Eq. (1.37). Moreover, the rectangular case in this conjecture has been proved in [DFŚ14, Ben22]. Despite these special cases, Conjecture 8 remained unproven for the last 15 years. The proof of this conjecture is one of the main results of this thesis.

1.3.3 A connection between the two families of conjectures: structure coefficients of characters

Throughout this thesis, we will see that Goulden–Jackson's conjectures stated in Section 1.3.1 and the dual problems presented in Section 1.3.2 turn out to be closely related. An illustration of this connection is given by the structure coefficients of Jack characters.

Structure coefficients of Jack characters

One of the nice properties satisfied by Jack characters $\theta_\mu^{(\alpha)}(\lambda)$, is that they are *shifted symmetric* in the parts of λ (see Section 4.1 for a precise definition). Moreover, $(\theta_\mu^{(\alpha)})_{\mu\in\mathbb{Y}}$ form a linear basis of the space of shifted symmetric functions. As a consequence, their structure coefficients $g_{\mu,\nu}^\pi(\alpha)$ are well defined:

$$\theta_\mu^{(\alpha)}\theta_\nu^{(\alpha)} = \sum_\pi g_{\mu,\nu}^\pi(\alpha)\theta_\pi^{(\alpha)}. \tag{1.41}$$

The following proposition has been proved by Dołęga and Féray [DF16, Proposition B.1].

Proposition 1.3.21. *If π, μ and ν are of the same size then*

$$c_{\mu,\nu}^\pi = g_{\mu,\nu}^\pi,$$

where $c_{\mu,\nu}^\pi$ are the coefficients of the Matching-Jack conjecture (see Eq. (1.29)).

Śniady's conjecture

The following conjecture due to Śniady, can be thought of as a generalization of the Matching-Jack conjecture to coefficients $g_{\mu,\nu}^\pi$ indexed by partitions of arbitrary sizes.

Conjecture 9 ([Śni19, Conjecture 2.2])**.** *For any π, μ and ν partitions, $g_{\mu,\nu}^\pi$ is a polynomial in $b := \alpha - 1$ with non-negative integer coefficients.*

In [DF16], Dołęga and Féray have proved that the coefficients $g_{\mu,\nu}^\pi$ are polynomial in the deformation parameter b.

We get from Eq. (1.41) and Proposition 1.3.21 that the coefficients $c_{\mu,\nu}^\pi$ of the Matching-Jack conjecture can be obtained as a special case of the structure coefficients of Jack characters. This suggests that understanding the combinatorial structure of these characters might be useful to study the coefficients $c_{\mu,\nu}^\pi$. This observation will be the starting point of Chapter 5.

1.4 Main tool: differential operators

The main tool used in this work to connect generating series of maps to Jack polynomials is differential operators, and more precisely the operators introduced in [CD22]. The interest of these operators is that, on the one hand they can be used to encode combinatorial operations on maps, and on the other hand they have a nice action on Jack polynomials. The purpose of Section 1.4.1 is to give a brief introduction to these techniques on simple examples. The operators of Chapuy–Dołęga themselves will be defined in Section 1.4.2.

1.4.1 First examples

We fix a bipartite map M, orientable or not (possibly disconnected). In this section the weight associated to M will be $p_{\lambda^\diamond(M)}$, *i.e.* the weight controlling only the face-type of M. We explain here how to use differential operators to add one edge to the map M. Concretely, these operators will act on the weight of M.

We consider the three operators on $\mathcal{S}_\alpha$;

$$A_1 := p_1,$$

$$A_2 := \sum_{i\geq 1} p_{i+1}\frac{i\partial}{\partial p_i},$$

$$A_3 := \sum_{i,j\geq 1} p_{i+j+1}\frac{ij\partial^2}{\partial p_i \partial p_j} + \sum_{i,j\geq 1} p_i p_j \frac{(i+j-1)\partial}{\partial p_{i+j-1}} + \sum_{i\geq 1} p_{i+1}\frac{i^2\partial}{\partial p_i}.$$

It is straightforward that

$$A_1 \cdot p_{\lambda^\diamond(M)} = p_{\lambda^\diamond(N)}, \tag{1.42}$$

where N is obtained from M by adding an isolated edge. Moreover,

$$A_2 \cdot p_{\lambda^\diamond(M)} = \sum_N p_{\lambda^\diamond(N)}, \tag{1.43}$$

where the sum is taken over maps N obtained by choosing a black corner of M and connecting to it a new white leaf (*i.e.* a white vertex of degree 1). Indeed, the action $\frac{i\partial}{\partial p_i}$ on the weight of M can be interpreted as marking a black corner incident to a face of degree $2i$, and the multiplication by p_{i+1} corresponds to the fact that we increase the degree of this face by 2.

Finally, we have

$$A_3 \cdot p_{\lambda^\diamond(M)} = \sum_e p_{\lambda^\diamond(M\cup e)}, \tag{1.44}$$

the sum being taken over all ways to add an edge e between two corners of the map M (without adding new vertices). The three terms of A_3 correspond to the three ways of adding such an edge; a straight or a twisted edge between two corners of the same face, or an edge between two corners of different faces. See Section 2.1 and Fig. 2.1 for more details.

Actually, these operators are closely related to the case $\alpha = 2$ of the Laplace-Beltrami operator $D^{(\alpha)}$ given in Eq. (1.17). More precisely, one can check that

$$A_2 = [D^{(2)}, A_1], \quad \text{and} \quad A_3 = [D^{(2)}, A_2].$$

where where $[\cdot,\cdot]$ denotes the usual algebra commutator;

$$[X,Y] := XY - YX. \tag{1.45}$$

Similarly, one can define using the same commutation relations a deformation of these operators:

$$A_2^{(\alpha)} = [D^{(\alpha)}, A_1], \quad \text{and} \quad A_3^{(\alpha)} = [D^{(\alpha)}, A_2^{(\alpha)}].$$

It is actually possible to give a similar combinatorial interpretation of an α-deformation of these operators, which consists in associating a non-orientability weight to the added edge, see Definition 2.1.3.

The commutation expressions given above are crucial to understand the action of the operators $A_2^{(\alpha)}$ and $A_3^{(\alpha)}$ on Jack polynomials; the action of p_1 is given by the *Pieri rule*, and the operator $D^{(\alpha)}$ is diagonal on these polynomials (see Proposition 1.2.9). We refer to [CD22, Corollary 5.4] for precise formulas.

We recall that the weight used in this section controls only the face-type of a map M. To keep track of more information, it turns out that one can use a family of operators of the same flavor with more refined weights, namely the operators introduced by Chapuy–Dołęga.

1.4.2 Chapuy–Dołęga operators

We recall from Section 1.2.2 that $\mathcal{P}$ is the polynomial algebra defined by

$$\mathcal{P} := \mathrm{Span}_{\mathbb{Q}(b)}\{p_\lambda\}_{\lambda\in\mathbb{Y}}.$$

We also consider an alphabet $Y := \{y_0, y_1, \dots\}$ and the space[5] $\mathcal{P}_Y$

$$\mathcal{P}_Y := \mathrm{Span}_{\mathbb{Q}(b)}\left\{y_i p_\lambda\right\}_{i\in\mathbb{N},\lambda\in\mathbb{Y}},$$

where $\mathbb{N} := \{0,1,2,\dots\}$ is the set of non-negative integers. A monomial $y_i p_\lambda$ will play the role of the weight of a rooted map, for which i is the degree of the root face and λ contains the degrees of non-root faces.

We use the "catalytic" operators $Y_+, \Gamma_Y : \mathcal{P}_Y \to \mathcal{P}_Y$ introduced in [CD22]:

$$Y_+ = \sum_{i\geq 0} y_{i+1}\frac{\partial}{\partial y_i}, \tag{1.46}$$

$$\Gamma_Y = (1+b)\cdot\sum_{i,j\geq 1} y_{i+j}\frac{j\partial^2}{\partial y_{i-1}\partial p_j} + \sum_{i,j\geq 1} y_i p_j \frac{\partial}{\partial y_{i+j-1}} + b\cdot\sum_{i\geq 1} y_{i+1}\frac{i\partial}{\partial y_i}. \tag{1.47}$$

Note that the operators Y_+ and Γ_Y have similar structures as the ones of $A_2^{(\alpha)}$ and $A_3^{(\alpha)}$ respectively, but they additionally use variables from the family Y. This justifies the name "catalytic", as these variables will play the same role as the catalytic variable in the classical Tutte decomposition [Tut62b], well-known to the combinatorial community. Indeed, when the operators Y_+ and Γ_Y act on the weight of a rooted map they also add edges as

[5] This notation is slightly different from the one used in [CD22].

in Eqs. (1.43) and (1.44) respectively, with the additional condition that the added edge is incident to the root corner.

We consider the operator $\Theta_Y : \mathcal{P}_Y \to \mathcal{P}$, defined by

$$\Theta_Y := \sum_{i\geq 1} p_i \frac{\partial}{\partial y_i}.$$

For $n \geq 0$, and variable u, the operator $\mathcal{B}_n^{(\alpha)} : \mathcal{P} \to \mathcal{P}[u]$ is defined by

$$\mathcal{B}_n^{(\alpha)}(\boldsymbol{p}, u) := \Theta_Y \left(\Gamma_Y + uY_+\right)^n \frac{y_0}{1+b}. \tag{1.48}$$

In this equation, the alphabet Y is only used in intermediate steps, and is "forgotten" at the end by the operator Θ_Y. Roughly, the operator $\mathcal{B}_n$ corresponds to the operation of adding a black vertex of degree n; first we add an isolated black vertex which plays the role of the root. To this vertex we attach n edges, using possibly new white vertices, and at the end we "forget" the root to obtain an unrooted map. In Section 2.2, we give a precise statement of this combinatorial interpretation.

Example 1.4.1. We give here non-catalytic expressions for the two first operators $\mathcal{B}_n$ *i.e.* expressions which do not involve the alphabet Y:

$$\mathcal{B}_1^{(\alpha)}(\boldsymbol{p}, u) = \frac{up_1}{\alpha} + \sum_{i\geq 1} p_{i+1} \frac{i\partial}{\partial p_i},$$

$$\mathcal{B}_2^{(\alpha)}(\boldsymbol{p}, u) = \frac{u^2 p_2}{\alpha} + \sum_{i\geq 1} \left(\big(2u + (i+1)(\alpha-1)\big) p_{i+2} + \sum_{\substack{j+k=i+2\\ j,k\geq 1}} p_j p_k \right) \frac{i\partial}{\partial p_i} + \frac{u}{\alpha}\big((\alpha-1)p_2 + p_{1,1}\big) + \alpha \sum_{i,j\geq 1} p_{i+j+2} \frac{i\partial}{\partial p_i}\frac{j\partial}{\partial p_j}.$$

Finally, we introduce the operator

$$\mathcal{B}_\infty^{(\alpha)}(t, \boldsymbol{p}, u) := \sum_{n\geq 1} \frac{t^n}{n} \mathcal{B}_n^{(\alpha)}(\boldsymbol{p}, u) : \mathcal{P} \to \mathcal{P}[u][\![t]\!].$$

When there is no confusion, we will use the simplified notation for these operators

$$\mathcal{B}_\infty \equiv \mathcal{B}_\infty^{(\alpha)}, \qquad \mathcal{B}_n \equiv \mathcal{B}_n^{(\alpha)}.$$

In this thesis we use the operators $\mathcal{B}_n$ to apprehend several problems involving Jack polynomials and generating series of maps:

- In Chapter 2, we use these operators with a weight $p_{\lambda^\bullet(M)} q_{\lambda^\diamond(M)} u^{|\mathcal{V}_\circ(M)|}$ controlling the face-type, the black-type and the number of white vertices.
- In Chapter 4, they are used to construct generating series of layered-maps.
- In Chapter 5, the same family of operators is involved in differential equations on generating series with control of the full profile, *i.e.* maps counted with weights $p_{\lambda^\bullet(M)} q_{\lambda^\diamond(M)} r_{\lambda^\circ(M)}$.

These results are detailed in the following section.

1.5 Main results

We now briefly state the main results of this thesis.

1.5.1 Chapter 2: Marginal sums in the Matching-Jack conjecture

The main result of Chapter 2 is based on [Ben22], the proof given here is however different.

We recall that the case of marginal-sums in the b-conjecture has been treated by Chapuy and Dołęga (see Theorem 1.3.10). We prove here an analog for the Matching-Jack conjecture.

First, we consider the marginal sums of coefficients $c^{\pi}_{\mu,\nu}$ defined in Eq. (1.29); for any partitions π and μ of the same size n and for any integer $1 \leq m \leq n$ let $\overline{c}^{\pi}_{\mu,m}(\alpha)$ be the coefficient defined by

$$\overline{c}^{\pi}_{\mu,m}(\alpha) := \sum_{\substack{\nu\vdash n\\ \ell(\nu)=m}} c^{\pi}_{\mu,\nu}(\alpha).$$

Equivalently, these coefficients are given by the expansion of $\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\underline{u})$ in the power-sum bases;

$$\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\underline{u}) = 1 + \sum_{n\geq 1} t^n \sum_{\pi,\mu\vdash n} \sum_{1\leq m\leq n} \frac{\overline{c}^{\pi}_{\mu,m}(\alpha)}{z_\pi \alpha^{\ell(\pi)}} p_\pi q_\mu u^m. \tag{1.49}$$

The following is an analog of Theorem 1.3.10 for the Matching-Jack conjecture; see also Theorem 2.3.2 and Theorem 2.4.1.

Theorem 1.5.1. *We have,*

$$\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\underline{u}) = \exp\left(\sum_{m\geq 1} \frac{t^m q_m}{m} \mathcal{B}_m(\boldsymbol{p},u)\right)\cdot 1.$$

Moreover, for any $n,m \geq 1$ and for any partitions $\pi,\mu \vdash n$, the marginal coefficient $\overline{c}^{\pi}_{\mu,m}$ is polynomial in b with non-negative integer coefficients. Moreover, there exists a SSON $\boldsymbol{\vartheta}$ *on vertex-labelled bipartite maps, such that*

$$\overline{c}^{\pi}_{\mu,m} = \sum_{M} b^{\boldsymbol{\vartheta}(M)},$$

where the sum is taken over vertex-labelled maps M such that $\lambda^\circ(M)=\pi$, $\lambda^\bullet(M)=\mu$ and $\ell(\lambda^\circ(M))=m$.

This theorem covers other partial results in the direction of the Matching-Jack conjecture (see [KV16, KPV18]).

Theorem 1.5.1 has been proved in [Ben22] by adopting the statistic of non-orientability used by Chapuy–Dołęga in Theorem 1.5.1 to vertex-labelled maps, the object of the Matching-Jack conjecture. The proof is based on some symmetry properties satisfied by these statistics.

We give in Section 2.4 a more direct proof of this theorem which uses a differential equation (a Tutte-like equation) satisfied by the specialized function $\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\underline{u})$. The statistic used has a simpler description than the one used in [Ben22].

1.5.2 Chapter 3: Integrality on the Matching-Jack conjecture

Chapter 3 is based on the work of [Ben23a].

Based on the result of Chapter 2, we prove the integrality part in the Matching-Jack conjecture.

Theorem 1.5.2. *For any $n \geq 1$, and any $\pi, \mu, \nu \vdash n$, the coefficient $c^{\pi}_{\mu,\nu}$ is a polynomial in b with integer coefficients.*

Since the approach used here is independent from the one considered in [DF16], it gives a new proof of the polynomiality of the coefficients $c^{\pi}_{\mu,\nu}$ in b (Theorem 1.3.14). Unfortunately, the non-negativity of the coefficients of $c^{\pi}_{\mu,\nu}$ as polynomials in b –the remaining part of the Matching-Jack conjecture– seems to be out of reach with our approach and requires new ideas.

Our proof of Theorem 1.5.2 strongly relies on the case of marginal sums (Theorem 1.5.1). In fact, we deduce the integrality of the coefficients $c^{\pi}_{\mu,\nu}$ from the integrality of their marginal sums $\overline{c}^{\pi}_{\mu,m}$, thanks to a new connection between these coefficients and the *Farahat–Higman algebra*. This two-step method is a key feature of our proof; the approach used to prove Theorem 1.5.1 can a priori not be used to say something on Theorem 1.5.2. Indeed, the differential equation satisfied by the specialization $\tau^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \underline{u})$ and used to obtain Section 1.5.1 does not apply for the function $\tau^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r})$ itself.

Let us explain the idea of this proof. Our starting point is a family of equations relating the coefficients $c^{\pi}_{\mu,\nu}$ to their marginal sums, which we call the *multiplicativity property* (or also the associativity property) of these coefficients; for any $\pi, \mu, \nu \vdash n \geq 1$ and for any $l \geq 1$ we have

$$\sum_{\kappa \vdash n} c^{\pi}_{\mu,\kappa}\, \overline{c}^{\kappa}_{\nu,l} = \sum_{\theta \vdash n} \overline{c}^{\pi}_{\theta,l}\, c^{\theta}_{\mu,\nu}.$$

See also Eq. (3.1). This property, observed by Chapuy and Dołęga (private communication), is a consequence of the orthogonality of Jack polynomials.

In the proof we treat these equations as a linear system, in which the coefficients $c^{\pi}_{\mu,\kappa}$ are the "unknown". We prove that this system completely determines the coefficients $c^{\pi}_{\mu,\nu}$; see Proposition 3.3.4. Moreover, for a well chosen subfamily of equations, the system obtained is encoded by a square matrix who is invertible over $\mathbb{Z}$ (Theorem 3.2.12). This last result is obtained by interpreting the coefficients of the matrix as structure coefficients in the Farahat–Higman algebra. This connection with the Farahat–Higman algebra seems new in this context.

1.5.3 Chapter 4: Combinatorial formula for Jack characters and proof of Lassalle's conjecture

Chapter 4 is based on a joint work with Maciej Dołęga [BD23].

We establish an explicit combinatorial expression of the expansion of Jack polynomials in the power-sum basis in terms of weighted maps. A more general formula is given for Jack characters and is used to prove Lassalle's conjecture (Conjecture 8).

Theorem 1.5.3. *For any partitions μ and $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_k]$,*

$$\theta_\mu^{(\alpha)}(\lambda) = [t^{|\mu|} p_\mu] \exp\left(\mathcal{B}_\infty(-t, \boldsymbol{p}, -\alpha\lambda_1)\right) \dots \exp\left(\mathcal{B}_\infty(-t, \boldsymbol{p}, -\alpha\lambda_k)\right) \cdot 1. \tag{1.50}$$

Moreover, there exists a statistic of non-orientability ϑ on layered maps such that for any such partitions μ and λ, we have

$$\theta_\mu^{(\alpha)}(\lambda) = (-1)^{|\mu|} \sum_M \frac{b^{\vartheta(M)}}{2^{|\mathcal{V}_\bullet(M)| - \mathrm{cc}(M)} \alpha^{\mathrm{cc}(M)}} \prod_{i \geq 1} \frac{(-\alpha\lambda_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}} \tag{1.51}$$

where the sum is taken over all vertex-labelled k-layered maps of face-type μ.

Actually, we prove that this theorem holds for a family of statistics ϑ. Although the two parameters α and b are related, we prefer to keep both of them in the previous formula since they play different roles. In particular, one may notice that the quantity $1/\left(2^{(|\mathcal{V}_\bullet(M)| - \mathrm{cc}(M))} \alpha^{\mathrm{cc}(M)}\right) \prod_{1 \leq i \leq k} \frac{(-\alpha\lambda_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}$ depends only on the underlying "k-layered graph" of M and is rather straightforward. This will not be the case for the *non-orientability weight* of the map $b^{\vartheta(M)}$.

In the case $b = 0$, and by definition of a statistic of non-orientability, only bipartite maps appear in Eq. (1.51), so that we recover the Stanley–Féray formula (1.36). Similarly, Eq. (1.51) coincides when $b = 1$ with the expression given in [FŚ11b, Thm 1.2].

As a direct consequence of Theorem 1.5.3, we obtain the following interpretation of Jack polynomials in the basis of power-sum functions.

Theorem 1.5.4. *Let n be a positive integer and let λ be a partition of n. There exists a statistic of non-orientability ϑ such that*

$$J_\lambda^{(\alpha)} = (-1)^n \sum_M p_{\lambda^\diamond(M)} \frac{b^{\vartheta(M)}}{2^{|\mathcal{V}_\bullet(M)| - \mathrm{cc}(M)} \alpha^{\mathrm{cc}(M)}} \prod_{1 \leq i \leq \ell(\lambda)} \frac{(-\alpha\lambda_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}, \tag{1.52}$$

where the sum is taken over all $\ell(\lambda)$-layered maps M of size n.

Another application of Theorem 1.5.3 is a formula for the expansion of Jack polynomials in the power-sum basis using creation operators (see also Theorem 4.7.2).

Theorem 1.5.5. *Fix a partition $\lambda = [\lambda_1, \dots, \lambda_k]$. Then*

$$J_\lambda^{(\alpha)} = \mathcal{B}_{\lambda_1}^{(+)} \cdots \mathcal{B}_{\lambda_k}^{(+)} \cdot 1,$$

where $\mathcal{B}_n^{(+)} := [t^n] \exp\left(\mathcal{B}_\infty(-t, \boldsymbol{p}, -\alpha n)\right)$.

One may notice that this formula is simpler than the one obtained in Eq. (1.50). In fact, it is obtained from the latter using some vanishing properties of the differential operators proved in Section 4.4 (see Section 4.7.2 for more details).

As in Eq. (1.36), the sum in Theorem 1.5.4 can be interpreted using maps whose edges are embedded in a non-bijective way in the Young diagram of λ (see also Remark 1.3.17).

The second main result of Chapter 4 gives an answer to Lassalle's conjecture (Conjecture 8).

Theorem 1.5.6. *The normalized Jack characters expressed in the Stanley coordinates* $(-1)^{|\mu|}z_\mu\tilde{\theta}^{(\alpha)}_\mu(\boldsymbol{s},\boldsymbol{r})$ *are polynomials in the variables* $b, -s_1, -s_2, \dots, r_1, r_2, \dots$ *with non-negative integer coefficients, where* $b := \alpha - 1$.

Actually, Theorem 1.5.6 does not immediately follow from Theorem 1.5.3: we use the combinatorial formula of Theorem 1.5.3 to prove the polynomiality and the positivity parts in Conjecture 8, but to prove the integrality part we use an other approach given by a family of operators related to an integrable system of Nazarov–Sklyanin [NS13]. This approach also allows us to obtain integrality in another conjecture of Lassalle related to Kerov polynomials (see Conjecture 12).

The following table summarizes the main results of Chapters 2 to 4.

Conjecture	Integrality	Positivity	Combinatorial interpretation
Marginal Matching-Jack conjecture	Theorem 2.4.1		
Matching-Jack conjecture (Conjecture 3)	Theorem 1.5.2	-	-
Lassalle's conjecture on Jack characters (Conjecture 8)	Theorem 4.2.12	Theorem 1.5.3	
Lassalle's conjecture on Kerov polynomials (Conjecture 12)	Theorem 4.2.12	-	-

1.5.4 Chapter 5: Differential equations for the series of bipartite maps

Chapter 5 is based on the work of [Ben24].

We establish a differential equation for the generating series of bipartite maps with control of the full profile, as well as for their α-deformed series. This result is new even in the orientable case for which we give a combinatorial proof. The approach used here is different from the one given by Tutte-like equations, in which we cannot keep track of the three alphabets of the profile. However, unlike Tutte equations, the equations we obtain here are signed.

We also prove integrality in Conjecture 9 on structure coefficients of Jack characters, as well as some cases related to the top coefficients in this conjecture.

We start by introducing the series of the structure coefficients $g^\pi_{\mu,\nu}$ (see Eq. (1.41));

$$G^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) := \sum_{\pi,\mu,\nu} \frac{g^\pi_{\mu,\nu}(\alpha)}{z_\pi \alpha^{\ell(\pi)}} t^{|\mu|+|\nu|-|\pi|} p_\pi q_\mu r_\nu. \tag{1.53}$$

Given Proposition 1.3.21, it is easy to see that the function $\tau^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r})$ corresponds to the homogeneous part of $G^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r})$. Conversely, we prove that $G^{(\alpha)}$ can be obtained from $\tau^{(\alpha)}$ by simple operations (see Eq. (5.16)). Moreover, for $\alpha = 1$ and $\alpha = 2$, the series $G^{(\alpha)}$ corresponds to the series of bipartite maps with some "forgotten" vertices of degree 1; see Proposition 5.1.5.

Let $\mathcal{B}_\infty^\perp(t, \boldsymbol{p}, u)$ denote the adjoint operator of $\mathcal{B}_\infty(t, \boldsymbol{p}, u)$ with respect to the scalar product of $\mathcal{S}_\alpha$, see Eq. (1.9). In Section 5.6.2, we provide a catalytic expression for this dual operator.

We now state the main result of Chapter 5.

Theorem 1.5.7. *The function $G^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r})$ satisfies the following equation*

$$\left(\mathcal{B}_\infty(-t, \boldsymbol{q}, u) + \mathcal{B}_\infty(-t, \boldsymbol{r}, u)\right) \cdot G^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) = \mathcal{B}_\infty^\perp(-t, \boldsymbol{p}, u) \cdot G^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}). \quad (1.54)$$

The proof of this result is based on the differential formula of Jack characters given in Eq. (1.50). We also prove in Proposition 5.5.2 that Eq. (1.54) characterizes the series $G^{(\alpha)}$.

By extracting coefficients in the variable u, Eq. (1.54) can be alternatively written as a family of equations (independent of u) which are indexed by non-negative integers; see Section 5.3.4.

In Theorem 5.5.1, we solve these differential equations and give an explicit expression of the coefficients $g^\pi_{\mu,\nu}$ using some coefficients a^λ_μ which are obtained from the operator $\mathcal{B}_\infty$ and which are known to count maps.

Remark 1.5.8. The results of Chapters 2, 3 and 5 can be extended to the case of constellations using mainly the same proofs. In each one of these cases, we will state the corresponding results without detailed proofs. We refer to the corresponding papers for more details.

1.5.5 Other works

We conclude the introduction by briefly mentioning other works which are not presented in this thesis.

- As explained in Remark 1.3.17, layered-maps which appear in the combinatorial formula of the Jack polynomial $J^{(\alpha)}_\lambda$ of Theorem 1.5.4 can be seen as maps whose edges are decorated in a non-bijective way with the cells of λ.

  In [Ben23b], we conjecture a variant of this formula with "bijective decorations", and we prove it when λ is a 2-column partition. Actually, this variant has been a key step in the proof of the cases $\alpha \in \{1, 2\}$ of Theorem 1.5.4 in [FŚ11a, FŚ11b].

- Macdonald polynomials, introduced by Macdonald in 1989, are symmetric polynomials which depend on two parameters q and t from which Jack polynomials are obtained by taking an appropriate limit; see [Mac95, Chapter VI].

  In a joint work with Michele D'Adderio [BD24], we introduce a Macdonald version of Jack characters obtained by a construction formula similar to the one given in Eq. (1.50). We also consider a new parametrization for these polynomials which allows us to generalize several conjectures about Jack polynomials to the Macdonald case. We give more details about these conjectures in Section 6.3.

Notation Throughout the thesis, we use straight letters to denote series (H,G,...), and calligraphic letters to denote operators ($\mathcal{B}_n$, $\mathcal{C}_\ell$, $\mathcal{G}$,...) or linear spaces ($\mathcal{A}$, $\mathcal{S}_\alpha$,$\mathcal{S}_\alpha^*$...).

Chapter 2

Statistics of non-orientability, differential operators and marginal sums in the Matching-Jack conjecture

The result of this chapter is based on [Ben22], the proof is however different.

The main purpose of this chapter is to prove the Matching-Jack conjecture for marginal sums, which can be viewed as an "averaged version" of the full conjecture; see Theorem 1.5.1. We recall that this is a disconnected version of the result of Chapuy–Dołęga [CD22] on the b-conjecture, which we "transfer" here to the Matching-Jack conjecture. More precisely, we deduce from their formula about connected series of maps a result for disconnected series. This deduction is not simple because of the presence of the non-orientability weights and the different normalization between the two conjectures.

In the proof provided in [Ben22] we start from the statistic defined in [CD22] for connected maps to define a statistic on vertex labelled disconnected maps. The proof uses symmetry properties of these statistics. It turned out that a simpler proof can be given using directly a differential equation for the series $\tau^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \underline{u})$ obtained in [CD22]. In this chapter, we present this second proof.

Structure of the Chapter

In Section 2.1, we introduce a family of strong statistics of non-orientability on vertex-labelled maps. We then explain in Section 2.2 how these statistics can be used to give an interpretation of Chapuy-Dołęga operators in terms of combinatorial operations on maps. Using these operators, we establish in Section 2.3 a differential construction formula for the function $\tau^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \underline{u})$. Finally, we combine in Section 2.4 the last two ingredients in order to prove the main theorem of the chapter.

2.1 Strong statistics of non-orientability

The purpose of this section is to describe a general method to define strong statistics of non-orientablity (SSON) on vertex-labelled maps (see Definition 1.3.12). We recall that the term "strong" refers to the fact that these statistics depend on the orientation of black vertices given by the labelling. In particular, the statistics given in this section are "strong" variants of the ones used in [La 09, DFŚ14, Doł17, CD22] which we introduce here in accordance with the normalization in Goulden–Jackson's Matching-Jack conjecture.

A statistic of non-orientability is thought of as a quantification of the non-orientability of a map. Such a question is unusual from a topological point of view; a map is either orientable or not. However, we can give a meaning to this notion by considering combinatorial decompositions of maps (Tutte like decompositions). More precisely, we decompose the map by deleting the edges one by one, and decide for each edge whether it contributes to the non-orientability of the map or not. We start with some general definitions related to non-orientable maps.

Definition 2.1.1 (Edge types)**.** *Fix a vertex-labelled map M with a distinguished edge e. Let $N := M\setminus\{e\}$, and let c_1 and c_2 be respectively the black and the white corners of N connected by e. Finally let v and w be the associated vertices and let $\deg(v)$ and $\deg(w)$ denote their respective degrees in M. We then say that;*

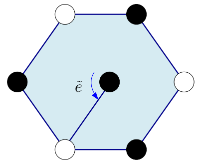
1. *e is an isolated edge if $\deg(v) = \deg(w) = 1$.*
2. *e is a white-leaf edge if $\deg(w) = 1$ and $\deg(v) > 1$.*
3. *e is a black-leaf edge if $\deg(v) = 1$ and $\deg(w) > 1$.*
4. *e is a border if the corners c_1 and c_2 are incident to the same face of N, and the number of faces increases by 1 by adding the edge e to N.*
5. *e is a twist if the corners c_1 and c_2 are incident to the same face of N, and the number of faces does not change by adding the edge e to N.*
6. *e is a handle if the corners c_1 and c_2 are incident to two different faces in the same connected component of N.*
7. *e is a bridge if the corners c_1 and c_2 are incident to two different faces in two different connected components of N.*

If e is an edge of one of the types 3–7, then we have a second way to add an edge between c_1 and c_2 which consists in twisting e. We denote $\widetilde{e}$ this second edge. We say that the pair $(e, \tilde{e})$ is a pair of twisted edges on the map N. For a given map with a distinguished edge (M, e), we denote $(\widetilde{M}, \tilde{e})$ the map obtained by twisting the edge e.

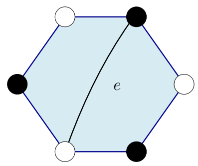
See Fig. 2.1 for examples.

One may notice that in the previous definition we distinguish white and black-leaf edges. This is related to the fact that black vertices are oriented by the labelling, so that we have two ways to add a black-leaf edge on a white corner; see Fig. 2.1a.

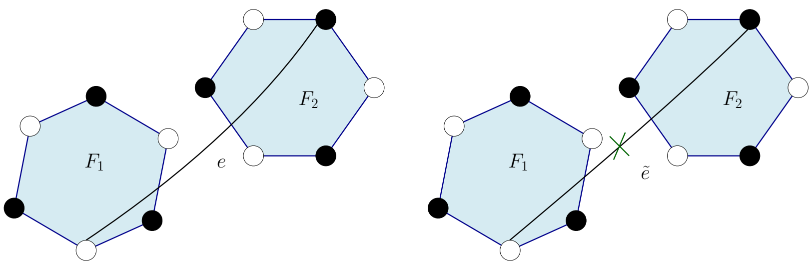
Remark 2.1.2. If N is connected and orientable, then exactly one of the maps M and $\widetilde{M}$ is orientable. However, a map obtained by connecting two different connected orientable maps is always orientable.



(a) A pair of twisted black-leaf edges. The blue arrow gives the orientation of the root of the added black vertex.



(b) A pair of twisted edges $(e, \tilde{e})$ between two corners of the same face; e is a border while $\tilde{e}$ is a twist.



(c) A pair of twisted edges $(e, \tilde{e})$ between two corners of different faces F_1 and F_2. If F_1 and F_2 are on the same connected component then the edges e and $\tilde{e}$ are handles, otherwise they are bridges.

Figure 2.1: The different ways of adding an edge connecting two corners of a vertex-labelled map.

However, when we consider vertex-labelled maps (an orientation is chosen for each black vertex of the map), then for each oriented map N exactly one of the maps M and $\widetilde{M}$ is oriented (even when M is not connected).

Before defining statistics of non-orientability on maps, we start by defining them on edges. This is given by *strong measures of non-orientability* (this is a variant of the definition given in [La 09], see also Definition 4.3.1). We consider a parameter b. This parameter will be later related to the Jack parameter α by $b = \alpha - 1$.

Definition 2.1.3. *We call* strong measure of non-orientability (SMON) *a function $\boldsymbol{\rho}$ defined on the set of connected vertex-labelled maps with a distinguished edge (M, e), with values in $\{1, b\}$, satisfying the following conditions:*

1. *$\boldsymbol{\rho}(M, e) = b$ if e is a twist.*
2. *if e is a handle, a bridge or a black-leaf edge then $\boldsymbol{\rho}$ satisfies the condition*

$$\left\{\boldsymbol{\rho}(M, e), \boldsymbol{\rho}(\widetilde{M}, \tilde{e})\right\} = \{1, b\}\,.$$

Moreover, if M is oriented then $\boldsymbol{\rho}(M, e) = 1$.

3. $\boldsymbol{\rho}(M, e) = 1$ *otherwise.*

More generally, if M is a vertex-labelled map (not necessarily connected), and e is an edge of M, then we set

$$\boldsymbol{\rho}(M, e) := \boldsymbol{\rho}(M_e, e),$$

where M_e is the connected component of M containing e.

It is actually possible to define a SMON directly on disconnected maps instead of defining them on connected maps and then extending them to disconnected maps. However, the definition used here makes the link with the connected generating series easier (see Remark 2.4.5). Indeed, a SMON $\boldsymbol{\rho}$ is not uniquely defined, since we have two choices for some edge types (item 2 in the previous definition). The convention we use here insures that these choices depend only on the connected component containing the edge.

Let M be a vertex-labelled map (connected or not) and let $e_1, e_2, \ldots, e_d$ be d distinct edges of M. For $0 \leq j \leq d-1$, we denote M_j the map obtained by deleting the edges e_d, e_{d-1},..., e_{d-j+1} from M. We define $\boldsymbol{\rho}(M, e_d, e_{d-1}, \ldots, e_1)$ as the weight obtained by deleting the edges e_d, $e_{d-1}, \ldots e_1$ successively:

$$\boldsymbol{\rho}(M, e_d, e_{d-1}, \ldots, e_1) := \boldsymbol{\rho}(M_0, e_d)\boldsymbol{\rho}(M_1, e_{d-1}) \ldots \boldsymbol{\rho}(M_{d-1}, e_1).$$

Given a SMON $\boldsymbol{\rho}$ we can define a SSON as follows. First we start by fixing a decomposition algorithm which associates to each map M a total order $\prec_M$ on its edges; $e_1 \prec_M e_2 \prec_M \cdots \prec_M e_{|M|}$, where $(e_i)_{1 \leq i \leq |M|}$ denote the edges of M. Then we define the *non-orientability weight* of M associated to $\boldsymbol{\rho}$ by

$$\boldsymbol{\rho}(M) := \boldsymbol{\rho}(M, e_{|M|}, e_{|M|-1}, \ldots, e_1).$$

Hence, $e \prec_M e'$ is to be understood as "e has appeared before e' in the construction of M". Note that $\boldsymbol{\rho}(M)$ is necessarily of the form b^r for some non-negative integer r, since we always have $\boldsymbol{\rho}(M, e) \in \{1, b\}$ by definition.

In [La 09, Doł17, CD22], several statistics of non-orientability were introduced which are all of the form explained above. However, the decomposition algorithm varies depending on the class of maps involved.

In this chapter, we use the following decomposition algorithm for vertex-labelled maps. We first define an order on vertices.

Fix a vertex-labelled map M. To each black vertex v of M we associate the pair of positive integers (d, j), where d is the degree of the vertex and j is the number given to v by the labelling of the map. By definition, the pairs associated to two distinct black vertices are different. We define then a linear order $\prec_M$ on the black vertices of M given by the lexicographic order on the pairs (d, j). In particular, the maximal black vertex with respect to $\prec_M$ is the vertex with maximal label among vertices of maximal degree.

This order on vertices induces an order on edges as follows. Let v_m denote the maximal black vertex of M and let d be its degree. We label the edges incident to v by $e_{|M|}, \ldots e_{|M|-d+1}$ as they appear when we turn around v starting from the vertex root. We then start again with the map obtained from M by forgetting v and all the edges incident to it. The order on the edges of M is then defined by $e_1 \prec_M e_2 \prec_M \cdots \prec_M e_{|M|}$. In other terms, this order on

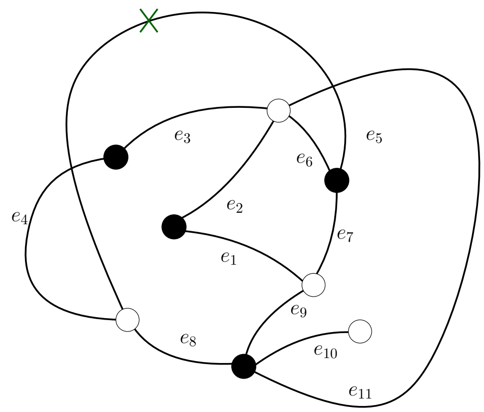


Figure 2.2: The order on edges obtained from the vertex-labelled map of Fig. 1.9a.

edges corresponds to decomposing the map starting at the root of the maximal black vertex and iterating.

Note that when we delete a black vertex and the edges incident to it the degrees of the other black vertices do not change. In particular, the map obtained by keeping the same labels on these black vertices is "well-labelled" and has the same order on its black vertices.

Conversely, vertex-labelled maps can be constructed recursively as follows; a map with $k+1$ black vertices is obtained in a unique way by adding the maximal black vertex $v_{\max}$ (and possibly some new white vertices incident to it) to a map with k black vertices. In this operation, the edges are added one by one while turning around $v_{\max}$, such that the last edge added is the edge to which points the root of $v_{\max}$.

Example 2.1.4. In Fig. 2.2, we give the order on edges obtained from the vertex-labelled map of Fig. 1.9a. One can then check that

- e_{11} and e_7 are borders,
- e_{10}, e_4 and e_2 are white-leaf edges,
- e_8, e_5 and e_3 are black-leaf edges,
- e_9 and e_6 are twists,
- e_1 is an isolated edge.

Remark 2.1.5. Note that if M is a vertex-labelled map, and N is one of its connected components then N inherits a structure of a vertex-labelled map from M. Moreover, the order $\prec_N$ is the order induced by $\prec_M$.

We now define a family of strong statistics of non-orientability on vertex-labelled maps.

Definition 2.1.6 (Strong Statistic of Non-Orientability for vertex-labelled maps)**.** *Let $\boldsymbol{\rho}$ be a* SMON *and let M be a vertex-labelled map with n edges. We define the non-orientability weight $\boldsymbol{\rho}(M)$ of M by*

$$\boldsymbol{\rho}(M) := \boldsymbol{\rho}(M, e_n, e_{n-1}, \ldots, e_1).$$

where e_i denote the edges of M ordered such that $e_1 \prec_M e_2 \prec_M \cdots \prec_M e_n$.

We then define the statistic $\boldsymbol{\vartheta}_\rho$ on vertex-labelled maps with non-negative integer values, given for every M by $\boldsymbol{\rho}(M) = b^{\boldsymbol{\vartheta}_\rho(M)}$.

Remark 2.1.7. Let M be a vertex-labelled map with connected components $M_1, \dots, M_m$. Since SMONs are defined independently on each connected component of a given map M, we have

$$\boldsymbol{\rho}(M) := \prod_{1 \le i \le m} \boldsymbol{\rho}(M_i),$$

where each one of the connected components is seen as vertex-labelled (see Remark 2.1.5).

Example 2.1.8. Applying the definition Definition 1.3.12 on the vertex-labelled maps of size 3 given in Fig. 1.12b, one can check any SSON $\boldsymbol{\vartheta}_\rho$ associates to the maps rooted by c_1 and c_3 the value 2 and the map rooted by c_2 the value 1 (we use here the notation of Fig. 1.12).

2.2 Combinatorial interpretation of the Chapuy–Dołęga operators

The purpose of this section is to give the combinatorial interpretation of the differential operators introduced by Chapuy and Dołęga in [CD22] (see Section 1.4.2) and their connection to the enumeration of maps considered with a non-orientability weight. We recall the notation;

$$\mathcal{P} := \mathrm{Span}_{\mathbb{Q}(\alpha)}\{p_\lambda\}_{\lambda \in \mathbb{Y}}, \qquad \text{and} \qquad \mathcal{P}_Y := \mathrm{Span}_{\mathbb{Q}(b)}\left\{y_i p_\lambda\right\}_{i \in \mathbb{N}, \lambda \in \mathbb{Y}},$$

where $(y_i)_{i \ge 0}$ is an infinite family of variables.

Let M be a bipartite map (connected or not). We define its *weight* by

$$\mathrm{weight}(M) := \frac{p_{\lambda^\diamond(M)}}{\alpha^{|\mathcal{V}_\bullet(M)|}} = \frac{1}{\alpha^{|\mathcal{V}_\bullet(M)|}} \prod_f p_{\deg(f)} \in \mathcal{P},$$

where $\lambda^\diamond(M)$ is the face-type of M defined in Section 1.1, and the product is taken over all faces f of M (here $\deg(f)$ denotes the degree of the face divided by 2). We now introduce the notion of rooting for vertex-labelled maps.

Definition 2.2.1. *We recall that a map M (connected or not) is rooted if it has a distinguished oriented black corner c, called the root of the map. A rooted map is vertex-labelled if all the black vertices are numbered as in Definition 1.1.1 except for the root vertex v_c. In this definition, we allow the root vertex to be of degree 0.*

As in Definition 2.1.6, we define an order $\prec_{(M,c)}$ on the black vertices of a vertex-labelled rooted map (M, c) with the convention that the root vertex is always maximal. In other terms:

- *for any non-root vertex v of M, we have $v \prec_{(M,c)} v_c$,*
- *for any non-root vertices v and v', $v \prec_{(M,c)} v'$ if and only if $v \prec_{M'} v'$, where M' is the vertex-labelled map obtained from M by deleting v_c and all the edges incident to it.*

Finally we denote $\boldsymbol{\rho}(M, c)$ the non-orientability weight associated to (M, c) with respect to $\prec_{(M,c)}$.

We associate to a rooted vertex-labelled map (M,c) the weight defined by:

$$\text{weight}(M,c) := \frac{1}{\alpha^{|\mathcal{V}_\bullet(M)|}} y_{\deg(f_c)} \prod_{f \neq f_c} p_{\deg(f)} \in \mathcal{P}_Y,$$

where $\deg(f_c)$ is the degree of the root face divided by 2, and the product runs over the faces of M different from the root face.

We recall the definition of the catalytic operators given in Section 2.2;

$$Y_+ = \sum_{i \geq 0} y_{i+1} \frac{\partial}{\partial y_i},$$

$$\Gamma_Y = (1+b) \cdot \sum_{i,j \geq 1} y_{i+j} \frac{j \partial^2}{\partial y_{i-1} \partial p_j} + \sum_{i,j \geq 1} y_i p_j \frac{\partial}{\partial y_{i+j-1}} + b \cdot \sum_{i \geq 1} y_{i+1} \frac{i \partial}{\partial y_i}.$$

The following proposition gives a combinatorial interpretation for the operators Y_+ and Γ_Y, which corresponds to the particular case $k = 1$ in [CD22, Proposition 4.4].

Proposition 2.2.2 ([CD22, Proposition 4.4]). *Let (M,c) be a vertex-labelled rooted map (connected or not). Then,*

$$Y_+ \cdot \text{weight}(M,c) = \text{weight}(M \cup \{e\}, c_2), \tag{2.1}$$

where e is an edge connecting c to a new white vertex. Moreover, for every SMON $\boldsymbol{\rho}$, we have

$$\Gamma_Y \cdot \text{weight}(M,c) = \sum_e \boldsymbol{\rho}(M \cup \{e\}, e) \text{weight}(M \cup \{e\}, c_2), \tag{2.2}$$

where the sum is taken over all ways to add an edge e connecting c to some white corner c' (without adding a new white vertex).

In the two cases, the obtained map $M \cup \{e\}$ is rooted such that the new root c_2 is pointing to the added edge e (see Fig. 2.3). With this rooting, $M \cup \{e\}$ inherits a vertex-labelling form the map M.

Proof. When we add a white leaf in a root face of degree i we obtain a root face of degree $i+1$. This gives the first equation. Note that in this case e is either an isolated edge or a white-leaf edge, and as a consequence, Eq. (2.1) can be rewritten as follows;

$$Y_+ \cdot \text{weight}(M,c) = \boldsymbol{\rho}(M \cup \{e\}, e) \text{weight}(M \cup \{e\}, c_2).$$

Let us now prove Eq. (2.2). When we add an edge e on the root corner, we distinguish three cases.

- The two corners c and c' lie in distinct faces of respective sizes $i-1$ and j, for $i,j \geq 1$. Let us show that this case corresponds to the first term of the operator Γ_Y. First, notice that once we fix a face of degree j, we have j ways to choose the corner c' and when we add the edge e we form a face of size $i+j$. This explains the term $\sum_{i,j \geq 1} y_{i+j} \frac{j\partial^2}{\partial y_{i-1} \partial p_j}$.

  Let $\tilde{e}$ denote the edge obtained by twisting e, see Section 2.1. In this case, we have by Definition 2.1.3 item 2 that

  $$\boldsymbol{\rho}(M \cup \{e\}, e) + \boldsymbol{\rho}(M \cup \{\tilde{e}\}, \tilde{e}) = 1 + b,$$

  and this explains the factor $1+b$ in the first term of Γ_Y.

Figure 2.3: The root c_2 of a map obtained by adding an edge e on a map rooted in c.

- The two corners c and c' lie in the same face of degree $i+j-1$ and the added edge e is a border, which splits the face into two faces of respective degrees i and j, with $i, j \geq 1$. Since we fix the degrees of the formed faces we only have one choice for the corner c'. Then $\boldsymbol{\rho}(M \cup \{e\}, e) = 1$, by Definition 2.1.3 item 3. Hence, this case corresponds to the second term of Γ_Y.
- The two corners c and c' lie in the same face of degree i and the added edge e is a twist. In this case we have i ways to choose the corner c'. By adding e we form a face of degree $i+1$ and $\boldsymbol{\rho}(M \cup \{e\}, e) = b$; see Definition 2.1.3 item 1. Hence, this case corresponds to the third term of Γ_Y. □

Moreover, if M is a vertex-labelled map, then

$$\frac{y_0}{\alpha}\text{weight}(M) = \text{weight}(M', c),$$

where M' is obtained from M by adding an isolated root black vertex v_c. Here we divide by α since we increase the number of the black vertices of the map by 1.

Finally, applying

$$\Theta_Y := \sum_{i \geq 1} p_i \frac{\partial}{\partial y_i} : \mathcal{P}_Y \to \mathcal{P}.$$

on the weight of the map corresponds to forgetting the root and to obtain the marking of an unrooted map. In other words, if (M, c) is a vertex-labelled rooted map with vertex root v_c, and $\deg(v_c)$ is maximal, then

$$\Theta_Y \cdot \text{weight}(M, c) = \text{weight}(M). \tag{2.3}$$

Moreover, M is a vertex-labelled map with the convention that v_c is the maximal vertex with respect to $\prec_M$.

To conclude, to add a black vertex of degree m on a map M we proceed as follows. We first apply $\frac{y_0}{\alpha}$ to add an isolated vertex. We then attach to it m edges. For each of these edges we choose to add a new white leaf (by applying Y_+) or to connect the edge to an existing white vertex (by applying Γ_Y). Finally, we forget the root by applying Θ_Y. This is precisely given by the operator $\mathcal{B}_m$:

$$\mathcal{B}_m^{(\alpha)}(\boldsymbol{p}, u) := \Theta_Y \left(\Gamma_Y + uY_+\right)^m \frac{y_0}{1+b}. \tag{2.4}$$

We then have the following proposition. We refer to [CD22, BD23] for a more formal proof.

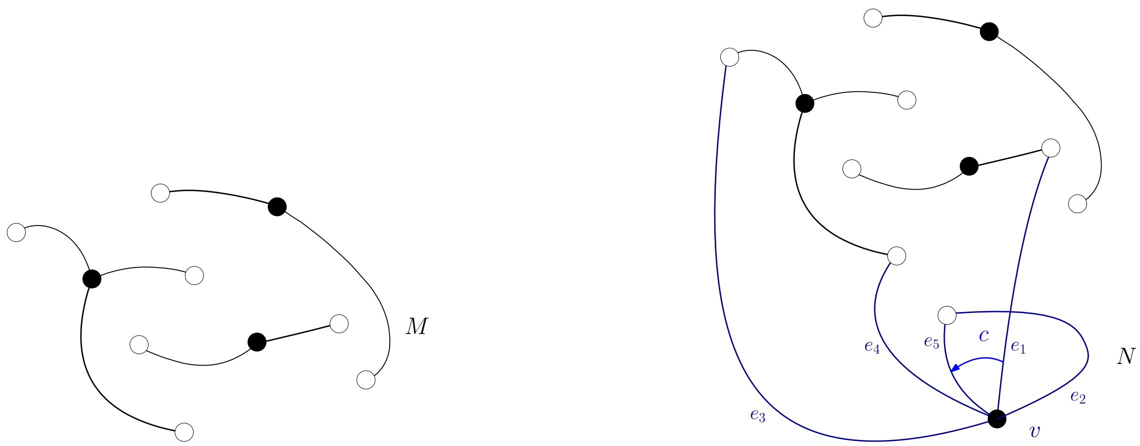


Figure 2.4: On the left a map M and on the right an example of a map N obtained by the action of $\mathcal{B}_5$. The new edges are represented in blue.

Proposition 2.2.3. *Fix a* SMON $\boldsymbol{\rho}$. *Let $m \geq 1$ and let M be a vertex-labelled map such that all the black vertices of M have degree less or equal to m. Then*

$$\mathcal{B}_m(\boldsymbol{p}, u) \cdot \text{weight}(M) u^{|\mathcal{V}_\circ(M)|} = \sum_N \boldsymbol{\rho}(N, e_m, e_{m-1}, \ldots, e_1) \text{weight}(N) u^{|\mathcal{V}_\circ(N)|},$$

where the sum is taken over all vertex-labelled maps N obtained by adding to M a maximal vertex v of degree m rooted in an oriented corner c, using potentially new white vertices, and where $e_m, e_{m-1}, \ldots, e_1$ are the added edges as they appear when we turn around v starting from the root c.

We recall that $|\mathcal{V}_\circ(M)|$ is the number of white vertices in M. See Fig. 2.4 for an example of the action of $\mathcal{B}_5$.

Remark 2.2.4. In [CD22], Chapuy and Dołęga give this combinatorial construction for k-constellations using a more general family of operators $\mathcal{B}_m(\boldsymbol{p}, u_1, \ldots, u_k)$.

2.3 A differential construction formula for $\tau^{(\alpha)}$ and commutation relation for operators $\mathcal{B}_n$

We recall that $\tau^{(\alpha)}(r, \boldsymbol{p}, \boldsymbol{q}, \underline{u})$ is the specialized function defined in Eq. (1.27). The following differential equation, due to Chapuy and Dołęga, will play a crucial role in this section.

Theorem 2.3.1 ([CD22, Theorem 5.7])**.** *For any $m \geq 1$, we have*

$$t^m \frac{\mathcal{B}_m(\boldsymbol{p}, u)}{m} \cdot \tau^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \underline{u}) = \frac{\partial}{\partial q_m} \tau^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \underline{u}).$$

We deduce a differential expression of the function $\tau^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \underline{u})$ using the operators $\mathcal{B}_m(\boldsymbol{p}, u)$.

Theorem 2.3.2. *The function $\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\underline{u})$ has the following expression:*

$$\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\underline{u}) = \exp\left(\sum_{m\geq 1}\frac{t^m q_m}{m}\mathcal{B}_m(\boldsymbol{p},u)\right)\cdot 1.$$

Remark 2.3.3. Recall that $\mathcal{B}_m(\boldsymbol{p},u)$ maps $\mathcal{P}$ to $\mathcal{P}[u]$, therefore

$$\exp\left(\sum_{m\geq 1}\frac{t^m q_m}{m}\mathcal{B}_m(\boldsymbol{p},u)\right) : \mathcal{P}\to\mathbb{Q}(\alpha)[\boldsymbol{p},\boldsymbol{q},u][\![t]\!]$$

is a well-defined operator.

Proof. Fix an integer $n\geq 1$ and a partition $\mu\vdash n$. We want to prove that

$$[t^n q_\mu]\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\underline{u}) = \frac{1}{\ell(\mu)!}\sum_{\gamma\models\mu}\frac{\mathcal{B}_{\gamma_{\ell(\mu)}}(\boldsymbol{p},u)}{\gamma_{\ell(\mu)}}\cdots\frac{\mathcal{B}_{\gamma_1}(\boldsymbol{p},u)}{\gamma_1}\cdot 1, \tag{2.5}$$

where the sum is taken over all the reorderings γ of μ. We start by noticing that[1], for any reordering γ of μ,

$$[t^n q_\mu]\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\underline{u}) = [t^n q_\emptyset]\left(\prod_{j\geq 1}\frac{1}{m_j(\mu)!}\right)\left(\prod_{1\leq i\leq\ell(\mu)}\frac{\partial}{\partial q_{\gamma_i}}\right)\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\underline{u}), \tag{2.6}$$

where $[q_\emptyset]$ denotes the extraction of the constant term in the variables q_i. Since there are $\ell(\mu)!\prod_{j\geq 1}\frac{1}{m_j(\mu)!}$ reorderings γ of μ, we can rewrite the last equation as follows

$$[t^n q_\mu]\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\underline{u}) = [t^n q_\emptyset]\frac{1}{\ell(\mu)!}\sum_{\gamma\models\mu}\frac{\partial}{\partial q_{\gamma_1}}\cdots\frac{\partial}{\partial q_{\gamma_{\ell(\mu)}}}\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\underline{u}).$$

Using Theorem 2.3.1 and the fact that the operators $\frac{\partial}{\partial q_i}$ commute with the operators $\mathcal{B}_j(\boldsymbol{p},u)$ we obtain

$$\begin{aligned}
[t^n q_\mu]\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\underline{u}) &= [t^n q_\emptyset]\frac{1}{\ell(\mu)!}\sum_{\gamma\models\mu}\frac{t^{\gamma_{\ell(\mu)}}\mathcal{B}_{\gamma_{\ell(\mu)}}(\boldsymbol{p},u)}{\gamma_{\ell(\mu)}}\cdots\frac{t^{\gamma_1}\mathcal{B}_{\gamma_1}(\boldsymbol{p},u)}{\gamma_1}\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},u)\\
&= \frac{1}{\ell(\mu)!}\sum_{\gamma\models\mu}\frac{\mathcal{B}_{\gamma_{\ell(\mu)}}(\boldsymbol{p},u)}{\gamma_{\ell(\mu)}}\cdots\frac{\mathcal{B}_{\gamma_1}(\boldsymbol{p},u)}{\gamma_1}[t^0 q_\emptyset]\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\underline{u})\\
&= \frac{1}{\ell(\mu)!}\sum_{\gamma\models\mu}\frac{\mathcal{B}_{\gamma_{\ell(\mu)}}(\boldsymbol{p},u)}{\gamma_{\ell(\mu)}}\cdots\frac{\mathcal{B}_{\gamma_1}(\boldsymbol{p},u)}{\gamma_1}\cdot 1.
\end{aligned}$$

This concludes the proof of Eq. (2.5) and hence the proof of the theorem. □

[1] This equation holds for any formal power-series in $\boldsymbol{q}$, and in particular is independent from the definition of $\tau^{(\alpha)}$.

A second consequence of Theorem 2.3.1 is the following commutation relation which will be useful in Chapter 4. The key idea of the proof is to consider the action of the commutators of $\mathcal{B}_\ell(\boldsymbol{p}, u)$ and $\mathcal{B}_m(\boldsymbol{p}, u)$ on the function $\tau^{(\alpha)}(\boldsymbol{p}, \boldsymbol{q}, \underline{u})$ and then extract some coefficient.

Proposition 2.3.4. *Let $m, \ell \geq 1$. Then,*

$$[\mathcal{B}_\ell(\boldsymbol{p}, u), \mathcal{B}_m(\boldsymbol{p}, u)] = 0.$$

Proof. Theorem 2.3.1 implies that

$$[\mathcal{B}_\ell(\boldsymbol{p}, u), \mathcal{B}_m(\boldsymbol{p}, u)] \cdot \tau^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \underline{u}) = 0, \tag{2.7}$$

where $[\cdot, \cdot]$ denotes the commutator. Indeed,

$$\begin{aligned} t^{\ell+m}\mathcal{B}_\ell(\boldsymbol{p}, u)\mathcal{B}_m(\boldsymbol{p}, u) \cdot \tau^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \underline{u}) &= \frac{m\partial}{\partial q_m}\frac{\ell\partial}{\partial q_\ell}\tau^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \underline{u}) \\ &= \frac{\ell\partial}{\partial q_\ell}\frac{m\partial}{\partial q_m}\tau^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \underline{u}) = t^{\ell+m}\mathcal{B}_m(\boldsymbol{p}, u)\mathcal{B}_\ell(\boldsymbol{p}, u) \cdot \tau^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \underline{u}), \end{aligned}$$

where the first and third equalities are consequences of Theorem 2.3.1 and the fact that $\frac{\partial}{\partial q_m}$ and $\mathcal{B}_\ell(\boldsymbol{p}, \underline{u})$ commute. By extracting the coefficient of $J^{(\alpha)}_\xi(\boldsymbol{q})$ in Eq. (2.7):

$$[\mathcal{B}_\ell(\boldsymbol{p}, u), \mathcal{B}_m(\boldsymbol{p}, u)] \cdot \frac{J^{(\alpha)}_\xi(\boldsymbol{p})J^{(\alpha)}_\xi(\underline{u})}{j^{(\alpha)}_\xi} = 0.$$

This concludes the proof, since Jack polynomials form a basis of $\mathcal{P}$, and $\frac{J^{(\alpha)}_\xi(\underline{u})}{j^{(\alpha)}_\xi} \neq 0$ by Theorem 1.2.6. □

The commutation relations of Proposition 2.3.4 combined with Proposition 2.2.3 have the following combinatorial interpretation; given a bipartite map N, the fact of adding two black vertices of respective degrees ℓ and m in one order or the other does not change the contribution of the non-orientability weight. More precisely,

$$\sum_M \boldsymbol{\rho}(M, e_1 \ldots, e_\ell, e'_1, \ldots, e'_m) = \sum_M \boldsymbol{\rho}(M, e'_1, \ldots, e'_m, e_1, \ldots, e_\ell)$$

where the two sums run over maps M obtained by adding two black vertices of respective degrees ℓ and m incident to the edges $(e_i)_{1\leq i\leq \ell}$ and $(e'_i)_{1\leq i\leq \ell}$ respectively. In particular, this commutation is obvious combinatorially for $b \in \{0, 1\}$, but a combinatorial proof for general b seems more challenging.

2.4 Matching-Jack conjecture for marginal sums

We recall that the marginal sum coefficients $c^\pi_{\mu,m}$ are defined by

$$\tau^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \underline{u}) = 1 + \sum_{n\geq 1} t^n \sum_{\pi,\mu \vdash n} \sum_{1\leq m\leq n} \frac{\overline{c}^\pi_{\mu,m}(\alpha)}{z_\pi \alpha^{\ell(\pi)}} p_\pi q_\mu u^m.$$

We now state the main theorem of this chapter.

Theorem 2.4.1. *For any* SMON $\boldsymbol{\rho}$*, we have*

$$\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\underline{u}) = \sum_{M} \frac{b^{\boldsymbol{\vartheta}_{\boldsymbol{\rho}}(M)}}{\alpha^{|\mathcal{V}_\bullet(M)|} z_{\lambda^\bullet(M)}} p_{\lambda^\bullet(M)} q_{\lambda^\diamond(M)} u^{|\mathcal{V}_\circ(M)|}, \tag{2.8}$$

where the sum is taken over all vertex-labelled maps, oriented or not.

Equivalently, for any partitions π,μ of the same size n and any integer $m \leq n$ we have

$$\overline{c}^{\pi}_{\mu,m} = \sum_{M} b^{\boldsymbol{\vartheta}_{\boldsymbol{\rho}}(M)},$$

the sum being taken over vertex-labelled maps M such that $\lambda^\bullet(M) = \pi$, $\lambda^\diamond(M) = \mu$ and $|\mathcal{V}_\circ(M)| = m$.

The proof provided in [Ben22] for Theorem 2.4.1 starts from the result of [CD22] on the marginal sums in the b-conjecture and uses a non-trivial symmetry property satisfied by the statistics of non-orientability used in this case. It turns out that this symmetry is related to the commutation relation of Proposition 2.3.4. We give here a proof which uses directly this commutation.

Proof. Using the symmetry of the function $\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\underline{u})$ in the alphabets $\boldsymbol{p}$ and $\boldsymbol{q}$, we write

$$\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\underline{u}) = 1 + \sum_{n\geq 1} t^n \sum_{\pi,\mu\vdash n} \sum_{1\leq m\leq n} \frac{\overline{c}^{\pi}_{\mu,m}(\alpha)}{z_\pi \alpha^{\ell(\pi)}} q_\pi p_\mu u^m.$$

We get that for any partition $\pi \vdash n \geq 1$,

$$\sum_{\substack{\mu\vdash n\\ 1\leq m\leq n}} \overline{c}^{\pi}_{\mu,m}\, p_\mu u^m = z_\pi \alpha^{\ell(\pi)} [t^{|\pi|} q_\pi] \tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\underline{u}).$$

We now use Eq. (2.6) and Theorem 2.3.1 to write[2]

$$\begin{aligned}
\sum_{\substack{\mu\vdash n\\ 1\leq m\leq n}} \overline{c}^{\pi}_{\mu,m}\, p_\mu u^m &= z_\pi \alpha^{\ell(\pi)} [t^n q_\emptyset] \left(\prod_{j\geq 1} \frac{1}{m_j(\pi)!}\right) \left(\prod_{1\leq i\leq \ell(\pi)} \frac{\partial}{\partial q_{\pi_i}}\right) \tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\underline{u}) \\
&= z_\pi \alpha^{\ell(\pi)} \left(\prod_{j\geq 1} \frac{1}{m_j(\pi)!}\right) \left(\prod_{1\leq i\leq \ell(\pi)} \frac{\mathcal{B}_{\pi_i}(\boldsymbol{p},u)}{\pi_i}\right) \cdot 1 \\
&= \alpha^{\ell(\pi)} \mathcal{B}_{\pi_1}(\boldsymbol{p},u) \cdots \mathcal{B}_{\pi_{\ell(\pi)}}(\boldsymbol{p},u) \cdot 1.
\end{aligned} \tag{2.9}$$

We also use here the fact that the $\frac{\partial}{\partial q_i}$ commute with the operators $\mathcal{B}_j(\boldsymbol{p})$. Applying Proposition 2.2.3 inductively we get

$$\sum_{\substack{\mu\vdash n\\ 1\leq m\leq n}} \overline{c}^{\pi}_{\mu,m}\, p_\mu u^m = \alpha^{\ell(\pi)} \sum_{M} \boldsymbol{\rho}(M) \text{weight}(M) u^{|\mathcal{V}_\circ(M)|}, \tag{2.10}$$

[2] One can also obtain Eq. (2.9) from Theorem 2.3.2 and Proposition 2.3.4.

where the sum is taken over vertex-labelled maps of black-type π, and

$$\mathrm{weight}(M) := \frac{p_{\lambda^\diamond(M)}}{\alpha^{|\mathcal{V}_\bullet(M)|}}.$$

We conclude by extracting the coefficient of $p_\mu u^m$ on both sides of Eq. (2.10). $\square$

Note that Theorem 2.3.2 and Theorem 2.4.1 imply Theorem 1.5.1.

Remark 2.4.2. As observed in the proof above, the left hand-side of Eq. (2.8) is clearly symmetric in $\boldsymbol{p}$ and $\boldsymbol{q}$. This symmetry is however not clear on the right-hand side. Indeed, exchanging the two alphabets corresponds the duality on maps which exchanges black vertices and faces, but the non-orientability weight is not invariant under this operation in general.

Using the correspondence between bipartite maps and matchings (Proposition 1.1.10) we deduce the following corollary of Theorem 2.4.1, which is a special case of Conjecture 4.

Corollary 2.4.3. *Fix a partition $\pi \vdash n \geq 1$, and two bipartite matchings δ_1 and δ_2 on $\mathcal{N}_n$ such that $\Lambda(\delta_1, \delta_2) = \pi$. There exists a statistic $\mathrm{st}_{\delta_1,\delta_2}$ on the matchings of $\mathcal{N}_n$ with non-negative integer values, such that*

- $\mathrm{st}_{\delta_1,\delta_2}(\delta) = 0$ *if and only if δ is bipartite.*
- *for any partition $\mu \vdash n$ and any integer $1 \leq m \leq n$, we have*

$$\overline{c}^{\pi}_{\mu,m} = \sum_{\substack{\nu \vdash n \\ \ell(\nu)=m}} \sum_{\delta \in \mathfrak{F}^{\delta_1,\delta_2}_{\mu,\nu}} b^{\mathrm{st}_{\delta_1,\delta_2}(\delta)}.$$

Remark 2.4.4. Note that by definition $\mathrm{st}_{\delta_1,\delta_2}$ is obtained from a strong statistic of non-orientability via the bijection from maps to matchings. However, the construction of these statistics by edge deletion as explained in Section 2.1 makes it easier to understand them on maps than on matchings.

Remark 2.4.5. We can deduce from Theorem 2.4.1 that

$$\log\left(\tau^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, u)\right) = \sum_{M} \frac{b^{\vartheta(M)}}{\alpha^{|\mathcal{V}_\bullet(M)|} z_{\lambda^\bullet(M)}} p_{\lambda^\diamond(M)} q_{\lambda^\bullet(M)} u^{|\mathcal{V}_\circ(M)|}, \tag{2.11}$$

where the sum is taken over all vertex-labelled **connected** maps, oriented or not.

We use here the fact that $\frac{b^{\vartheta(M)}}{\alpha^{|\mathcal{V}_\bullet(M)|} z_{\lambda^\bullet(M)}} p_{\lambda^\diamond(M)} q_{\lambda^\bullet(M)} u^{|\mathcal{V}_\circ(M)|}$ is multiplicative on the connected components of M; see Remark 2.1.7. Hence, we can apply the logarithm in order to obtain the generating series of connected maps (we use here a variant of the exponential formula for labelled combinatorial classes see e.g. [FS09, Chapter II]).

One may notice that the normalization in Eq. (2.11) does not directly give the similar result on marginal sums for the b-conjecture (see Theorem 1.3.10); indeed the sum should be taken over rooted maps instead of vertex-labelled maps in order to obtain the right normalization factors. As a consequence, a different algorithm of decomposition is required to define statistics of non-orientability on rooted maps (see [CD22, Section 3.2]).

2.5 Generalization to constellations

As mentioned in Remark 2.2.4, the combinatorial construction of this chapter also applies to the case of constellations. We state here a generalized version of Theorem 2.4.1.

Fix an integer $k \geq 1$. For an integer $n \geq 1$, two partitions $\pi, \mu \vdash n$, and integers $1 \leq m_1, \dots, m_k \leq n$, we define the generalized marginal coefficient

$$\overline{c}^{\pi}_{\mu,m_1,\dots,m_k}(\alpha) = \sum_{\substack{\nu^{(1)},\dots,\nu^{(k)} \vdash n \\ \ell(\nu^{(1)})=m_1 \dots \ell(\nu^{(k)})=m_k}} c^{\pi}_{\mu,\nu^{(1)},\dots,\nu^{(k)}}(\alpha),$$

where $c^{\pi}_{\mu,\nu^{(1)},\dots,\nu^{(k)}}$ are the coefficients obtained by the power-sum expansion of $\tau^{(\alpha)}_k(t, \boldsymbol{p}, \boldsymbol{q}^{(0)}, \boldsymbol{q}^{(1)}, \dots, \boldsymbol{q}^{(k)})$ defined in Eq. (1.33). We then have the following theorem.

Theorem 2.5.1. *The coefficient $\overline{c}^{\pi}_{\mu,m_1,\dots,m_k}$ is a polynomial in b with non-negative integer coefficients.*

As in the case $k = 1$, this theorem has a combinatorial reformulation; the coefficient $\overline{c}^{\pi}_{\mu,m_1,\dots,m_k}$ counts constellations with control of the degrees of faces and vertices of color 0, and the number of vertices of color i for $1 \leq i \leq k$; see [Ben22, Theorem 1.4].

Chapter 3

Integrality in the Matching-Jack conjecture and the Farahat-Higman algebra

This chapter is based on the work of [Ben23a].

The main purpose of this section is to prove the integrality in the Matching-Jack conjecture (Theorem 1.5.2) using the integrality of the marginal sum coefficients proved in the previous chapter. The starting point of this proof is *the multiplicativity property* (Eq. (3.1)), which provides a system of equations relating the coefficients $c^{\pi}_{\mu,\nu}$ to the marginal coefficients $\overline{c}^{\pi}_{\mu,m}$. Another key tool of the proof is a connection between the coefficients of the Matching-Jack conjecture and the Farahat-Higman algebra.

The Farahat-Higman algebra was introduced in [FH59] in order to study the structure coefficients of the conjugacy classes $C_\mu(n)$ in the center of the symmetric group algebra $Z(\mathbb{Z}\mathfrak{S}_n)$. It has been shown that the Farahat-Higman algebra is isomorphic to the algebra of integral symmetric functions; see [GJ94, CGS04]. It is also related to the algebra of partial permutations introduced by Ivanov and Kerov in [IK99].

In Section 3.4, we will consider a graded version of the Farahat-Higman algebra that we denote $\mathcal{Z}_\infty$ and that has been introduced in [Mac95, Example 24, page 131]. We study the structure coefficients of this algebra, called *top coefficients* of the Farahat-Higman algebra. This leads to a new basis of $\mathcal{Z}_\infty$, which is useful in the proof of the main theorem. Since the Farahat-Higman is of independent interest, Theorem 3.4.6 might reveal itself useful in the future, independently of its application in this chapter.

Structure of the chapter

In Section 3.1, we explain the main ideas of the proof and we give some consequences of the main result. In Section 3.2, we introduce the top coefficients t^{ρ}_{π} and we formulate Theorem 3.2.12 which is a key step in the proof of the main result. In Section 3.3 we prove that Theorem 3.2.12 implies Theorem 1.5.2. We consider a graded version of the Farahat-Higman algebra in Section 3.4 and we use it to prove Theorem 3.2.12. In Section 3.5, we give a combinatorial interpretation for the multiplicativity property when $b = 0$ and $b = 1$ and we use it to obtain a new proof for the Matching-Jack conjecture in these cases. In Section 3.6, we give other consequences of the main result related to the generalized Goulden–Jackson conjectures.

3.1 Idea of the proof and some other consequences

3.1.1 Steps of the proof

A key tool of the proof of Theorem 1.5.2 is the following multiplicativity property which is a consequence of the orthogonality of Jack polynomials (see also Proposition 3.3.1); for all partitions λ, μ, ν of the same size $n \geq 1$ and for every $l \geq 1$, one has

$$\sum_{\kappa \vdash n} c^{\lambda}_{\mu,\kappa}\, \overline{c}^{\kappa}_{\nu,l} = \sum_{\theta \vdash n} \overline{c}^{\lambda}_{\theta,l}\, c^{\theta}_{\mu,\nu}. \tag{3.1}$$

The equations (3.1) will be considered as a system of linear equations that allows us to recover $c^{\lambda}_{\mu,\nu}$ from the marginal coefficients $\overline{c}^{\lambda}_{\mu,l}$ (see Proposition 3.3.4). We now give the key steps of the proof of Theorem 1.5.2.

- We prove that for a particular choice of instances of Equation (3.1) (*i.e.* for a subset of quadruples (λ, μ, ν, l)), we obtain a square linear system. The matrix of this system is block triangular.
- We prove that the diagonal blocks of the matrix encoding this system, denoted $(\mathcal{Q}^{(r)})_{1\leq r\leq n-1}$, contain some coefficients t^{ρ}_{π}, that are independent of b (see Section 3.2).
- We prove that the matrices $\mathcal{Q}^{(r)}$ are invertible in $\mathbb{Z}$ by seeing them as change-of-basis matrices in the graded Farahat-Higman algebra $\mathcal{Z}_\infty$ (see Proposition 3.4.5 and Theorem 3.4.6).

This connection with the Farahat–Higman algebra seems to be new in this context. Unfortunately, we were not able to use this approach to get the positivity of the coefficients $c^{\pi}_{\mu,\nu}$ as polynomials in b.

3.1.2 Multiplicativity property for matchings and other consequences

As a consequence of our techniques, we obtain a new proof for the "Matching-Jack conjecture"[1] for $\alpha = 1$ and $\alpha = 2$, *i.e.* a formula for the generating series of orientable and non-orientable maps with control of the full profile, in terms of Schur and Zonal functions. Unlike the proof given in [GJ96b], the proof we give here does not use representation theory[2]. This new proof, detailed in Section 3.5, relies on the following three ingredients:

- We observe that, at a combinatorial level, matchings satisfy a multiplicativity property of the same form as in Equation (3.1).

[1] We recall that the cases $\alpha = 1$ and $\alpha = 2$ have preceded the conjecture [GJ96b], hence the quotes.

[2] A more intricate representation-free proof can also be obtained using the arguments provided in [CD22] (private communication with Guillaume Chapuy and Maciej Dołęga).

- We use the combinatorial interpretation of the coefficients $\overline{c}^{\lambda}_{\mu,l}$ in terms of matchings given in Corollary 2.4.3.
- As in the proof of the main result, we use the fact that Equation (3.1) entirely determines the coefficients $c^{\lambda}_{\mu,\nu}$ from their marginal sums, see Proposition 3.3.1.

As explained in Section 1.3.1, the Matching-Jack conjecture is closely related to the b-conjecture. In Section 3.6.1, we use the integrality of the coefficients $c^{\lambda}_{\mu,\nu}$ to obtain the integrality of the coefficients $h^{\pi}_{\mu,\nu}$ of the b-conjecture up to a rescaling factor; see Theorem 3.6.4. We also give in Section 3.6.2 similar integrality results for the generalized Goulden–Jackson introduced in Section 1.3.1.

Remark 3.1.1. We recall that the parameter b is related to α by $b := \alpha - 1$. Since the integrality of the coefficients of $c^{\lambda}_{\mu,\nu}$ as polynomials in α or in b are equivalent, we will be using in this chapter the parameter α rather than b.

3.2 Preliminaries

3.2.1 Some notation

We define the *rank* of a partition λ by

$$\mathrm{rk}(\lambda) := |\lambda| - \ell(\lambda).$$

Note that if $\lambda \vdash n$, then $0 \leq \mathrm{rk}(\lambda) \leq n-1$. If λ and μ are two partitions, then their *entry-wise sum* is defined by

$$\lambda \oplus \mu = [\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots].$$

We also define the *union* partition $\lambda \cup \mu$ as the partition whose parts are obtained by taking the union of the parts of λ and μ. In other words, for every $i \geq 1$, one has

$$m_i(\lambda \cup \mu) = m_i(\lambda) + m_i(\mu).$$

Finally, we set

$$\lambda - \mathbf{1} := [\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_{\ell(\lambda)} - 1, 0, \dots]. \tag{3.2}$$

Note that for any λ, one has

$$\mathrm{rk}(\lambda) = |\lambda - \mathbf{1}|.$$

Moreover, $\rho = \lambda - \mathbf{1}$ if and only if $\lambda = \rho \oplus 1^k$ for some $k \geq \ell(\rho)$.

We recall that the *dominance* order is the partial order defined on partitions of the same size by

$$\mu \leq \lambda \iff \mu_1 + \dots + \mu_i \leq \lambda_1 + \dots + \lambda_i \text{ for } i \geq 1.$$

Definition 3.2.1. *For every $n \geq 0$, we consider a total order $\preceq$ on the partitions of size n, with the two following properties:*

1. *If $\mu \leq \lambda$, then $\mu \preceq \lambda$.*
2. *If $\ell(\lambda) < \ell(\mu)$ then $\mu \preceq \lambda$.*

Such order is well defined since $\mu \leq \lambda$ implies $\ell(\lambda) \leq \ell(\mu)$. We define the dual order $\preceq'$ as the total order given by

$$\mu \preceq' \lambda \iff \mu' \preceq \lambda',$$

where λ' is the conjugate of λ defined in Eq. (1.5). Notice that $\lambda_1 < \mu_1$ implies $\mu \preceq' \lambda$.

Remark 3.2.2. The order $\preceq$ is not unique in general. Here, we fix once and for all such an order for each $n \geq 0$.

3.2.2 Elementary symmetric functions

In addition to the families of symmetric functions introduced in Section 1.2.2, another family will be useful in this chapter; *elementary symmetric functions*. They are defined as follows; for any $n \geq 1$;

$$e_n(\boldsymbol{x}) := \sum_{1 \leq i_1 < i_2 < \cdots < i_n} x_{i_1} \ldots x_{i_n},$$

and for any partition $\lambda := [\lambda, \ldots, \lambda_k]$,

$$e_\lambda(\boldsymbol{x}) := e_{\lambda_1}(\boldsymbol{x}) \ldots e_{\lambda_k}(\boldsymbol{x}).$$

It is well known that elementary symmetric functions form a basis of $\mathcal{S}_\alpha$, and that they are triangular in the monomial basis; see [Mac95, Equation (2.3)].

Theorem 3.2.3 ([Mac95])**.** *For any partition λ,*

$$e_\lambda = \sum_{\mu \leq \lambda'} a_{\lambda,\mu} m_\mu, \tag{3.3}$$

for some non-negative integer coefficients $a_{\lambda,\mu}$. Moreover, $a_{\lambda,\lambda'} = 1$.

Example 3.2.4. For $\lambda = [n, 1]$, one has

$$\begin{aligned} e_{[n,1]}(\boldsymbol{x}) &= \left(\sum_{1 \leq i_1 < \cdots < i_n} x_{i_1} \ldots x_{i_n} \right) \left(\sum_{j \geq 1} x_j \right) \\ &= m_{[2,1^n]}(\boldsymbol{x}) + (n+1) m_{1^{n+1}}(\boldsymbol{x}). \end{aligned}$$

3.2.3 Top coefficients t_π^ρ

As announced above, the proof of Theorem 1.5.2 involves the resolution of a linear system satisfied by the coefficients $c_{\mu,\nu}^\lambda$. In this section, we introduce the matrices $\mathcal{Q}^{(r)}$ that encode this system.

We start by recalling the following consequence of Theorem 1.5.1.

Corollary 3.2.5. *For any partitions $|\lambda| = |\mu|$ and integer $l \geq 1$, the coefficient $\overline{c}_{\mu,l}^\lambda$ is polynomial in α with integer coefficients.*

The following lemma gives an upper bound on the degree of $\overline{c}_{\mu,l}^\lambda$.

Lemma 3.2.6. *For any partitions $\lambda, \mu \vdash n \geq 1$ and $l \geq 1$, we have the following bound on the degree of $\overline{c}^{\lambda}_{\mu,l}$ as a polynomial in α:*

$$\deg_{\alpha}(\overline{c}^{\lambda}_{\mu,l}) \leq n - l + \ell(\lambda) - \ell(\mu).$$

Proof. For any partitions λ and ν of size n, we have that (see [GJ96a, Lemma 3.2])

$$\sum_{\mu \vdash n} c^{\lambda}_{\mu,\nu} = \frac{n!}{z_{\nu}} \alpha^{n-\ell(\nu)}.$$

We take the sum over the partitions ν of length l and of size n:

$$\sum_{\mu \vdash n} \overline{c}^{\lambda}_{\mu,l} = \sum_{\ell(\nu)=l} \frac{n!}{z_{\nu}} \alpha^{n-\ell(\nu)}.$$

Since the polynomials $\overline{c}^{\lambda}_{\mu,l}$ have non-negative coefficients in b (see Theorem 1.5.1), we deduce that $\deg_{\alpha}\left(\overline{c}^{\lambda}_{\mu,l}\right) \leq n - l$. Moreover, we see from the definition of the coefficients $c^{\lambda}_{\mu,\nu}$ (Eq. (1.29)) that

$$\frac{c^{\lambda}_{\mu,\nu}}{z_{\lambda}\alpha^{\ell(\lambda)}} = \frac{c^{\mu}_{\lambda,\nu}}{z_{\mu}\alpha^{\ell(\mu)}},$$

and by taking the sum over ν of length l

$$\frac{\overline{c}^{\lambda}_{\mu,l}}{z_{\lambda}\alpha^{\ell(\lambda)}} = \frac{\overline{c}^{\mu}_{\lambda,l}}{z_{\mu}\alpha^{\ell(\mu)}}.$$

Hence $\deg_{\alpha}\left(\overline{c}^{\lambda}_{\mu,l}\right) = \deg_{\alpha}\left(\frac{z_{\lambda}\alpha^{\ell(\lambda)}}{z_{\mu}\alpha^{\ell(\mu)}}\, \overline{c}^{\mu}_{\lambda,l}\right) \leq n - l + \ell(\lambda) - \ell(\mu)$. □

In this section, we are interested in the coefficients $\overline{c}^{\lambda}_{\mu,l}$ for which the bound given in Lemma 3.2.6 is zero. Proposition 3.2.9 gives a stability property for these coefficients when adding parts of size 1. We start by proving this property for the coefficients $c^{\kappa}_{\nu,\theta}$.

Lemma 3.2.7. *Fix an integer $N > 0$ and let $\kappa, \nu, \theta \vdash N \geq 1$, such that $\mathrm{rk}(\kappa) = \mathrm{rk}(\nu) + \mathrm{rk}(\theta)$. Then*

1. *for every $n \geq 1$, we have*
$$c^{\kappa}_{\nu,\theta}(1) = c^{\kappa\cup 1^n}_{\nu\cup 1^n,\theta\cup 1^n}(1),$$
where the coefficients are evaluated at $\alpha = 1$.

2. *if $c^{\kappa}_{\nu,\theta}(1) \neq 0$, then $m_1(\kappa) \leq m_1(\theta)$, where $m_1(\cdot)$ denotes the number of parts equal to 1.*

Proof. We know from Eq. (1.30), that $c^{\kappa}_{\nu,\theta}(1)$ counts oriented vertex-labelled maps of profile (κ, ν, θ). Similarly, since $\mathrm{rk}(\kappa \cup 1^n) = \mathrm{rk}(\nu \cup 1^n) + \mathrm{rk}(\theta \cup 1^n)$, we get that $c^{\kappa\cup 1^n}_{\nu\cup 1^n,\theta\cup 1^n}(1)$ count oriented maps with profile $(\kappa \cup 1^n, \nu \cup 1^n, \theta \cup 1^n)$.

We define the genus of a map, as the sum of genera of all its connected components. The Euler formula (see Eq. (1.1)) can be then extended to (not necessarily connected) maps as follows; for any map M of profile (κ, ν, θ), one has

$$2g(M) - 2\mathrm{cc}(M) = N - \ell(\kappa) - \ell(\nu) - \ell(\theta),$$

where $g(M)$ is the genus of M and $\mathrm{cc}(M)$ is its number of connected components. We also recall that $\ell(\nu)$ is the number of faces and $\ell(\kappa)$ and $\ell(\theta)$ are respectively the number of black and white vertices. The last equation can be rewritten as follows;

$$2g(M) - 2\mathrm{cc}(M) = \mathrm{rk}(\nu) + \mathrm{rk}(\theta) - \mathrm{rk}(\kappa) - 2\ell(\kappa).$$

Using the assumption of the lemma, we get

$$2g(M) - 2\mathrm{cc}(M) = -2\ell(\kappa).$$

But we know that $g(M) \geq 0$, hence

$$2\mathrm{cc}(M) - 2\ell(\kappa) \geq 0.$$

Moreover, the number of connected components of a map cannot exceed its number of black vertices. We deduce that $\mathrm{cc}(M) = \ell(\kappa)$ and $g(M) = 0$.

We deduce that a map of profile (κ, ν, θ) satisfies the condition that each one of its connected components is planar and contains exactly one black vertex. In particular, a black vertex of degree 1 corresponds to an isolated edge. Hence, adding a part of size 1 to each one of the partitions κ, ν and θ corresponds to adding isolated edges and does not change the number of counted maps[3]. This gives item 1 of the lemma. Similarly, we obtain item 2 by noticing that the number white vertices of degree 1 is at least the number of isolated edges, or equivalently the number of black vertices of degree 1. ☐

Remark 3.2.8. Actually, Lemma 3.2.7 holds without the specialization at $\alpha = 1$, since under the condition $\mathrm{rk}(\kappa) = \mathrm{rk}(\nu) + \mathrm{rk}(\theta)$, the coefficient $c^{\kappa}_{\nu,\theta}$ is independent of α. This is a consequence of a generalized version of Lemma 3.2.6 (see [DF16, Corollary 4.2]).

We now deduce the following proposition.

Proposition 3.2.9. *For any $\kappa, \nu \vdash N \geq 1$, and $l \geq 1$ such that $\mathrm{rk}(\kappa) = \mathrm{rk}(\nu) + N - l$, we have*

$$\overline{c}^{\kappa}_{\nu,l} = \overline{c}^{\kappa\cup 1^n}_{\nu\cup 1^n, l+n}, \quad \textit{for every } n \geq 1. \tag{3.4}$$

Proof. First, notice that from Lemma 3.2.6 we have for any α that $\overline{c}^{\kappa}_{\nu,l}(\alpha) = \overline{c}^{\kappa}_{\nu,l}(1)$ and similarly $\overline{c}^{\kappa\cup 1^n}_{\nu\cup 1^n, l+n}(\alpha) = \overline{c}^{\kappa\cup 1^n}_{\nu\cup 1^n, l+n}(1)$. Moreover,

$$\overline{c}^{\kappa\cup 1^n}_{\nu\cup 1^n, l+n}(1) = \sum_{\substack{\theta\vdash N+n \\ \ell(\theta)=l+n}} c^{\kappa\cup 1^n}_{\nu\cup 1^n,\theta}(1).$$

[3]One can check that when we add isolated edges to a such map, we have a unique way (up to symmetries) to number the new black vertices in order to obtain vertex-labelled maps.

From Lemma 3.2.7 item 2, the partitions θ which contribute to this sum are of the form $\theta = \widetilde{\theta} \cup 1^n$, where $\widetilde{\theta} \vdash N$ and $\ell(\widetilde{\theta}) = l$. Combining this fact with Lemma 3.2.7 item 1, we obtain

$$\begin{aligned}
\overline{c}^{\kappa\cup 1^n}_{\nu\cup 1^n, l+n}(1) &= \sum_{\substack{\widetilde{\theta}\vdash N \\ \ell(\widetilde{\theta})=l}} c^{\kappa\cup 1^n}_{\nu\cup 1^n, \widetilde{\theta}\cup 1^n}(1) \\
&= \sum_{\substack{\widetilde{\theta}\vdash N \\ \ell(\widetilde{\theta})=l}} c^{\kappa}_{\nu,\widetilde{\theta}}(1) \\
&= \overline{c}^{\kappa}_{\nu,l}(1).
\end{aligned}$$

This finishes the proof of the proposition. □

This allows us to define the top coefficients t^{ρ}_{π}.

Definition 3.2.10. *Let ρ and π be two partitions of size $r \geq 1$. We define the top coefficient*[4] t^{ρ}_{π} *by*

$$\mathrm{t}^{\rho}_{\pi} := \overline{c}^{\kappa}_{\nu,l},$$

where κ and ν are two partitions of the same size $n \geq r + \ell(\rho)$, such that

$$\begin{cases} \kappa := \rho \oplus 1^{n-r} \text{ (or equivalently } \rho = \kappa - \mathbf{1}) \\ \nu := \pi \cup 1^{n-r} \\ l \text{ is such that } n - l + \mathrm{rk}(\pi) = r. \end{cases}$$

Note that given Proposition 3.2.9 this definition does not depend on n. We consider the matrix of top coefficients defined for any $r \geq 1$ by $\mathcal{Q}^{(r)} := (\mathrm{t}^{\rho}_{\pi})_{\pi,\rho\vdash r}$.

Example 3.2.11. For $r = 3$, the matrix $\mathcal{Q}^{(r)}$ is given by

$\pi \setminus \rho$	$[3]$	$[2,1]$	$[1^3]$
$[3]$	4	1	0
$[2,1]$	6	4	3
$[1^3]$	1	1	1

.

The following theorem will be proved in Section 3.4 (see also Theorem 3.4.6).

Theorem 3.2.12. *The matrix $\mathcal{Q}^{(r)} = (\mathrm{t}^{\rho}_{\pi})_{\rho,\pi\vdash r}$ is invertible in $\mathbb{Z}$ for every $r \geq 1$.*

There exists an explicit expression of the top coefficients t^{ρ}_{π}, see [BG92, GS98]. However, for the proof of Theorem 3.2.12, it will be more natural to consider the algebraic definition of these coefficients and see the matrix $\mathcal{Q}^{(r)}$ as a change-of-basis matrix (see Proposition 3.4.5).

[4]This terminology will be justified in Section 3.4.

3.3 Proof of Theorem 1.5.2

The main purpose of this section is to prove that Theorem 3.2.12 implies Theorem 1.5.2. We start by proving the multiplicativity property satisfied by the coefficients $c^{\lambda}_{\mu,\nu}$ (see also Equation (3.1)):

Proposition 3.3.1. *For any $\lambda, \mu, \nu, \rho \vdash n \geq 1$, we have*

$$\sum_{\kappa \vdash n} c^{\lambda}_{\mu,\kappa} c^{\kappa}_{\nu,\rho} = \sum_{\theta \vdash n} c^{\lambda}_{\theta,\rho} c^{\theta}_{\mu,\nu}.$$

In particular, for any $\lambda, \mu, \nu \vdash n \geq 1$ and $l \geq 1$,

$$\sum_{\kappa \vdash n} c^{\lambda}_{\mu,\kappa}\, \overline{c}^{\kappa}_{\nu,l} = \sum_{\theta \vdash n} \overline{c}^{\lambda}_{\theta,l}\, c^{\theta}_{\mu,\nu}. \tag{3.5}$$

Proof. We recall that $c^{\lambda}_{\mu,\nu,\rho}$ are a four parameter version of the coefficients $c^{\lambda}_{\mu,\nu}$, introduced in Eq. (1.33) and defined by

$$\tau_2^{(\alpha)}(\boldsymbol{p}, \boldsymbol{q}^{(0)}, \boldsymbol{q}^{(1)}, \boldsymbol{q}^{(2)}) = \sum_{n \geq 0} t^n \sum_{\lambda,\mu,\nu,\rho \vdash n} \frac{c^{\lambda}_{\mu,\nu,\rho}(\alpha)}{z_\lambda \alpha^{\ell(\lambda)}} p_\lambda q^{(0)}_\mu q^{(1)}_\nu q^{(2)}_\rho,$$

with

$$\tau_2^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}^{(0)}, \boldsymbol{q}^{(1)}, \boldsymbol{q}^{(2)}) = \sum_{\xi \in \mathbb{Y}} t^{|\xi|} \frac{J^{(\alpha)}_\xi(\boldsymbol{p}) J^{(\alpha)}_\xi(\boldsymbol{q}^{(0)}) J^{(\alpha)}_\xi(\boldsymbol{q}^{(1)}) J^{(\alpha)}_\xi(\boldsymbol{q}^{(2)})}{j^{(\alpha)}_\xi}.$$

From now on, we take the specialization $t = 1$. Let $\boldsymbol{r} := (r_1, r_2, ..)$ be an additional sequence of power-sum variables. We consider the two functions $\tau^{(\alpha)}(1, \boldsymbol{p}, \boldsymbol{q}^{(0)}, \boldsymbol{r})$ and $\tau^{(\alpha)}(1, \boldsymbol{r}, \boldsymbol{q}^{(1)}, \boldsymbol{q}^{(2)})$, and we take their scalar product with respect to the variables $\boldsymbol{r}$. Since

$$\langle J^{(\alpha)}_{\xi^1}(\boldsymbol{r}), J^{(\alpha)}_{\xi^2}(\boldsymbol{r}) \rangle_{\boldsymbol{r}} = \delta_{\xi^1,\xi^2} j^{(\alpha)}_{\xi^1},$$

we get

$$\langle \tau^{(\alpha)}(1, \boldsymbol{p}, \boldsymbol{q}^{(0)}, \boldsymbol{r}), \tau^{(\alpha)}(1, \boldsymbol{r}, \boldsymbol{q}^{(1)}, \boldsymbol{q}^{(2)}) \rangle_{\boldsymbol{r}} = \tau_2^{(\alpha)}(1, \boldsymbol{p}, \boldsymbol{q}^{(0)}, \boldsymbol{q}^{(1)}, \boldsymbol{q}^{(2)}).$$

Expanding the two sides of the last equation in the power-sum bases and using the orthogonality of the power-sum bases, we get

$$\sum_{n \geq 0} \sum_{\lambda,\mu,\nu,\rho,\kappa \vdash n} \frac{c^{\lambda}_{\mu,\kappa}}{z_\lambda \alpha^{\ell(\lambda)}} p_\lambda q^{(0)}_\mu \frac{c^{\kappa}_{\nu,\rho}}{z_\kappa \alpha^{\ell(\kappa)}} q^{(1)}_\nu q^{(2)}_\rho \,\langle r_\kappa, r_\kappa \rangle_{\boldsymbol{r}} = \sum_{n \geq 0} \sum_{\lambda,\mu,\nu,\rho \vdash n} \frac{c^{\lambda}_{\mu,\nu,\rho}}{z_\lambda \alpha^{\ell(\lambda)}} p_\lambda q^{(0)}_\mu q^{(1)}_\nu q^{(2)}_\rho.$$

Using

$$\langle r_\kappa, r_\kappa \rangle_{\boldsymbol{r}} = z_\kappa \alpha^{\ell(\kappa)}$$

and extracting the coefficient of $p_\lambda q^{(0)}_\mu q^{(1)}_\nu q^{(2)}_\rho$, we get

$$c^{\lambda}_{\mu,\nu,\rho} = \sum_{\kappa \vdash n} c^{\lambda}_{\mu,\kappa} c^{\kappa}_{\nu,\rho}, \tag{3.6}$$

for any partitions λ, μ, ν, ρ of size n.

Finally, it is easy to see from the definition, that the coefficients $c^{\lambda}_{\mu,\nu,\rho}$ are symmetric in the parameters μ, ν and ρ. In particular, we have

$$c^{\lambda}_{\mu,\nu,\rho} = c^{\lambda}_{\rho,\mu,\nu}. \tag{3.7}$$

By combining Eqs. (3.6) and (3.7), we obtain the first equation of the proposition. We deduce the second one by taking the sum over all partitions ρ of size n and length l. □

We fix $n > 0$. For $0 \leq r < n$, we introduce the assertion $\mathcal{A}^{(r)}_n$:

$$\mathcal{A}^{(r)}_n : \text{for any } \lambda, \mu, \kappa \vdash n \text{ such that } \operatorname{rk}(\kappa) = r, \text{ the coefficient } c^{\lambda}_{\mu,\kappa} \text{ is an integer polynomial in } \alpha.$$

Our purpose is to prove $\mathcal{A}^{(r)}_n$ by induction on r. We start by the following lemma.

Lemma 3.3.2. *We fix $1 \leq r < n$, and we assume that the assertions $\mathcal{A}^{(i)}_n$ hold for $i < r$. Let $\lambda, \mu \vdash n$ and let (ν, l) be a pair satisfying the condition*

$$\nu \vdash n, \ \operatorname{rk}(\nu) < r, \ \textit{and } n - l + \operatorname{rk}(\nu) = r. \tag{C1}$$

Then the quantity

$$P^{(r)}_{\lambda,\mu,\nu,l} := \sum_{\operatorname{rk}(\kappa)=r} c^{\lambda}_{\mu,\kappa}\, \overline{c}^{\kappa}_{\nu,l}, \tag{3.8}$$

is an integer polynomial in α.

Proof. Note that with the conditions of the lemma, the right hand-side $\sum_{\theta\vdash n} \overline{c}^{\lambda}_{\theta,l}\, c^{\theta}_{\mu,\nu}$ in Equation (3.5) is an integer polynomial (we use the induction hypothesis and Corollary 3.2.5). This implies that the left hand-side $\sum_{\kappa\vdash n} c^{\lambda}_{\mu,\kappa}\, \overline{c}^{\kappa}_{\nu,l}$ in Equation (3.5) is an integer polynomial. We conclude using the two following facts:

- if $\operatorname{rk}(\kappa) > r$ then $\overline{c}^{\kappa}_{\nu,l} = 0$. Indeed, from Lemma 3.2.6, we get that

$$\deg_{\alpha}(\overline{c}^{\kappa}_{\nu,l}) \leq n - l + \operatorname{rk}(\nu) - \operatorname{rk}(\kappa) = r - \operatorname{rk}(\kappa) < 0.$$

- if $\operatorname{rk}(\kappa) < r$ then we know that $c^{\lambda}_{\mu,\kappa}$ is an integer polynomial from the induction hypothesis. □

For fixed $\lambda, \mu \vdash n$ and $r < n$, we get from the previous lemma more equations of type (3.8) than variables $c^{\lambda}_{\mu,\kappa}$, where $\operatorname{rk}(\kappa) = r$. In order to obtain a square system, we start by considering the equations (3.8) that are indexed by pairs (ν, l) satisfying the following condition refining (C1):

$$(\nu, l) = (\pi \cup 1^{n-r}, l), \text{ where } 1 \leq l \leq n, \pi \vdash r \text{ and } n - l + \operatorname{rk}(\pi) = r. \tag{C2}$$

We denote by $\mathcal{S}^{(r)}_{\lambda,\mu}$ the linear system obtained by taking the equations (3.8) for (ν, l) satisfying condition (C2), and we denote by $\mathcal{Q}^{(r)}_n$ the matrix associated to this system. In other terms $\mathcal{Q}_n^{(r)} = (\overline{c}^{\kappa}_{\nu,l})$ where indices κ of columns are partitions of n of rank r, and indices of rows are pairs (ν, l) satisfying condition (C2). Note that this matrix is independent of λ and μ.

The system $\mathcal{S}^{(r)}_{\lambda,\mu}$ thus obtained is a square system for $n = |\lambda| = |\mu|$ large enough compared to r. In general, we have the following proposition relating the matrices $\mathcal{Q}_n^{(r)}$ for $n > r$ to the square matrix $\mathcal{Q}^{(r)}$ (see Definition 3.2.10).

Proposition 3.3.3. *Fix $1 \leq r < n$. The following hold.*

- *If $2r \leq n$, then $\mathcal{Q}^{(r)} = \mathcal{Q}_n^{(r)}$.*
- *If $2r > n$, then the matrix $\mathcal{Q}_n^{(r)}$ is a submatrix of $\mathcal{Q}^{(r)}$, obtained by erasing only columns. More precisely, $\mathcal{Q}_n^{(r)} = (t^{\rho}_{\pi})$ where the row index π is a partition of r, and the column index ρ is a partition of r, such that $\ell(\rho) \leq n - r$.*

Proof. First, we have the following bijection

$$\begin{aligned}\{\kappa \text{ partition of } n \text{ with } \operatorname{rk}(\kappa) = r\} &\xrightarrow{\sim} \{\rho \text{ partition of } r \text{ with } \ell(\rho) \leq n - r\} \\ \kappa &\longmapsto \kappa - \mathbf{1} \\ \rho \oplus 1^{n-r} &\longleftarrow\!\!| \rho,\end{aligned} \tag{3.9}$$

where $\kappa - \mathbf{1}$ is defined in Eq. (3.2). Moreover, we recall that $\mathrm{t}^{\rho}_{\pi} = \overline{c}^{\kappa}_{\pi\cup 1^{n-r}, n-r+\operatorname{rk}(\pi)}$, where $\kappa = \rho \oplus 1^{n-r}$ (see Definition 3.2.10). This gives the second item of the proposition.

In order to obtain the first one, notice that when $2r \leq n$, all partitions ρ of size r satisfy $\ell(\rho) \leq r \leq n - r$ and $\mathcal{Q}_n^{(r)}$ contains all the columns of $\mathcal{Q}^{(r)}$. □

We now prove Theorem 1.5.2.

Proof of Theorem 1.5.2. We prove $\mathcal{A}_n^{(r)}$ by induction on r. For $r = 0$, the only partition of rank 0 is $\kappa = [1^n]$. In this case,

$$c^{\lambda}_{\mu,[1^n]} = \delta_{\lambda,\mu}$$

for all partitions λ, μ, where $\delta_{\lambda,\mu}$ is the Kronecker delta; see [GJ96a, Lemma 3.3] (this can also be seen as a special case of Theorem 1.5.1).

Now we fix $r > 0$ and we assume that $\mathcal{A}_n^{(j)}$ holds for each $j \leq r-1$. We fix two partitions $\lambda, \mu \vdash n \geq 1$, and we consider the system $\mathcal{S}^{(r)}_{\lambda,\mu}$. It can be written as follows

$$\mathcal{Q}_n^{(r)} X^{(r)}_{\lambda,\mu} = Y^{(r)}_{\lambda,\mu},$$

where $Y^{(r)}_{\lambda,\mu}$ is the column vector containing the polynomials $P^{(r)}_{\lambda,\mu,\nu,l}$ for (ν, l) satisfying condition (C2), and $X^{(r)}_{\lambda,\mu}$ is the column vector containing $c^{\lambda}_{\mu,\kappa}$ for $\kappa \vdash n$ of rank r. We define the column vector $\widetilde{X}^{(r)}_{\lambda,\mu} := (x^{\lambda}_{\mu,\rho})$ for $\rho \vdash r$, where

$$x^{\lambda}_{\mu,\rho} := \begin{cases} c^{\lambda}_{\mu,\rho\oplus 1^{n-r}} & \text{if } \ell(\rho) \leq n - r \\ 0 & \text{otherwise.} \end{cases}$$

In other terms, $\widetilde{X}^{(r)}_{\lambda,\mu}$ is the column vector obtained from $X^{(r)}_{\lambda,\mu}$ using the bijection of Eq. (3.9) and adding zeroes to the entries indexed by partitions $\rho \vdash r$ not in the image of this bijection. Using Proposition 3.3.3, the previous system can be rewritten as follows

$$\mathcal{Q}^{(r)}\widetilde{X}^{(r)}_{\lambda,\mu} = Y^{(r)}_{\lambda,\mu}.$$

Indeed, adding some columns to $\mathcal{Q}^{(r)}_n$ and zeroes in the corresponding entries of $X^{(r)}_{\lambda,\mu}$ does not affect the right-hand side. But we know from Theorem 3.2.12 that $\mathcal{Q}^{(r)}$ is invertible in $\mathbb{Z}$, and since the entries of $Y_{\lambda,\mu}$ are integer polynomials in α (see Lemma 3.3.2), we deduce that this is also the case for the entries of $\widetilde{X}^{(r)}_{\lambda,\mu}$. Thus the coefficients $c^{\lambda}_{\mu,\kappa}$ are integer polynomials in α, when the partition κ has rank r. This gives the assertion $\mathcal{A}^{(r)}_n$. □

Note that the previous proof implies that Equation (3.1) allows to recover the coefficients $c^{\lambda}_{\mu,\nu}$ from their marginal sums. More precisely, we have the following proposition:

Proposition 3.3.4. *Fix a real α and let $(y^{\lambda}_{\mu,\nu})_{\lambda,\mu,\nu\vdash n}$ be a family of numbers indexed by partitions of size $n \geq 1$, satisfying*

$$\begin{cases} y^{\lambda}_{\mu,[1^n]} = \delta_{\lambda,\mu} \\ \sum_{\kappa\vdash n} y^{\lambda}_{\mu,\kappa}\, \overline{c}^{\kappa}_{\nu,l}(\alpha) = \sum_{\theta\vdash n} \overline{c}^{\lambda}_{\theta,l}(\alpha) y^{\theta}_{\mu,\nu}. \end{cases}$$

Then we have that $y^{\lambda}_{\mu,\nu} = c^{\lambda}_{\mu,\nu}(\alpha)$ for all partitions $\lambda, \mu, \nu \vdash n$.

3.4 Graded Farahat-Higman Algebra

This section is dedicated to the proof of Theorem 3.2.12. We start by some notation related to permutations. If σ is a permutation of $\mathfrak{S}_n$, then it can also be seen as a permutation of $\mathfrak{S}_{n+1}$ by adding $n+1$ as a fixed point.

If σ is a permutation of cycle type λ, we define its *reduced cycle type* as the partition $\lambda - \mathbf{1}$. Hence if $\sigma \in \mathfrak{S}_n \subset \mathfrak{S}_{n+1}...$, then its reduced cycle type does not depend on n. For any partition λ, we define $C_\lambda(n) \in \mathbb{Z}\mathfrak{S}_n$ as the sum of all permutations in $\mathfrak{S}_n$ of reduced cycle type λ. Note that $C_\lambda(n) = 0$ if $|\lambda| + \ell(\lambda) > n$.

For every $n \geq 0$, the family $(C_\lambda(n))_{|\lambda|+\ell(\lambda)\leq n}$ form a basis of the center of the group algebra of $\mathfrak{S}_n$. The multiplication in this algebra is given by

$$C_\lambda(n)C_\mu(n) = \sum_{|\kappa|+\ell(\kappa)\leq n} \rho^{\kappa}_{\lambda,\mu}(n)C_\kappa(n), \tag{3.10}$$

for some structure coefficients $\rho^{\kappa}_{\lambda,\mu}(n)$. The latter are linked to the coefficients $c^{\lambda}_{\mu,\nu}$ evaluated at $\alpha = 1$ as follows (see [GJ96a, Proposition 3.1]):

$$\rho^{\kappa}_{\lambda,\mu}(n) = \begin{cases} c^{\kappa\oplus 1^{n-|\kappa|}}_{\lambda\oplus 1^{n-|\lambda|},\mu\oplus 1^{n-|\mu|}}(1) & \text{if } \max\left(|\lambda|+\ell(\lambda), |\mu|+\ell(\mu), |\kappa|+\ell(\kappa)\right) \leq n, \\ 0 & \text{otherwise.} \end{cases} \tag{3.11}$$

The following proposition is due to Farahat and Higman.

Proposition 3.4.1 ([FH59]). *The structure coefficients $\rho^{\kappa}_{\lambda,\mu}$ are polynomials in n, and satisfy the following properties:*

1. $\rho^{\kappa}_{\lambda,\mu} = 0$ *if* $|\kappa| > |\lambda| + |\mu|$,
2. $\rho^{\kappa}_{\lambda,\mu}$ *is independent of* n *if* $|\kappa| = |\lambda| + |\mu|$.

Example 3.4.2. If $\lambda = \mu = [1]$, then for any $n \geq 2$,

$$C_\lambda(n) = C_\mu(n) = \sum_{1 \leq i < j \leq n} (i, j),$$

is the sum of all transpositions in $\mathfrak{S}_n$. It can be checked that

$$C_{[1]}(n) C_{[1]}(n) = \binom{n}{2} C_\emptyset(n) + 3C_{[2]}(n) + 2C_{[1,1]}(n).$$

The Farahat-Higman algebra introduced in [FH59] is the algebra generated by $(C_\lambda(n))_{\lambda \in \mathbb{Y}}$, and in which the structure coefficients are the polynomials $\rho^{\kappa}_{\lambda,\mu}(n)$.

We are here interested in the structure coefficients $\rho^{\kappa}_{\lambda,\mu}$ in the case $|\kappa| = |\lambda| + |\mu|$. They are called *the top connection coefficients* of the Farahat-Higman algebra. To study these coefficients, we consider the graded algebra $\mathcal{Z}_n$ associated to $Z(\mathbb{Z}\mathfrak{S}_n)$ with respect to the filtration

$$\deg(C_\lambda(n)) = |\lambda|.$$

We denote by $\mathfrak{c}_\lambda(n)$ the image of $C_\lambda(n)$ in $\mathcal{Z}_n$. Concretely, $\mathcal{Z}_n = \bigoplus_{1 \leq r \leq n-1} \mathcal{Z}_n^{(r)}$, where

$$\mathcal{Z}_n^{(r)} := \mathrm{Span}_{\mathbb{Z}} \left\{ \mathfrak{c}_\lambda(n); \lambda \vdash r \text{ and } \ell(\lambda) \leq n - r \right\},$$

and the multiplication in $\mathcal{Z}_n$ is defined by

$$\mathfrak{c}_\lambda(n)\, \mathfrak{c}_\mu(n) = \sum_{\kappa \vdash |\lambda| + |\mu|} \rho^{\kappa}_{\lambda,\mu}(n)\, \mathfrak{c}_\kappa(n). \tag{3.12}$$

Note that compared to Equation (3.10), we keep only the top degree terms. The graded algebra $\mathcal{Z}_n$ comes with a linear isomorphism $\phi_n : Z(\mathbb{Z}\mathfrak{S}_n) \xrightarrow{\sim} \mathcal{Z}_n$, that sends $C_\lambda(n)$ to $\mathfrak{c}_\lambda(n)$ (which is obviously not an algebra isomorphism).

For any $f \in \mathcal{Z}_n$ let $[f] \in \mathcal{Z}_n$ denote the top degree term of f. We deduce from Eq. (3.12) that if f and g are two elements of $Z(\mathbb{Z}\mathfrak{S}_n)$ with homogeneous degree then

$$\phi_n(f) \cdot \phi_n(g) = [\phi_n(fg)]. \tag{3.13}$$

Since the structure coefficients in $\mathcal{Z}_n$ are independent of n, we can define a family of $\mathbb{Z}$-algebra morphisms:

$$\begin{aligned} \psi_n : \mathcal{Z}_{n+1} &\longrightarrow \mathcal{Z}_n \\ \mathfrak{c}_\lambda(n+1) &\longmapsto \begin{cases} \mathfrak{c}_\lambda(n) & \text{if } |\lambda| + \ell(\lambda) \leq n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Let $\mathcal{Z}_\infty := \varprojlim \mathcal{Z}_n$ be the projective limit of the $\mathcal{Z}_n$'s, and let $\mathfrak{c}_\lambda := \varprojlim \mathfrak{c}_\lambda(n)$ in $\mathcal{Z}_\infty$. We also define $\mathcal{Z}_\infty^{(r)} := \mathrm{Span}_\mathbb{Z}\{\mathfrak{c}_\lambda; \lambda \vdash r\}$, hence $\mathcal{Z}_\infty = \bigoplus_{r \geq 1} \mathcal{Z}_\infty^{(r)}$. The algebra $\mathcal{Z}_\infty$ is *the graded Farahat-Higman* algebra (see [Mac95, Example 24, page 131] for more details about the construction of the algebra $\mathcal{Z}_\infty$).

We define for every $r \geq 1$, $\mathfrak{f}_r(n) := \sum_{\lambda \vdash r} \mathfrak{c}_\lambda(n)$, and for any partition μ, $\mathfrak{f}_\mu(n) := \prod_i \mathfrak{f}_{\mu_i}(n)$. We also define $\mathfrak{g}_\pi(n) := \mathfrak{c}_{\pi-\mathbf{1}}(n)\, \mathfrak{f}_{\ell(\pi)}(n)$, for any partition π. Note that

$$\deg(\mathfrak{f}_\mu(n)) = \deg(\mathfrak{g}_\mu(n)) = |\mu|,$$

for n large enough. Finally we define the limits $\mathfrak{f}_\mu := \varprojlim \mathfrak{f}_\mu(n)$ and $\mathfrak{g}_\pi := \varprojlim \mathfrak{g}_\pi(n)$ in $\mathcal{Z}_\infty$. We have the following theorem due to Farahat and Higman.

Theorem 3.4.3 ([FH59]). *For every $r \geq 0$, $(\mathfrak{f}_\lambda)_{\lambda \vdash r}$ is a $\mathbb{Z}$-basis of $\mathcal{Z}_\infty^{(r)}$.*

The following lemma relates the top coefficients t_π^ρ (see Definition 3.2.10) to change of bases coefficients in the graded algebra $\mathcal{Z}_\infty$.

Lemma 3.4.4. *For all partitions $\rho, \pi \vdash r \geq 0$, we have $\mathrm{t}_\pi^\rho = [\mathfrak{c}_\rho]\, \mathfrak{g}_\pi$.*

Proof. From the definition $\mathrm{t}_\pi^\rho = \overline{c}_{\nu,l}^\kappa$, where κ and ν are the two partitions of the same size $n \geq r + \ell(\rho)$, such that $\kappa = \rho \oplus 1^{n-r}$, $\nu = \pi \cup 1^{n-r}$ and l is such that $n - l + \mathrm{rk}(\nu) = r$. Moreover,

$$\overline{c}_{\nu,l}^\kappa = [C_{\kappa-\mathbf{1}}(n)] \left(\sum_{\lambda \vdash n-l} C_\lambda(n) \right) C_{\nu-\mathbf{1}}(n), \tag{3.14}$$

where we use Equation (3.11) and the fact that under the condition

$$n - l + \mathrm{rk}(\nu) = r = \mathrm{rk}(\kappa)$$

we have $\overline{c}_{\nu,l}^\kappa = \overline{c}_{\nu,l}^\kappa(1)$. From the last equation, Eq. (3.14) corresponds to a top degree coefficient extraction. As a consequence, we can consider it in the graded algebra $\mathcal{Z}_n$:

$$\mathrm{t}_\pi^\rho = \overline{c}_{\nu,l}^\kappa = [\mathfrak{c}_{\kappa-\mathbf{1}}(n)]\, \mathfrak{f}_{n-l}(n)\, \mathfrak{c}_{\nu-\mathbf{1}}(n).$$

To conclude, note that $\nu - \mathbf{1} = \pi - \mathbf{1}$, that $\kappa - \mathbf{1} = \rho$ and that

$$n - l = r - \mathrm{rk}(\nu) = r - \mathrm{rk}(\pi) = \ell(\pi).$$ □

We deduce the following proposition.

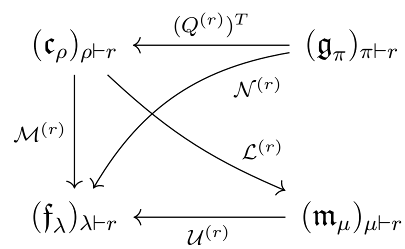
Proposition 3.4.5. *For every $r \geq 0$, the matrix $(\mathcal{Q}^{(r)})^T$ is the matrix of $(\mathfrak{g}_\pi)_{\pi \vdash r}$ in the basis $(\mathfrak{c}_\rho)_{\rho \vdash r}$.*

Hence, our goal is to prove the following theorem that implies Theorem 3.2.12:

Theorem 3.4.6. *For every $r \geq 0$, the family $(\mathfrak{g}_\pi)_{\pi \vdash r}$ is a $\mathbb{Z}$-basis for $\mathcal{Z}_\infty^{(r)}$.*

The rest of this section is dedicated to the proof of this theorem. Let us explain the steps of this proof; instead of studying the matrix of $(\mathfrak{g}_\pi)$ in the basis $(\mathfrak{c}_\rho)$, we consider its matrix $\mathcal{N}$ in the basis $(\mathfrak{f}_\lambda)$. This matrix satisfies some block triangularity property. Its diagonal blocks are submatrices of $\mathcal{M}$, defined as the matrix of $(\mathfrak{c}_\rho)$ in the basis $(\mathfrak{f}_\lambda)$. In order to prove that these submatrices of $\mathcal{M}$ are invertible[5] in $\mathbb{Z}$, we will need to introduce an intermediate basis $(\mathfrak{m}_\mu)$ (see definition below). This basis has the property that its transition matrices to both bases $(\mathfrak{c}_\rho)$ and $(\mathfrak{f}_\lambda)$ (denoted respectively $\mathcal{L}$ and $\mathcal{U}$) are triangular (Proposition 3.4.9 and Theorem 3.4.11).

The following diagram illustrates the different bases of $\mathcal{Z}_\infty^{(r)}$ that will be useful for the proof of Theorem 3.4.6, and the associated transition matrices.



We denote by $(\mathcal{J}_i)_{i\geq 2}$ the Jucys-Murphy elements:

$$\mathcal{J}_i := (1,i) + ... + (i-1,i) \in \mathbb{Z}\mathfrak{S}_n, \text{ for every } n \geq i.$$

We denote for any symmetric function f,

$$f(\Xi_n) = f(\mathcal{J}_2, \mathcal{J}_3, ..., \mathcal{J}_n, 0, 0, ..).$$

This evaluation is well defined since the Jucys–Murphy elements commute (see [Mur81]). Moreover, the evaluation of the elementary symmetric function in the Jucys-Murphy elements has the following expression (see [Juc74]): for any $l \geq 1$,

$$e_l(\Xi_n) = \sum_{\lambda\vdash l} C_\lambda(n).$$

Since the elementary functions form a basis of the symmetric functions algebra, we get that $f(\Xi_n) \in Z(\mathbb{Z}\mathfrak{S}_n)$ for any symmetric function f. Using Eq. (3.13), we also get that the top degree term of the image of $e_\lambda(\Xi_n)$ in $\mathcal{Z}_n$ is given by

$$[\phi_n(e_\lambda(\Xi_n))] = \mathfrak{f}_\lambda(n).$$

We now consider the evaluation of the monomial symmetric functions in the Jucys-Murphy elements.

Definition 3.4.7. *For any partition $\mu \vdash r \geq 0$, we define $\mathfrak{m}_\mu(n) \in \mathcal{Z}_n$ by*

$$\mathfrak{m}_\mu(n) := [\phi_n(m_\mu(\Xi_n))],$$

where m_μ denotes the monomial symmetric function associated to μ. We also introduce their limit in $\mathcal{Z}_\infty$; $\mathfrak{m}_\mu := \varprojlim \mathfrak{m}_\mu(n)$.

[5]Note that Theorem 3.4.3 implies that the matrix $\mathcal{M}^{(r)}$ has determinant ± 1, however we will need to obtain this result for some submatrices of $\mathcal{M}^{(r)}$ called South-East blocks, see Corollary 3.4.14.

Example 3.4.8. When $\mu = [2,1]$ and $n \geq 4$ we have

$$m_{[2,1]}\left(m_\mu(\Xi_n)\right) = \sum_{\substack{2\leq i,j\leq n \\ i\neq j}} \mathcal{J}_i^2 \mathcal{J}_j$$
$$= 3C_{[3]}(n) + C_{[2,1]}(n) + \left(\binom{n}{2} - 1\right) C_{[1]}(n).$$

Hence

$$\mathfrak{m}_{[2,1]}(n) = \left[3\,\mathfrak{c}_{[3]}(n) + \mathfrak{c}_{[2,1]}(n) + \left(\binom{n}{2} - 1\right)\mathfrak{c}_{[1]}(n)\right] = 3\,\mathfrak{c}_{[3]}(n) + \mathfrak{c}_{[2,1]}(n)$$

and

$$\mathfrak{m}_{[2,1]} = 3\,\mathfrak{c}_{[3]} + \mathfrak{c}_{[2,1]}\,.$$

Note that since the elementary functions form a basis of the symmetric functions algebra, $\mathfrak{m}_\mu(n)$ is a linear combination of $\mathfrak{f}_\lambda(n)$, and the previous limit is well defined. The elements $\mathfrak{m}_\mu(n)$ have been studied in [MN13], and an explicit expression of their expansions in the basis $(\mathfrak{c}_\rho(n))$ has been given.

In the following, we will study some triangularity properties of the different transition matrices in $\mathcal{Z}_\infty$, where the three families $(\mathfrak{c}_\rho)$, $(\mathfrak{m}_\mu)$ and $(\mathfrak{g}_\pi)$ will be indexed with partitions with respect to the total order $\preceq$, and the basis $(\mathfrak{f}_\lambda)$ will be indexed with partitions with respect to the dual order $\preceq'$ (see Section 3.2.1 for the definition of these orders). Moreover, we order rows and columns in increasing order (see examples below).

Proposition 3.4.9. *The matrix $\mathcal{U}^{(r)}$ of $(\mathfrak{m}_\mu)_{\mu\vdash r,\preceq}$ in the basis $(\mathfrak{f}_\lambda)_{\mu\vdash r,\preceq'}$ is upper triangular, with diagonal coefficients equal to* 1*, i.e*

$$[\mathfrak{f}_\lambda]\,\mathfrak{m}_\mu = 0 \textit{ if } \mu \prec \lambda' \textit{ and } [\mathfrak{f}_{\mu'}]\,\mathfrak{m}_\mu = 1.$$

Example 3.4.10. For $r = 3$, the matrix $\mathcal{U}^{(r)}$ is given by

$\mathfrak{f}_\lambda \setminus \mathfrak{m}_\mu$	$[1^3]$	$[2,1]$	$[3]$
$[3]$	1	-3	3
$[2,1]$	0	1	-3
$[1^3]$	0	0	1

Proof. By inverting Eq. (3.3), we get that for any μ

$$m_\mu = \sum_{\lambda\vdash r} u_{\lambda,\mu} e_\lambda,$$

where $u_{\lambda,\mu}$ are integers such that $u_{\mu',\mu} = 1$ and $u_{\lambda,\mu} = 0$ if $\mu < \lambda'$. In particular $u_{\lambda,\mu} = 0$ if $\mu \prec \lambda'$. By evaluating at the Jucys-Murphy elements and applying ϕ_n we obtain

$$\phi_n(m_\mu(\Xi_n)) = \sum_{\lambda\vdash r} u_{\lambda,\mu}\phi_n(e_\lambda(\Xi_n)). \tag{3.15}$$

But we know that for any $\lambda \vdash r$

$$\deg(\phi_n(e_\lambda(\Xi_n))) = \deg([\phi_n(e_\lambda(\Xi_n))]) = \deg(\mathfrak{f}_\lambda(n)) = |\lambda| = r,$$

for n large enough. Hence, by taking the top degree term in Eq. (3.15) we obtain

$$\mathfrak{m}_\mu(n) = \sum_{\lambda \vdash r} u_{\lambda,\mu}\, \mathfrak{f}_\lambda(n).$$

We conclude by taking the limit in $\mathcal{Z}_\infty$. □

The following theorem is due to Matsumoto and Novak.

Theorem 3.4.11 ([MN13, Theorem 2.4])**.** *For every $r \geq 0$, the matrix $\mathcal{L}^{(r)}$ of $(\mathfrak{c}_\rho)_{\rho\vdash r,\preceq}$ in $(\mathfrak{m}_\mu)_{\mu\vdash r,\preceq}$ is lower triangular, with diagonal coefficients equal to 1, i.e.*

$$[\mathfrak{m}_\mu]\, \mathfrak{c}_\rho = 0 \textit{ if } \mu < \rho \textit{ and } [\mathfrak{m}_\rho]\, \mathfrak{c}_\rho = 1.$$

Example 3.4.12. For $r = 3$, $\mathcal{L}^{(r)}$ is given by

$\mathfrak{m}_\mu \setminus \mathfrak{c}_\rho$	$[1^3]$	$[2,1]$	$[3]$
$[1^3]$	1	0	0
$[2,1]$	-1	1	0
$[3]$	2	-3	1

.

For every $r \geq 0$, we define $\mathcal{M}^{(r)}$ as the matrix of $(\mathfrak{c}_\rho)_{\rho\vdash r,\preceq}$ in $(\mathfrak{f}_\lambda)_{\lambda\vdash r,\preceq'}$. Hence

$$\mathcal{M}^{(r)} = \mathcal{U}^{(r)}\mathcal{L}^{(r)}.$$

For every $1 \leq i \leq r$, we define $\mathcal{M}^{(r,i)}$ as the submatrix of $\mathcal{M}^{(r)}$ obtained by keeping the indices (λ, ρ) such that the rows index λ satisfies $\lambda_1 \leq i$ and the columns index ρ satisfies $\ell(\rho) \leq i$. This matrix is a South-East block of $\mathcal{M}^{(r)}$ (this is a consequence of item (2) in Definition 3.2.1).

Example 3.4.13. For $r = 3$,

$\mathcal{M}^{(3)}$ is given by

$\mathfrak{f}_\lambda \setminus \mathfrak{c}_\rho$	$[1^3]$	$[2,1]$	$[3]$
$[3]$	10	-12	3
$[2,1]$	-7	10	-3
$[1^3]$	2	-3	1

and $\mathcal{M}^{(3,2)}$ by

$\mathfrak{f}_\lambda \setminus \mathfrak{c}_\rho$	$[2,1]$	$[3]$
$[2,1]$	10	-3
$[1^3]$	-3	1

.

Similarly, we define $\mathcal{U}^{(r,i)}$ as the submatrix of $\mathcal{U}^{(r)}$ obtained by keeping the indices (λ, μ) such that $\lambda_1 \leq i$ and $\ell(\mu) \leq i$, and $\mathcal{L}^{(r,i)}$ as the submatrix of $\mathcal{L}^{(r)}$ obtained by keeping the indices (μ, ρ) such that $\ell(\mu) \leq i$ and $\ell(\rho) \leq i$.

Corollary 3.4.14. *For every $r \geq 1$ and $1 \leq i \leq r$, the submatrix $\mathcal{M}^{(r,i)}$ has determinant* 1.

Proof. Using the triangularity properties of $\mathcal{L}^{(r)}$ and $\mathcal{U}^{(r)}$ one can check that

$$\mathcal{M}^{(r,i)} = \mathcal{U}^{(r,i)}\mathcal{L}^{(r,i)},$$

and that $\det\left(\mathcal{U}^{(r,i)}\right) = \det\left(\mathcal{L}^{(r,i)}\right) = 1.$ □

We now prove Theorem 3.4.6.

Proof of Theorem 3.4.6. Let $\mathcal{N}^{(r)}$ be the matrix of $(\mathfrak{g}_\pi)_{\pi\vdash r,\preceq}$ in $(\mathfrak{f}_\lambda)_{\lambda\vdash r,\preceq'}$. Our purpose is to prove that $\det(\mathcal{N}^{(r)})=1$. Let $\mathcal{N}^{(r)}_{\lambda,\pi}$ denote the coefficients of $\mathcal{N}^{(r)}$, namely

$$\mathfrak{g}_\pi=\sum_{\lambda\vdash r}\mathcal{N}^{(r)}_{\lambda,\pi}\,\mathfrak{f}_\lambda\,. \tag{3.16}$$

We start by proving that $\mathcal{N}^{(r)}$ is block-upper triangular. More precisely, if $\mathcal{N}^{(r)}_{\lambda,\pi}\neq 0$ then $\lambda_1\geq\ell(\pi)$. Indeed, from the definition of $\mathfrak{g}_\pi$ we have $\mathfrak{g}_\pi=\mathfrak{c}_{\pi-\mathbf{1}}\,\mathfrak{f}_{\ell(\pi)}$. We write $\mathfrak{c}_{\pi-\mathbf{1}}$ in the basis $(\mathfrak{f}_\kappa)_{\kappa\vdash r-\ell(\pi)}$,

$$\mathfrak{c}_{\pi-\mathbf{1}}=\sum_{\kappa\vdash r-\ell(\pi)}\mathcal{M}^{(r-\ell(\pi))}_{\kappa,\pi-\mathbf{1}}\,\mathfrak{f}_\kappa\,,$$

where $(\mathcal{M}^{(r-\ell(\pi))}_{\kappa,\nu})_{\kappa,\nu\vdash r-\ell(\pi)}$ are the coefficients of the matrix $\mathcal{M}^{(r-\ell(\pi))}$. Multiplying the last equation by $\mathfrak{f}_{\ell(\pi)}$ and comparing it to Equation (3.16), we get

$$\mathcal{N}^{(r)}_{\lambda,\pi}=\mathcal{M}^{(r-\ell(\pi))}_{\lambda\backslash\ell(\pi),\pi-\mathbf{1}}. \tag{3.17}$$

In particular if $\mathcal{N}^{(r)}_{\lambda,\pi}\neq 0$ then one has $\ell(\pi)$ is a part of λ, and then necessarily $\ell(\pi)\leq\lambda_1$.

This proves that that $\mathcal{N}^{(r)}$ is a block-upper triangular matrix. We define for $1\leq i\leq r$ the diagonal block $\mathcal{N}^{(r,i)}$ as the submatrix that contains the coefficients $\mathcal{N}^{(r)}_{\lambda,\pi}$ for $\ell(\pi)=\lambda_1=i$. Moreover, we have a bijection between pairs of partitions (λ,π) of size r such that $\ell(\pi)=\lambda_1=i$ and pairs of partitions (ν,κ) of size $r-i$ and such that $\nu_1\leq i$ and $\ell(\kappa)\leq i$, given by:

$$(\lambda,\pi)\longmapsto(\lambda\backslash\lambda_1,\pi-\mathbf{1})$$

Using Equation (3.17), we deduce that

$$\mathcal{N}^{(r,i)}=\mathcal{M}^{(r-i,i)}.$$

Corollary 3.4.14 then implies that $\det(\mathcal{N}^{(r,i)})=1$. Since $\mathcal{N}^{(r)}$ is block triangular, we deduce that $\det\left(\mathcal{N}^{(r)}\right)=1$, and that $(\mathfrak{g}_\pi)_{\pi\vdash r}$ is a basis for $\mathcal{Z}^{(r)}_\infty$. □

Example 3.4.15. We give here the matrix $\mathcal{N}^{(r)}$ for $r=5$. The diagonal blocks, defined by $\ell(\pi)=\lambda_1$, are colored in gray. Note that for $i=2$, the matrix $\mathcal{N}^{(5,2)}$ is equal to $\mathcal{M}^{(3,2)}$ given in Example 3.4.13.

$\mathfrak{f}_\lambda\setminus\mathfrak{g}_\pi$	$[1^5]$	$[2,1^3]$	$[2^2,1]$	$[3,1^2]$	$[3,2]$	$[4,1]$	$[5]$
$[5]$	1	0	0	0	0	0	0
$[4,1]$		1	0	0	0	0	-4
$[3,2]$			3	-1	-12	3	0
$[3,1^2]$			-2	1	0	0	4
$[2^2,1]$					10	-3	2
$[2,1^3]$					-3	1	-4
$[1^5]$							1

3.5 The cases $\alpha = 1$ and $\alpha = 2$ in the Matching-Jack conjecture via the multiplicativity property

In this section, we give a combinatorial interpretation of the multiplicativity property of Equation (3.1) in terms of matchings in the special cases $\alpha = 1$ and $\alpha = 2$. We use this interpretation to give a new proof of the Matching-Jack conjecture in these two cases. Unlike the classical proof given in [GJ96b], the approach that we use here does not use representation theory. A key step in the proof is the combinatorial interpretation of the marginal sum coefficients in terms of matchings obtained in Theorem 2.4.1; see also Theorem 3.5.2 and Remark 3.5.5.

3.5.1 Multiplicativity for Matchings

Let $\lambda \vdash n \geq 1$ and let δ_1 and δ_2 be two bipartite matchings on $\mathcal{N}_n$ such that $\Lambda(\delta_1, \delta_2) = \lambda$. We define for any partitions $\mu, \nu \vdash n$ the two quantities

$$a^{\lambda}_{\mu,\nu} = \left|\{\delta \text{ such that } \Lambda(\delta_1, \delta) = \mu \text{ and } \Lambda(\delta, \delta_2) = \nu\}\right|,$$

and

$$\widetilde{a}^{\lambda}_{\mu,\nu} = \left|\{\delta \text{ bipartite matching such that } \Lambda(\delta_1, \delta) = \mu \text{ and } \Lambda(\delta, \delta_2) = \nu\}\right|,$$

see Section 1.1.4 for the definition of the partition Λ. We recall that these quantities do not depend on the choice of the bipartite matchings δ_1 and δ_2 (see Remark 1.1.8). In particular, $a^{\lambda}_{\mu,\nu} = a^{\lambda}_{\nu,\nu}$ and $\widetilde{a}^{\lambda}_{\mu,\nu} = \widetilde{a}^{\lambda}_{\nu,\nu}$ Moreover, they satisfy the following multiplicativity property.

Proposition 3.5.1. *For all partitions $\lambda, \mu, \nu, \rho \vdash n \geq 1$, we have*

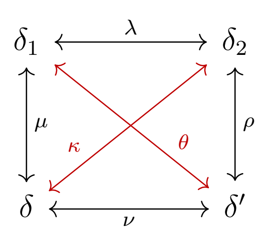
$$\sum_{\kappa \vdash n} a^{\lambda}_{\mu,\kappa} a^{\kappa}_{\nu,\rho} = \sum_{\theta \vdash n} a^{\lambda}_{\theta,\rho} a^{\theta}_{\mu,\nu}. \tag{3.18}$$

$$\sum_{\kappa \vdash n} \widetilde{a}^{\lambda}_{\mu,\kappa} \widetilde{a}^{\kappa}_{\nu,\rho} = \sum_{\theta \vdash n} \widetilde{a}^{\lambda}_{\theta,\rho} \widetilde{a}^{\theta}_{\mu,\nu}. \tag{3.19}$$

Proof. We fix two bipartite matchings δ_1 and δ_2 of $\mathcal{N}_n$ satisfying $\Lambda(\delta_1, \delta_2) = \lambda$. We introduce the set

$$\mathfrak{F}^{\delta_1,\delta_2}_{\mu,\nu,\rho} := \left\{(\delta, \delta') \text{ such that } \Lambda(\delta_1, \delta) = \mu, \Lambda(\delta, \delta') = \nu, \text{ and } \Lambda(\delta', \delta_2) = \rho\right\}.$$

For every $(\delta, \delta') \in \mathfrak{F}^{\delta_1,\delta_2}_{\mu,\nu,\rho}$ we have a diagram



for some partitions θ and κ of n, where the arrow between two matchings δ_a and δ_b is labelled by a partition ξ if $\Lambda(\delta_a, \delta_b) = \xi$. We have two ways to count the number of pairs (δ, δ') in $\mathfrak{F}^{\delta_1,\delta_2}_{\mu,\nu,\rho}$; either we start by choosing δ (this corresponds to the left-hand side in Equation (3.18)) or we start by choosing δ' (this corresponds to the right-hand side in Equation (3.18)).

Similarly, we obtain Equation (3.19) by considering the set

$$\widetilde{\mathfrak{F}}^{\delta_1,\delta_2}_{\mu,\nu,\rho} := \{(\delta, \delta') \text{ bipartite matchings such that } \Lambda(\delta_1, \delta) = \mu, \\ \Lambda(\delta, \delta') = \nu, \text{ and } \Lambda(\delta', \delta_2) = \rho\}\,. \qquad \square$$

3.5.2 The Matching-Jack conjecture for $\alpha = 1$ and $\alpha = 2$

With the notation of this section, the cases $\alpha = 1$ and $\alpha = 2$ of Theorem 1.5.1 (proved in Chapter 2), can be formulated as follows.

Theorem 3.5.2. *For any integers $n, l \geq 1$ and partitions $\lambda, \mu \vdash n$ we have:*

$$\overline{c}^{\lambda}_{\mu,l}(2) = \sum_{\ell(\nu)=l} a^{\lambda}_{\mu,\nu}, \quad \textit{and} \quad \overline{c}^{\lambda}_{\mu,l}(1) = \sum_{\ell(\nu)=l} \widetilde{a}^{\lambda}_{\mu,\nu}.$$

Proof. From Theorem 1.5.1, we get

$$\overline{c}^{\lambda}_{\mu,l}(2) = \sum_{\ell(\nu)=l} a^{\lambda}_{\nu,\mu}, \quad \text{and} \quad \overline{c}^{\lambda}_{\mu,l}(1) = \sum_{\ell(\nu)=l} \widetilde{a}^{\lambda}_{\nu,\mu}.$$

We conclude using the symmetry of $a^{\lambda}_{\nu,\mu}$ and $\widetilde{a}^{\lambda}_{\nu,\mu}$ in μ and ν. $\square$

In the following, we use the previous theorem and the multiplicativity property to give a new proof for the cases $b = 0$ and $b = 1$ of the Matching-Jack conjecture.

Theorem 3.5.3. *For all partitions $\lambda, \mu, \nu \vdash n \geq 1$, we have*

$$c^{\lambda}_{\mu,\nu}(2) = a^{\lambda}_{\mu,\nu}, \quad \textit{and} \quad c^{\lambda}_{\mu,\nu}(1) = \widetilde{a}^{\lambda}_{\mu,\nu}.$$

Proof. Taking the sum over partitions ρ of length l in Equations (3.18) and (3.19) and using Theorem 3.5.2 we obtain:

$$\sum_{\kappa \vdash n} a^{\lambda}_{\mu,\kappa}\, \overline{c}^{\kappa}_{\nu,l}(2) = \sum_{\theta \vdash n} \overline{c}^{\lambda}_{\theta,l}(2) a^{\theta}_{\mu,\nu},$$

$$\sum_{\kappa \vdash n} \widetilde{a}^{\lambda}_{\mu,\kappa}\, \overline{c}^{\kappa}_{\nu,l}(1) = \sum_{\theta \vdash n} \overline{c}^{\lambda}_{\theta,l}(1) \widetilde{a}^{\theta}_{\mu,\nu}.$$

We use Proposition 3.3.4 to conclude. $\square$

Remark 3.5.4. Let $n \geq 1$ and let $(\mathrm{st}_\lambda)_{\lambda \vdash n}$ be a family of statistic on matchings of $\mathcal{N}_n$. For all partitions $\lambda, \mu, \nu \vdash n$, we define

$$y^{\lambda}_{\mu,\nu} := \sum_{\delta \in \widetilde{\mathfrak{F}}^{\varepsilon,\delta_\lambda}_{\mu,\nu}} b^{\mathrm{st}_\lambda(\delta)},$$

where ε and δ_λ are the matchings defined in Section 1.1.4. One possible way to prove that the family (st_λ) is a solution for the Matching-Jack conjecture, is to prove that:

1. $y^\lambda_{\mu,\nu}$ satisfy the multiplicativity property of Proposition 3.3.4.
2. $y^\lambda_{\mu,\nu}$ are a solution for the conjecture in the case of marginal sums, namely

$$\sum_{\ell(\nu)=l} y^\lambda_{\mu,\nu} = \overline{c}^\lambda_{\mu,l}\,.$$

We can think of the two previous properties as generalizations of Proposition 3.5.1 and Theorem 3.5.2 given in this section in the cases $\alpha \in \{1,2\}$. We recall that the statistics st_λ defined in Chapter 2 satisfy the second property. Unfortunately, they do not to satisfy the first one (see also Section 6.1.1 for a detailed discussion).

Remark 3.5.5. It is worth mentioning that the proof of Theorem 3.5.2 corresponding to the case $\alpha \in \{1,2\}$ is simpler than the general case of Theorem 2.4.1. Indeed, we recall that the proof of the latter is based on the differential equation of Theorem 2.3.1 from [CD22]. This differential equation is obtained using some commutation relations, which turn out to have nice combinatorial proofs when $\alpha \in \{1,2\}$. For more details we refer to the discussion in [CD22, Section 4.3].

3.6 Some consequences of the main result

3.6.1 A partial integrality result for the b-conjecture

We consider a "connected" version $d^\lambda_{\mu,\nu}$ of the coefficients $c^\lambda_{\mu,\nu}$, defined by

$$\log\left(\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\right) = \sum_{n\geq 1} t^n \sum_{\lambda,\mu,\nu\vdash n} \frac{d^\lambda_{\mu,\nu}}{z_\lambda \alpha^{\ell(\lambda)}} p_\lambda q_\mu r_\nu. \tag{3.20}$$

These coefficients are related to the coefficients $h^\lambda_{\mu,\nu}$ of the b-conjecture (see Section 1.3.1) by

$$h^\lambda_{\mu,\nu} = \frac{n d^\lambda_{\mu,\nu}}{z_\lambda \alpha^{\ell(\lambda)-1}}, \tag{3.21}$$

for any $\lambda,\mu,\nu \vdash n \geq 1$.

In this section we prove that the coefficients $d^\lambda_{\mu,\nu}$ are integer polynomials in α (see Theorem 3.6.4). Using Theorem 1.3.9, this implies that $\frac{z_\lambda}{n} h^\lambda_{\mu,\nu}$ is an integer polynomial in α, however we do not have information about the divisibility of the coefficients by $\frac{z_\lambda}{n}$.

Definition 3.6.1. *Fix a set S. A set-partition of S is an unordered family of non-empty disjoint subsets of S whose union is S. We denote by $\mathcal{P}(S)$ the set of set-partitions of S. If π is a set-partition of S into s blocks then we denote $\ell(\pi) = s$. For every $s \geq 1$, we denote by $\mathcal{P}_s(S)$ the set of set-partitions of S into s blocks. Finally, we write $B \in \pi$ is B is a block of π.*

For any integer $n \geq 1$, we set $[\![n]\!] := \{1,2,...,n\}$. Fix a partition λ. For any subset $B \subset [\![\ell(\lambda)]\!]$, we denote by $\lambda_B := [\lambda_i; i \in B]$ the partition obtained by keeping the parts of λ with an index in B.

The following is a variant of [DF17, Lemma 5.2] where it is formulated in terms of cumulants. The proof is quite similar.

Lemma 3.6.2. *For any partitions $\lambda,\mu,\nu\vdash n\geq 1$, we have*

$$d^{\lambda}_{\mu,\nu}=\sum_{\pi\in\mathcal{P}(\llbracket\ell(\lambda)\rrbracket)}\left(\sum_{(\mu^B)_{B\in\pi},(\nu^B)_{B\in\pi}}\prod_{B\in\pi}(-1)^{\ell(\pi)-1}(\ell(\pi)-1)!c^{\lambda_B}_{\mu^B,\nu^B}\right), \tag{3.22}$$

where the second sum is taken over tuples of partitions $(\mu^B)_{B\in\pi}$ and $(\nu^B)_{B\in\pi}$ such that $\cup_{B\in\pi}\mu^B=\mu$, $\cup_{B\in\pi}\nu^B=\nu$, and for each $B\in\pi$ we have $|\mu^B|=|\nu^B|=|\lambda_B|$.

Example 3.6.3. Fix $n=4$, $\lambda=\mu=[2,2]$ and $\nu=[2,1,1]$. We then have two set partitions of $\llbracket\ell(\lambda)\rrbracket=\{1,2\}$, which are $\pi=\{\{1,2\}\}$, and $\pi=\{\{1\},\{2\}\}$. They have respectively 1 and 2 blocks. We then have

$$\begin{aligned}d^{[2^2]}_{[2^2],[2,1^2]}&=c^{[2^2]}_{[2^2],[2,1^2]}-\left(c^{[2]}_{[2],[2]}c^{[2]}_{[2],[1^2]}+c^{[2]}_{[2],[1^2]}c^{[2]}_{[2],[2]}\right)\\&=c^{[2^2]}_{[2^2],[2,1^2]}-2c^{[2]}_{[2],[2]}c^{[2]}_{[2],[1^2]}.\end{aligned}$$

Proof. From equations (1.29) and (3.20) we have

$$\sum_{n\geq1}t^n\sum_{\lambda,\mu,\nu\vdash n}\frac{d^{\lambda}_{\mu,\nu}}{z_\lambda\alpha^{\ell(\lambda)}}p_\lambda q_\mu r_\nu=\log\left(1+\sum_{n\geq1}t^n\sum_{\lambda,\mu,\nu\vdash n}\frac{c^{\lambda}_{\mu,\nu}}{z_\lambda\alpha^{\ell(\lambda)}}p_\lambda q_\mu r_\nu\right).$$

We fix three partitions λ,μ,ν of the same size. By expanding the logarithm in the previous equation we obtain

$$d^{\lambda}_{\mu,\nu}=\sum_{s\geq1}\frac{(-1)^{s-1}}{s}\sum_{(\lambda^i,\mu^i,\nu^i)_{1\leq i\leq s}}\frac{z_\lambda}{z_{\lambda^1}\dots z_{\lambda^s}}\prod_{1\leq i\leq s}c^{\lambda^i}_{\mu^i,\nu^i}, \tag{3.23}$$

where the second sum is taken over s-tuples $(\lambda^i,\mu^i,\nu^i)_{1\leq i\leq s}$ such that $\cup_{1\leq i\leq s}\lambda^i=\lambda$, $\cup_{1\leq i\leq s}\mu^i=\mu$ and $\cup_{1\leq i\leq s}\nu^i=\nu$. We will prove that for every $s\geq1$,

$$\frac{1}{s!}\sum_{(\lambda^i,\mu^i,\nu^i)_{1\leq i\leq s}}\frac{z_\lambda}{z_{\lambda^1}\dots z_{\lambda^s}}\prod_{1\leq i\leq s}c^{\lambda^i}_{\mu^i,\nu^i}=\sum_{\pi\in\mathcal{P}_s(\llbracket\ell(\lambda)\rrbracket)}\left(\sum_{(\mu^B)_{B\in\pi},(\nu^B)_{B\in\pi}}\prod_{B\in\pi}c^{\lambda_B}_{\mu^B,\nu^B}\right). \tag{3.24}$$

From an s-tuple of partitions $(\lambda^i)_{1\leq i\leq s}$ such that $\cup_i\lambda^i=\lambda$, we can obtain an ordered set-partition of $\llbracket\ell(\lambda)\rrbracket$ into s sets B_1,...,B_s by reordering the parts of $\lambda^1,...,\lambda^s$ to reconstruct λ. In this reordering we have $\binom{m_j(\lambda)}{m_j(\lambda^1),...,m_j(\lambda^s)}$ ways to reorder the parts of size j. Hence we have

$$\prod_{1\leq j}\binom{m_j(\lambda)}{m_j(\lambda^1),...,m_j(\lambda^s)}=\frac{z_\lambda}{z_{\lambda^1}\dots z_{\lambda^s}}$$

ways to obtain an ordered partition of $\llbracket\ell(\lambda)\rrbracket$. Finally, we divide by $s!$, since the first sum in the right hand-side of Equation (3.24) is taken over unordered set-partitions.

Combining Eq. (3.23) and Eq. (3.24) concludes the proof of the lemma. □

We deduce the main theorem of this section.

Theorem 3.6.4. *For all partitions $\lambda,\mu,\nu\vdash n\geq 1$, the coefficient $d^{\lambda}_{\mu,\nu}$ is a polynomial in b with integer coefficients.*

Proof. This is a direct consequence of Lemma 3.6.2 and Theorem 1.5.2. □

3.6.2 A generalization to coefficients with $k+2$ parameters

In this section we state some integrality consequences for the generalized coefficients $c^{\lambda}_{\mu^1,\dots,\mu^k}$ and $h^{\lambda}_{\mu^1,\dots,\mu^k}$ defined in Section 1.3.1. We start by the following multiplicativity property which generalizes Proposition 3.3.1 (see also [Ben22, Proposition 6.1]).

Proposition 3.6.5. *Let $k \geq 2$ and $\lambda, \mu^0, \dots, \mu^k \vdash n \geq 1$. We have*

$$c^{\lambda}_{\mu^0,\dots,\mu^k}(\alpha) = \sum_{\xi \vdash n} c^{\lambda}_{\mu^0,\dots\mu^{k-2},\xi}(\alpha) c^{\xi}_{\mu^{k-1},\mu^k}(\alpha).$$

The following corollary is an immediate consequence of the previous proposition and Theorem 1.5.2.

Corollary 3.6.6. *For any $k \geq 1$ and $\lambda, \mu^0, \dots, \mu^k \vdash n \geq 1$, the coefficient $c^{\lambda}_{\mu^0,\dots,\mu^k}$ is a polynomial in b with integer coefficients.*

One can see that the arguments[6] used in Section 3.6.1 can be generalized to obtain integrality information about the integrality of coefficients $h^{\lambda}_{\mu^0,\dots,\mu^k}$.

Proposition 3.6.7. *For any $k \geq 1$ and $\lambda, \mu^0, \dots, \mu^k \vdash n \geq 1$, we have that $\frac{z_\lambda}{n} h^{\lambda}_{\mu^0,\dots,\mu^k}$ is an integer polynomial in b.*

In particular, since $z_{[n]} = n$, we have the following corollary.

Corollary 3.6.8. *For any $k \geq 1$ and $\mu^0, \dots, \mu^k \vdash n \geq 1$, the coefficient $h^{[n]}_{\mu^0,\dots,\mu^k}$ is an integer polynomial in b.*

The case of one single part partition in the b- and the Matching-Jack conjecture has been considered in previous works and some partial results have been proved in this direction, see [KV16, Doł17, KPV18]. Corollary 3.6.8 establishes the integrality for the one single part partition case in the "generalized" b-conjecture. This result is known for $k = 1$. Indeed, [CD22, Theorem 5.10] implies that $h^{[n]}_{\mu,\nu,[n],\dots,[n]}$ is a non-negative integer polynomial in b. Corollary 3.6.8 is however new for general k.

[6] We use here a generalized version of Theorem 1.3.9 that gives the polynomiality of $h^{\lambda}_{\mu^0,\dots,\mu^k}$ for $k > 1$, see [Ben22, Theorem 6.6].

Chapter 4

Jack characters and a proof of Lassalle's conjecture

This chapter is based on the work of [BD23] joint with Maciej Dołęga.

The main purpose of this chapter is to prove the combinatorial formula of Jack characters given in Theorem 1.5.3 as well as Lassalle's conjecture (Theorem 1.5.6). We briefly explain the ideas of the proofs.

As we have already mentioned, the ideas used in the proofs of the special cases $b = 0$ and $b = 1$ of Theorem 1.5.3 do not apply to the general case. The key tool in our proof is the differential calculus approach developed by Chapuy and Dołęga in [CD22] and explained in Section 2.2. We combine this approach with an algebraic characterization of Jack characters as the unique shifted symmetric functions determined by vanishing conditions in the spirit of [KS96]; see Theorem 4.1.11.

More precisely;

- we give a construction of the series of weighted layered maps using the differential operators of Chapuy–Dołęga; see Proposition 4.3.5. This step can be seen as a multi-layer generalization of the construction of Sections 2.1 and 2.2.

- we use a characterization of Jack characters $\theta_\mu^{(\alpha)}$ as shifted symmetric functions due to Féray (Theorem 4.1.11), and prove that the generating series of layered maps satisfies the conditions of Theorem 4.1.11, see Theorem 4.4.1 and Theorem 4.6.1.

  To this purpose we prove several commutation relations between the differential operators obtained in the first step, using methods inspired by the theory of Lie algebras. We show that these algebraic relations reflect the desired combinatorial and algebraic properties of the right-hand side in Theorem 1.5.3.

The second main result of this chapter is the proof of Lassalle's conjecture on Jack characters (Theorem 1.5.6). This theorem does not immediately follow from Theorem 1.5.3. Indeed, the proof of Theorem 1.5.6 consists of two parts that are proved using very different techniques. In the first part we deduce positivity as a consequence of the combinatorial expression of Jack characters obtained in Theorem 1.5.3. In the second part, we obtain the integrality[1] using the integrable system of Nazarov–Sklyanin [NS13]. We relate their theory to Jack characters by proving an explicit combinatorial formula expressing a certain basis of shifted symmetric functions in terms of normalized Jack characters. We conclude by showing that the transition matrix between these two bases is invertible over $\mathbb{Z}$. As a byproduct, we prove that Kerov polynomials for Jack characters have integer coefficients, which was an open problem posed by Lassalle in [Las09] (see Section 4.2 for details).

Structure of the chapter

The chapter is organized as follows. In Section 4.1 we give a proof of Theorem 4.1.11 that characterizes the Jack characters. In Section 4.2, we prove the integrality in Theorem 1.5.6. In Section 4.3 we explain the combinatorial decomposition of layered maps and we give a differential expression for the associated generating series. Section 4.4 is dedicated to the proof of the first characterization property, namely the vanishing property. In Section 4.5 we prove a series of commutation relations for differential operators which are used to obtain the second characterization property in Section 4.6. In Section 4.7, we finish the proof of Theorem 1.5.3 and we prove the positivity in Theorem 1.5.6.

4.1 Characterization of Jack characters as shifted symmetric functions

In this section, we introduce shifted symmetric functions and we give some properties related to Jack characters. In particular, we recall a theorem of Féray which uniquely determines Jack characters as shifted symmetric functions with specific vanishing properties; see Theorem 4.1.11.

4.1.1 Shifted symmetric functions

We start by defining shifted symmetric functions and recalling some results related to them. Several of these results are based on the works of Lassalle and Knop–Sahi [KS96, Las98, Las08a].

Definition 4.1.1 ([Las08a]). *We say that a polynomial in k variables $(s_1, \dots, s_k)$ with coefficients in $\mathbb{Q}(\alpha)$ is α-shifted symmetric if it is symmetric in the variables $s_i - i/\alpha$. An α-shifted symmetric function (or simply a shifted symmetric function) is a sequence $(f_k)_{k\geq 1}$ of shifted symmetric polynomials of bounded degrees, such that for every $k \geq 1$, the function f_k is an α-shifted symmetric polynomial in k variables and*

$$f_{k+1}(s_1, \dots, s_k, 0) = f_k(s_1, \dots, s_k). \tag{4.1}$$

Moreover, f has degree n if f_k has degree n for k large enough.

We denote by $\mathcal{S}^*_\alpha$ the algebra of shifted symmetric functions.

[1] Here, the term integrality (reps. positivity) refers to the fact that characters are polynomials and that coefficient in the corresponding parameters are integers (resp. positive). In particular, we give two different proofs of the polynomiality part of the conjecture.

Example 4.1.2. The *shifted symmetric power-sum functions* are defined for any $n \geq 0$ by

$$p_n^{\#}(s_1, s_2, \dots) := \sum_{i \geq 1} \left(\left(s_i - \frac{i-1}{\alpha} \right)^n - \left(-\frac{i-1}{\alpha} \right)^n \right).$$

With the notation of Definition 4.1.1, the polynomials $(p_n^{\#})_k$ are then given by

$$(p_n^{\#})_k = \sum_{1 \leq i \leq k} \left(\left(s_i - \frac{i-1}{\alpha} \right)^n - \left(-\frac{i-1}{\alpha} \right)^n \right)$$

Note that the second item in the sum assures the stability property of Eq. (4.1). As in the symmetric case, we extend this definition by multiplicativity;

$$p_\mu^{\#} := p_{\mu_1}^{\#} \dots p_{\mu_{\ell(\mu)}}^{\#}.$$

Shifted symmetric power-sum symmetric functions $(p_\mu^{\#})_{\mu \in \mathbb{Y}}$ form a basis of $\mathcal{S}_\alpha^*$. One may notice that the top homogeneous part of $p_\mu^{\#}$ is p_μ.

A shifted symmetric function f can be seen as a function on Young diagrams as follows; if $\lambda = [\lambda_1, \dots, \lambda_k]$ is a partition, then we denote

$$f(\lambda) := f(\lambda_1, \dots, \lambda_k, 0, \dots).$$

Theorem 4.1.3 ([KS96])**.** *Let $n \geq 0$, and let g be a function on Young diagrams with values in $\mathbb{Q}(\alpha)$. There exists a unique shifted symmetric function f of degree less than or equal to n such that $f(\lambda) = g(\lambda)$ for any $|\lambda| \leq n$.*

In particular, a shifted symmetric function is completely determined by its evaluation on Young diagrams $(f(\lambda))_{\lambda \in \mathbb{Y}}$. As a consequence, for any partition μ, there exists a unique shifted symmetric function J_μ^* which satisfies the following properties:

1. $\deg(J_\mu^*) \leq |\mu|$,
2. $J_\mu^*(\mu) = \alpha^{-|\mu|} j_\mu^{(\alpha)}$,
3. $J_\mu^*(\lambda) = 0$, for any $|\lambda| \leq |\mu|$ and $\lambda \neq \mu$.

Knop and Sahi have proved the following remarkable result about of these functions; see [KS96, Theorem 2.1 and Corollary 4.7].

Theorem 4.1.4 ([KS96])**.** *For any partition μ, the function J_μ^* has degree μ, and its top homogeneous part is $J_\mu^{(\alpha)}$. Moreover, we have a stronger vanishing condition*

3'. $J_\mu^*(\lambda) = 0$, unless $\mu \subseteq \lambda$.

The functions J_μ^* are known as *Jack shifted symmetric functions*. Sometimes they are also referred to as *interpolation functions* since they are defined by their evaluation at some Young diagrams.

Example 4.1.5. When $\mu = 2$, we have

$$J_{[2]}^* = \alpha p_2^{\#} + \left(p_1^{\#}\right)^2 - (\alpha+1) p_1^{\#}.$$

4.1.2 Lassalle's isomorphism

As $(J^{(\alpha)}_\mu)_{\mu\in\mathbb{Y}}$ is a basis of $\mathcal{S}_\alpha$, using a triangularity argument we can see that $(J^*_\mu)_{\mu\in\mathbb{Y}}$ is a basis of $\mathcal{S}^*_\alpha$. We can then extend by linearity the map $J^{(\alpha)}_\mu \longmapsto J^*_\mu$, into an isomorphism

$$\begin{aligned} \mathcal{S}_\alpha &\longrightarrow \mathcal{S}^*_\alpha \\ J^{(\alpha)}_\mu &\longmapsto J^*_\mu. \end{aligned} \tag{4.2}$$

The following proposition follows from the previous construction and Theorem 4.1.4.

Proposition 4.1.6. *Let f be a homogeneous symmetric function. Then the top homogeneous part of f^* coincides with f.*

Lassalle established in [Las98] an explicit formula for the isomorphism of Eq. (4.2) (see also [Las08a, Eq. 3.1]).

Theorem 4.1.7 ([Las98])**.** *Let f be a symmetric function. Then for any partition λ the following holds*

$$f^*(\lambda) = \alpha^{-|\lambda|}\left\langle \exp(p_1)\cdot f, J^{(\alpha)}_\lambda \right\rangle,$$

where $\langle , \rangle$ denotes the scalar product on symmetric functions defined in Eq. (1.9).

Remark 4.1.8. Note that this theorem gives only a formula for f^* as a function on Young diagrams (the interpolation values). We give a more general formula for f^* as a shifted symmetric function in Theorem 4.7.4.

4.1.3 Jack characters

It turns out that Jack characters can be obtained from power-sum symmetric functions by applying Lassalle's isomorphism.

Lemma 4.1.9 ([Las08a, Proposition 2])**.** *For any partition μ, we have*

$$\alpha^{|\mu|-\ell(\mu)}/z_\mu \cdot p^*_\mu = \theta^{(\alpha)}_\mu.$$

Proof. From the definition of Jack characters Eq. (1.40), we have for any partition λ

$$\theta^{(\alpha)}_\mu(\lambda) = \frac{1}{z_\mu \alpha^{\ell(\mu)}}\left\langle p_\mu, \exp\left(\frac{\partial}{\partial p_1}\right) J^{(\alpha)}_\lambda \right\rangle.$$

Since the dual of $\frac{\partial}{\partial p_1}$ is $\frac{p_1}{\alpha}$ (see Lemma 1.2.4), we get

$$\begin{aligned} \theta^{(\alpha)}_\mu(\lambda) &= \frac{1}{z_\mu \alpha^{\ell(\mu)}}\left\langle \exp\left(\frac{p_1}{\alpha}\right) p_\mu, J^{(\alpha)}_\lambda \right\rangle \\ &= \frac{\alpha^{|\mu|-|\lambda|}}{z_\mu \alpha^{\ell(\mu)}}\left\langle \exp\left(p_1\right) p_\mu, J^{(\alpha)}_\lambda \right\rangle \\ &= \frac{\alpha^{|\mu|-\ell(\mu)}}{z_\mu} p^*_\mu(\lambda), \end{aligned}$$

as desired. □

Since power-sum functions $(p_\mu)_{\mu\in\mathbb{Y}}$ form a linear basis of $\mathcal{S}_\alpha$, Lemma 4.1.9 implies that Jack characters $(\theta^{(\alpha)}_\mu)_{\mu\in\mathbb{Y}}$ form a basis of $\mathcal{S}^*_\alpha$.

Example 4.1.10. Let us express some Jack characters in terms of the shifted symmetric functions $p^\#_n$. From Eq. (1.10), we have

$$\theta^{(\alpha)}_\emptyset(\lambda) = 1.$$

More generally,

$$\theta^{(\alpha)}_{1^k}(\lambda) = \frac{1}{k!}\binom{|\lambda|}{k} = p^\#_1(\lambda)(p^\#_1(\lambda)-1)\dots(p^\#_1(\lambda)-k+1),$$

for any $k \geq 0$.

Moreover, by definition

$$\theta^{(\alpha)}_2(\lambda) = [p_{2,1^{n-2}}]J^{(\alpha)}_\lambda,$$

for any $\lambda \vdash n$. From [Mac95, Chapter VI, Example 1.b], we have

$$\begin{aligned}\theta^{(\alpha)}_2(\lambda) &= \sum_{i\geq 1}\frac{\alpha}{2}\lambda_i(\lambda_i-1) - \sum_{i\geq 1}(i-1)\lambda_i \\ &= \frac{\alpha}{2}\left(p^\#_2(\lambda) - p^\#_1(\lambda)\right).\end{aligned}$$

We conclude this section by the following theorem due to Féray (private communication), which gives a characterization of Jack characters as shifted symmetric functions satisfying some properties.

Theorem 4.1.11. *(Féray) Fix a partition μ. The Jack character $\theta^{(\alpha)}_\mu$ is the unique α-shifted symmetric function of degree $|\mu|$ with top homogeneous part $\alpha^{|\mu|-\ell(\mu)}/z_\mu \cdot p_\mu$, such that $\theta^{(\alpha)}_\mu(\lambda) = 0$ for any partition λ such that $|\lambda| < |\mu|$.*

Proof. The fact that $\theta^{(\alpha)}_\mu(\lambda) = 0$ if $|\lambda| < |\mu|$ comes from the definition (see Eq. (1.39)). Its top homogeneous part is obtained from Proposition 4.1.6 and Lemma 4.1.9 above.

Uniqueness: Let F be an α-shifted symmetric function of degree $|\mu|$ with the same top degree part as $\theta^{(\alpha)}_\mu$, and such that $F(\lambda) = 0$ for any partition λ such that $|\lambda| < |\mu|$. Set $G := F - \theta^{(\alpha)}_\mu$. Then G is an α-shifted symmetric function of degree at most $|\mu| - 1$ with

$$G(\lambda) = 0 \text{ for } |\lambda| < |\mu|. \tag{4.3}$$

But we know from Theorem 4.1.3 that $G = 0$ is the unique function satisfying this two conditions. This finishes the proof of the theorem. □

4.2 Integrality in Lassalle's conjecture

Before we prove Theorem 1.5.3 we present the proof of Theorem 1.5.6. The positivity will follow (Section 4.7.3) from the combinatorial interpretation (in terms of layered maps) stated in Theorem 1.5.3, whose proof is technically involved and will occupy the main part of this chapter. Integrality, however, does not follow from Theorem 1.5.3 and requires different ideas. We prove it using an approach based on combinatorics of Nazarov–Sklyanin operators interpreted as lattice paths (we also obtain the polynomiality part of the conjecture). These developments are independent of the other sections, and they are also of independent interest, as we demonstrate by proving other problems stated in the literature as a byproduct.

4.2.1 Nazarov–Sklyanin operators and α-polynomial functions

Recall that $\theta_\mu^{(\alpha)}$ is a linear basis of the algebra $\mathcal{S}_\alpha^*$ of α-shifted symmetric functions. In this subsection we introduce new families of bases of $\mathcal{S}_\alpha^*$ related to Kerov's transition measure.

Kerov's transition measure

Kerov associated with a Young diagram λ a certain probability measure μ_λ on $\mathbb{R}$ that is very useful for studying asymptotic behaviour of random Young diagrams. This *transition measure* is uniquely characterized by its Cauchy transform:

$$G_{\mu_\lambda}(z) := \int_{\mathbb{R}} \frac{d\mu_\lambda(x)}{z-x} = \frac{1}{z+\ell(\lambda)} \prod_{i=1}^{\ell(\lambda)} \frac{z+i-\lambda_i}{z+i-1-\lambda_i}. \tag{4.4}$$

In particular the k-th *moment* $M_k(\lambda)$ of the transition measure μ_λ can be computed by applying a simple relation between the Cauchy transform expanded around infinity and the generating function of moments:

$$z^{-1} + \sum_{k\geq 1} M_k(\lambda) z^{-k-1} = G_{\mu_\lambda}(z).$$

Fix $\ell \geq 0$. For a partition λ of length less than or equal to ℓ, $M_k(\lambda)$ can be treated as a polynomial of $\lambda_1, \dots, \lambda_\ell$ via

$$M_k(\lambda) = M_k(\lambda_1, \dots, \lambda_\ell),$$

with $\lambda_i = 0$ if $i > \ell(\lambda)$. It follows from the definition that these polynomials are 1-shifted symmetric in ℓ variables. Moreover, M_k satisfies the stability property

$$M_k(\lambda_1, \dots, \lambda_\ell, 0) = M_k(\lambda_1, \dots, \lambda_\ell).$$

We deduce that M_k is a 1-shifted symmetric function for any $k \geq 0$.

Define[2] now the following α-deformation

$$M_k^{(\alpha)}(\lambda) := M_k(\alpha \cdot \lambda_1, \dots, \alpha \cdot \lambda_{\ell(\lambda)}), \tag{4.5}$$

which is a well-defined function on $\mathbb{Y}$. In a similar way, we get that $M_k^{(\alpha)}(\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)})$ is an α-shifted symmetric polynomial, and that it satisfies the stability property

$$M_k^{(\alpha)}(\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}, 0) = M_k^{(\alpha)}(\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}).$$

As a consequence, $M_k^{(\alpha)}$ defines an α-shifted symmetric function.

[2] The normalization used here is slightly different from the ones used in [Las09] and in [DF16].

Example 4.2.1. We have

$$\begin{aligned} M_1^{(\alpha)}(\lambda) &= 0, \\ M_2^{(\alpha)}(\lambda) &= \alpha|\lambda| = \alpha p_1^{\#}(\lambda), \\ M_3^{(\alpha)}(\lambda) &= \alpha^2 p_2^{\#}(\lambda) - \alpha p_1^{\#}(\lambda). \end{aligned}$$

It was proved by Dołęga and Féray that $M_k^{(\alpha)}$ is an algebraic basis of $\mathcal{S}_\alpha^*$.

Theorem 4.2.2 ([DF16]). *The family* $\left(M_k^{(\alpha)}\right)_{k\geq 2}$ *is a basis (over* $\mathbb{Q}(\alpha)$*) of the algebra* $\mathcal{S}_\alpha^*$*.*

The above theorem is a starting point for defining other interesting bases using other observables arising from classical and free probability. Besides the moments, we will use the *Boolean cumulants* $B_\ell^{(\alpha)}$, and the *free cumulants* $R_\ell^{(\alpha)}$. In our context, it is convenient to define them by the following recursive formulas that can be easily inverted over $\mathbb{Z}$, (see [DFŚ10, Proposition 2.2] and [CDM23, Proposition 3.2]).

Definition 4.2.3. *We define the boolean cumulants* $(B_k^{(\alpha)})_{k\geq 2}$ *and the free cumulants* $(R_k^{(\alpha)})_{k\geq 2}$ *by the following relations; for any integer* $\ell \geq 2$*,*

$$M_\ell^{(\alpha)} = \sum_{n\geq 1} \sum_{\substack{k_1,\dots,k_n\geq 2 \\ k_1+\dots+k_n=\ell}} B_{k_1}^{(\alpha)} \cdots B_{k_n}^{(\alpha)}, \tag{4.6}$$

$$M_\ell^{(\alpha)} = \sum_{n\geq 1} \frac{(\ell)_{n-1}}{n!} \sum_{\substack{k_1,\dots,k_n\geq 2 \\ k_1+\dots k_n=\ell}} R_{k_1}^{(\alpha)} \cdots R_{k_n}^{(\alpha)} \tag{4.7}$$

where

$$(\ell)_n = \ell(\ell-1)\dots(\ell-n+1).$$

Remark 4.2.4. Note that the previous relations define the families $B_\ell^{(\alpha)}$ and $R_\ell^{(\alpha)}$ by a triangularity argument, since the coefficient of $B_\ell^{(\alpha)}$ and $R_\ell^{(\alpha)}$ in $M_\ell^{(\alpha)}$ is equal to 1.

We then have the following consequence of Theorem 4.2.2.

Theorem 4.2.5. *The family* $\left(X_\ell^{(\alpha)}\right)_{\ell\geq 2}$ *is a basis (over* $\mathbb{Q}[\alpha]$*) algebra* $\mathcal{S}_\alpha$ *where* $X = M, B, R$*.*

Nazarov–Sklyanin operators

In the following, we use the notation

$$p_{-k} := p_k^\perp = \alpha \frac{k\partial}{\partial p_k},$$

for any $k \geq 1$. We also use the convention that $p_0 := 0$. Consider the (infinite) row vector

$$P = (P_{1,k})_{k\in\mathbb{N}_{\geq 1}}$$

and dually the column vector

$$P^\dagger = (P^\dagger_{k,1})_{k\in\mathbb{N}_{\geq 1}},$$

where $P_{1,k} := p_k$, and $P^\dagger_{k,1} := p_{-k}$, are regarded as operators on the algebra of symmetric functions $\mathcal{S}_\alpha$. Let $L = (L_{i,j})_{i,j\in\mathbb{N}_{\geq 1}}$ be the infinite matrix defined by

$$L_{i,j} := p_{j-i} + \delta_{i,j}\, ib,$$

for all $i, j \in \mathbb{N}_{\geq 1}$, where $b := \alpha - 1$ is the shifted Jack parameter. Namely, we have

$$L = \begin{bmatrix} b & P_{1,1} & P_{1,2} & \cdots \\ P^\dagger_{1,1} & 2b & P_{1,1} & \ddots \\ P^\dagger_{2,1} & P^\dagger_{1,1} & 3b & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

The result of Nazarov–Sklyanin from [NS13] can be reformulated as follows.

Theorem 4.2.6 ([NS13, Theorem 2])**.** *The following equality holds true for all $\ell \geq 0$ and all partitions λ:*

$$PL^\ell P^\dagger J^{(\alpha)}_\lambda = B^{(\alpha)}_{\ell+2}(\lambda) \cdot J^{(\alpha)}_\lambda, \tag{4.8}$$

where $B_{\ell+2}$ is the boolean cumulant defined in Eq. (4.6).

Nazarov and Sklyanin stated their theorem differently, as they did not realize the connection with the transition measure, the fact which is crucial for us. This connection was first noticed by Moll [Mol15], and we refer the reader to the proof of Theorem 4.2.6 presented in [CDM23, Theorem 3.9].

Example 4.2.7. When $\ell = 0$, we have

$$PP^\dagger = \sum_{i\geq 1} \alpha p_i \frac{i\partial}{\partial p_i},$$

is the degree operator, *i.e.* for any λ

$$PP^\dagger \cdot J^{(\alpha)}_\lambda = \alpha|\lambda| J^{(\alpha)}_\lambda.$$

For $\ell = 1$, the operator $PLP^\dagger$ is directly related to the Laplace–Beltrami operator D_α defined in Eq. (1.17). Indeed, with the notation of this section, D_α can be rewritten as follows,

$$\begin{aligned} D^{(\alpha)} &= \frac{1}{2\alpha}\left(\sum_{i,j\geq 1} p_{i+j}p_{-i}p_{-j} + \sum_{i,j\geq 1} p_i p_j p_{-(i+j)} + b \cdot \sum_{i\geq 1}(i-1)p_i p_{-i}\right) \\ &= \frac{1}{2\alpha}\left(\sum_{i,j\geq 1} P_{1,i+j}L_{i+j,j}P^\dagger_{j,1} + \sum_{i,j\geq 1} P_{1,i}L_{i,i+j}P^\dagger_{i+j,1} + \sum_{i\geq 1} P_{1,i}(L_{i,i} - b)P^\dagger_{i,1}\right) \\ &= \frac{1}{2\alpha}PLP^\dagger - \frac{b}{2\alpha}PP^\dagger. \end{aligned}$$

Hence, Theorem 4.2.6 can be thought of as a generalization of Proposition 1.2.9.

4.2.2 Integrality

To simplify notation, we will use in this section a slightly different normalization for Jack characters given by

$$\mathrm{Ch}_\mu^{(\alpha)}(\lambda) := z_\mu \theta_\mu^{(\alpha)}(\lambda). \tag{4.9}$$

The fact that Jack characters form a basis of $\mathcal{S}_\alpha^*$ (see Section 4.1.3) implies that for any $X = B, M, R$ and for any $\ell_1, \dots, \ell_k \geq 2$ we have

$$\alpha^{-k} X_{\ell_1}^{(\alpha)} \cdots X_{\ell_k}^{(\alpha)} = \sum_\mu x_\mu^{\ell_1,\dots,\ell_k}(\alpha)\, \mathrm{Ch}_\mu^{(\alpha)}\,. \tag{4.10}$$

for some coefficients $x_\mu^{\ell_1,\dots,\ell_k}(\alpha)$ in $\mathbb{Q}(\alpha)$. We prove that when $X = B, M$ these coefficients are polynomials in b and have non-negative integer coefficients and we provide their combinatorial interpretation in terms of the Łukasiewicz ribbon paths introduced in [CDM23].

Łukasiewicz ribbon paths

Informally speaking, in this chapter an *excursion* is a directed lattice path with steps of the form $(1, k)$ with $k \in \mathbb{Z}$, starting at $(0, 0)$, finishing at $(\ell, 0)$ that stays in the first quadrant and has no horizontal steps on the x-axis. More formally, an excursion Γ of length $\ell \geq 1$ is a sequence of points $w = (w_0, \dots, w_\ell)$ in $(\mathbb{N}_{\geq 0})^2$ such that $w_j = (j, y_j)$, with $y_0 = y_\ell = 0$, and if $w_i = (i, 0)$ for some i then $w_{i+1} \neq (i + 1, 0)$. It is uniquely encoded by the sequence of its *steps* $e_j := w_j - w_{j-1} = (1, y_{j+1} - y_j)$. For a step $e = (1, y)$ its degree $\deg(e)$ is equal to y. Steps of degree 0 are called *horizontal steps*.

For a given excursion $\Gamma = (w_0, \dots, w_\ell)$, the set of points $\mathbf{S}(\Gamma) := \{w_1, w_2, \dots, w_\ell\}$ (not counting the origin $w_0 = (0, 0)$) naturally decomposes as

$$\mathbf{S}(\Gamma) = \bigcup_{n \in \mathbb{Z}} \mathbf{S}_n(\Gamma),$$

where $\mathbf{S}_n(\Gamma)$ is a set of points preceded by a step $(1, n)$. We also denote by $\mathbf{S}^i(\Gamma) \subset \mathbf{S}(\Gamma)$ the subset of points with second coordinate equal to i, i.e.

$$\mathbf{S}^i(\Gamma) := \{w_j = (j, y_j) \colon y_j = i\}.$$

Additionally, we denote $\mathbf{S}_0(\Gamma)$ by $\mathbf{S}_\rightarrow(\Gamma)$ to remind that these points are preceded by horizontal steps, and we define $\mathbf{S}^i_\rightarrow(\Gamma) := \mathbf{S}_\rightarrow(\Gamma) \cap \mathbf{S}^i(\Gamma)$.

For an ordered tuple $\vec{\Gamma} = (\Gamma_1, \dots, \Gamma_k)$ of k excursions Γ_i, we will treat $\vec{\Gamma}$ itself as an excursion obtained by concatenating $\Gamma_1, \dots, \Gamma_k$, and we define $\mathbf{S}_n(\vec{\Gamma}), \mathbf{S}^i(\vec{\Gamma}), \mathbf{S}^i_\rightarrow(\vec{\Gamma})$, in the same way as before. We say that $p = (w_i, w_j)$ *is a pairing of degree* $n > 0$ if $w_i \in \mathbf{S}_{-n}(\vec{\Gamma}), w_j \in \mathbf{S}_n(\vec{\Gamma})$, and w_i appears before w_j in $\vec{\Gamma}$, i.e. $i < j$.

By definition, a *ribbon path on* k *sites of lengths* $\ell_1, \dots, \ell_k$ is a pair $\vec{\mathbf{\Gamma}} = (\vec{\Gamma}, \mathbf{P}(\vec{\mathbf{\Gamma}}))$ consisting of an ordered tuple $\vec{\Gamma}$ of k excursions $\Gamma_1, \dots, \Gamma_k$ of lengths $\ell_1, \dots, \ell_k$, respectively, and a set $\mathbf{P}(\vec{\mathbf{\Gamma}})$ of disjoint pairings $p_1, \dots, p_q$ on $\vec{\Gamma}$. This notion of ribbon paths was introduced by Moll in [Mol23]. We denote by $\mathbf{P}_n(\vec{\mathbf{\Gamma}}) \subset \mathbf{P}(\vec{\mathbf{\Gamma}})$ the subset of pairings of degree n, and define $\mathbf{S}_n(\vec{\mathbf{\Gamma}}) := \mathbf{S}_n(\vec{\Gamma}) \setminus \mathbf{P}_{|n|}(\vec{\mathbf{\Gamma}})$ as the set obtained by removing the points belonging to

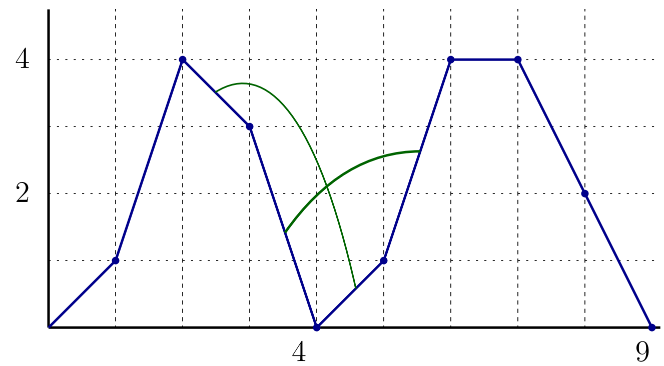


Figure 4.1: An example of a ribbon path $\vec{\mathbf{\Gamma}}$ of length 9. The green arcs illustrate the pairings of the path.

$\mathbf{P}_{|n|}(\vec{\mathbf{\Gamma}})$ from $\mathbf{S}_n(\vec{\Gamma})$. Also, we define the points preceded by horizontal steps of the ribbon graph $\vec{\mathbf{\Gamma}}$ as in the path $\vec{\Gamma}$, namely $\mathbf{S}^i_{\rightarrow}(\vec{\mathbf{\Gamma}}) := \mathbf{S}^i_{\rightarrow}(\vec{\Gamma})$. Then we have the decomposition[3]:

$$\mathbf{S}(\vec{\mathbf{\Gamma}}) = \bigcup_{i=0}^{\infty} \mathbf{S}^i_{\rightarrow}(\vec{\mathbf{\Gamma}}) \cup \bigcup_{n=1}^{\infty} \big(\mathbf{P}_n(\vec{\mathbf{\Gamma}}) \cup \mathbf{S}_{-n}(\vec{\mathbf{\Gamma}}) \cup \mathbf{S}_n(\vec{\mathbf{\Gamma}})\big). \tag{4.11}$$

We will denote $\mathbf{R}(\ell_1, \dots, \ell_k)$ the set of ribbon paths on k sites of lengths $\ell_1, \dots, \ell_k$.

Example 4.2.8. In Fig. 4.1, we give an example of a ribbon path $\vec{\mathbf{\Gamma}} = (\vec{\Gamma}, \mathbf{P}(\vec{\mathbf{\Gamma}}))$ of length 9, with

$$\mathbf{P}(\vec{\mathbf{\Gamma}}) = \{((3,3),(5,1)),((4,0),(6,4))\}.$$

The path $\vec{\mathbf{\Gamma}}$ can be seen as an element of $\mathbf{R}(9)$ or of $\mathbf{R}(4,5)$. Note that

$$\mathbf{S}^0(\vec{\mathbf{\Gamma}}) = \{(4,0),(9,0)\}, \quad \text{and} \quad \mathbf{S}_{\rightarrow}(\vec{\mathbf{\Gamma}}) = \mathbf{S}^4_{\rightarrow}(\vec{\mathbf{\Gamma}}) = \{(7,4)\}.$$

We also have

$$\mathbf{S}_3(\vec{\Gamma}) = \{(2,4),(6,4)\}, \quad \text{while} \quad \mathbf{S}_3(\vec{\mathbf{\Gamma}}) = \{(2,4)\},$$

since $(6,4)$ is paired.

The following theorem is a direct combinatorial interpretation of the operator $PL^{\ell_1-2}P^\dagger \cdots PL^{\ell_k-2}P^\dagger$ in terms of ribbon paths, and we leave its proof as a simple exercise (the full proof of its variant can be found in [Mol23]).

Theorem 4.2.9. *The following identity holds true:*

$$PL^{\ell_1-2}P^\dagger \cdots PL^{\ell_k-2}P^\dagger = \sum_{\substack{\vec{\mathbf{\Gamma}} \in \mathbf{R}(\ell_1,\dots,\ell_k),\\ |\mathbf{S}^0(\vec{\mathbf{\Gamma}})|=k}} \prod_{n=1}^{\infty} (\alpha \cdot n)^{|\mathbf{P}_n(\vec{\mathbf{\Gamma}})|} \prod_{i=1}^{\infty} (i \cdot b)^{|\mathbf{S}^i_{\rightarrow}(\vec{\mathbf{\Gamma}})|} \cdot \prod_{j=1}^{\infty} p_j^{|\mathbf{S}_j(\vec{\mathbf{\Gamma}})|} \cdot \prod_{m=1}^{\infty} p_{-m}^{|\mathbf{S}_{-m}(\vec{\mathbf{\Gamma}})|}.$$

[3] In this equation, we abused the notation: the set of pairings $\mathbf{P}_n(\vec{\mathbf{\Gamma}})$ contains pairs of distinct points (w_i, w_j), but we implicitly treated such pairs as the 2-element sets $\{w_i, w_j\}$, for simplicity of notation.

Let us briefly explain how this theorem is obtained from Theorem 4.2.6. First, one can rewrite the operator $PL^{\ell-2}P^{\dagger}$ as a sum over paths Γ of length ℓ and such that $|\mathbf{S}^0| = 1$. For a step of size k of such path is associated the operator p_k if $k \neq 0$, and the operator ib if $k = 0$ and the step is at height 0. The operator associated to Γ is then obtained by taking from left to right the product over all these step operators. The formula of Theorem 4.2.9 is then obtained by taking the normal ordering; we move the multiplicative operators ($(p_k)_{k>0}$) to the left, and the derivative operators ($(p_k)_{k<0}$) to the right. In such commutation, an operator p_k can be cancelled by p_{-k} and this corresponds to considering paths decorated with pairings.

We now introduce a particular family of ribbon paths. We call a *Łukasiewicz path* an excursion Γ which has only up steps of degree 1. Similarly, a ribbon path $\vec{\mathbf{\Gamma}}$ whose non-paired up steps are only of degree 1 is called a *Łukasiewicz ribbon path*, i.e. $\mathbf{S}_n(\vec{\mathbf{\Gamma}}) = \emptyset$, $\forall\, n \geq 2$, if $\vec{\mathbf{\Gamma}}$ is a Łukasiewicz ribbon path. We will denote $\mathbf{L}(\ell_1, \ldots, \ell_k)$ the set of Łukasiewicz ribbon paths on k sites of lengths $\ell_1, \ldots, \ell_k$.

If $\vec{\mathbf{\Gamma}}$ is a Łukasiewicz ribbon path, then we associate to it a partition $\mu(\vec{\mathbf{\Gamma}})$ obtained by reordering the degrees of non-paired down steps; in other terms

$$m_i(\mu(\vec{\mathbf{\Gamma}})) = |\mathbf{S}_{-i}(\vec{\mathbf{\Gamma}})|,$$

$m_i(\mu(\vec{\mathbf{\Gamma}}))$ being the number of parts of size i in $\mu(\vec{\mathbf{\Gamma}})$.

Łukasiewicz paths are classical objects in combinatorics[4]. We show here that Łukasiewicz ribbon paths naturally arise in studying the coefficients $x_{\mu}^{\ell_1,\ldots,\ell_k}$ of Eq. (4.10).

Theorem 4.2.10. *The following identities hold true:*

$$B_{\ell_1}^{(\alpha)} \cdots B_{\ell_k}^{(\alpha)} = \sum_{\substack{\vec{\mathbf{\Gamma}} \in \mathbf{L}(\ell_1,\ldots,\ell_k),\\ |\mathbf{S}^0(\vec{\mathbf{\Gamma}})| = k}} \prod_{n=1}^{\infty} (\alpha n)^{|\mathbf{P}_n(\vec{\mathbf{\Gamma}})|} \prod_{i=1}^{\infty} (i \cdot b)^{|\mathbf{S}^i_{\rightarrow}(\vec{\mathbf{\Gamma}})|} \cdot \alpha^{\ell(\mu(\vec{\mathbf{\Gamma}}))} \operatorname{Ch}^{(\alpha)}_{\mu(\vec{\mathbf{\Gamma}})}, \tag{4.12}$$

$$M_{\ell_1}^{(\alpha)} \cdots M_{\ell_k}^{(\alpha)} = \sum_{\vec{\mathbf{\Gamma}} \in \mathbf{L}(\ell_1,\ldots,\ell_k)} \prod_{n=1}^{\infty} (\alpha n)^{|\mathbf{P}_n(\vec{\mathbf{\Gamma}})|} \prod_{i=1}^{\infty} (i \cdot b)^{|\mathbf{S}^i_{\rightarrow}(\vec{\mathbf{\Gamma}})|} \cdot \alpha^{\ell(\mu(\vec{\mathbf{\Gamma}}))} \operatorname{Ch}^{(\alpha)}_{\mu(\vec{\mathbf{\Gamma}})}. \tag{4.13}$$

The following lemma will be useful for the proof of Theorem 4.2.10.

Lemma 4.2.11. *Let λ and μ be two partitions. Then,*

$$\left(\prod_{j \geq 1} \left(\frac{j\partial}{\partial p_j} \right)^{m_j(\mu)} \cdot J_{\lambda}^{(\alpha)} \right) \Bigg|_{p_i = \delta_{i,1}} = \operatorname{Ch}^{(\alpha)}_{\mu}(\lambda).$$

Proof. Let $k := |\lambda| - |\mu|$. If $k < 0$, then both sides of the equation are zero. We assume now that $k \geq 0$. Using the definition of the scalar product (see Eq. (1.9)) and Lemma 1.2.4, we

[4] See [FS09, Section I.5] for an explanation of their name and more background.

have

$$\begin{aligned}
\left(\prod_{j\geq 1}\left(\frac{j\partial}{\partial p_j}\right)^{m_j(\mu)}\cdot J_\lambda^{(\alpha)}\right)\Bigg|_{p_i=\delta_{i,1}} &= \frac{1}{\alpha^k k!}\left\langle \prod_{j\geq 1}\left(\frac{j\partial}{\partial p_j}\right)^{m_j(\mu)}\cdot J_\lambda^{(\alpha)}, p_{1^k}\right\rangle\\
&= \frac{1}{k!}\left\langle \left(\frac{\partial}{\partial p_1}\right)^k \prod_{j\geq 1}\left(\frac{j\partial}{\partial p_j}\right)^{m_j(\mu)}\cdot J_\lambda^{(\alpha)}, 1\right\rangle\\
&= \frac{1}{k!}\left\langle \left(\frac{\partial}{\partial p_1}\right)^k \cdot J_\lambda^{(\alpha)}, \frac{1}{\alpha^{\ell(\mu)}}p_\mu\right\rangle\\
&= z_\mu[p_\mu]\frac{1}{k!}\left(\frac{\partial}{\partial p_1}\right)^k\cdot J_\lambda^{(\alpha)}\\
&= z_\mu\theta_\mu^{(\alpha)}(\lambda). \qquad \square
\end{aligned}$$

To obtain the last line, we use the definition of Jack characters; see Eq. (1.40).

Proof of Theorem 4.2.10. We start by explaining that equation (4.13) is a direct consequence of (4.12) and relation (4.6). Indeed, take a Łukasiewicz ribbon path $\vec{\mathbf{\Gamma}} \in \mathbf{L}(\ell_1,\dots,\ell_k)$ and consider its points touching the x-axis $\mathbf{S}^0(\vec{\mathbf{\Gamma}})$. They satisfy $\mathbf{S}^0(\vec{\mathbf{\Gamma}}) \supseteq L$, where

$$L := \{\ell_1, \ell_1+\ell_2, \dots, \ell_1+\dots+\ell_k\}.$$

In other terms,

$$\mathbf{S}^0(\vec{\mathbf{\Gamma}}) = \{\ell_1^1,\dots,\ell_1^1+\dots+\ell_1^{n_1},\dots,\ell_1^1+\dots+\ell_{k-1}^{n_{k-1}}+\ell_k^1,\dots,\ell_1^1+\dots+\ell_k^{n_k}\},$$

for some $\ell_i^j \geq 2$ satisfying $\sum_{j=1}^{n_i}\ell_i^j = \ell_i$ for each $i = 1,\dots,k$. In particular, we can consider $\vec{\mathbf{\Gamma}}$ as an element of $\mathbf{L}(\ell_1^1,\dots,\ell_1^{n_1},\dots,\ell_k^1,\dots,\ell_k^{n_k})$. Using this decomposition we can rewrite the RHS of (4.13) as

$$\sum_{n_1,\dots,n_k\geq 1}\ \sum_{\substack{\ell_1^1,\dots,\ell_1^{n_1}\geq 2,\\ \ell_1^1+\dots+\ell_1^{n_1}=\ell_1}}\cdots\sum_{\substack{\ell_k^1,\dots,\ell_k^{n_k}\geq 2,\\ \ell_k^1+\dots+\ell_k^{n_k}=\ell_k}}\ \sum_{\substack{\vec{\mathbf{\Gamma}}\in\mathbf{L}(\ell_1^1,\dots,\ell_1^{n_1},\dots,\ell_k^1,\dots,\ell_k^{n_k}),\\ |\mathbf{S}^0(\vec{\mathbf{\Gamma}})|=n_1+\dots+n_k}}
\prod_{n=1}^{\infty}(\alpha n)^{|\mathbf{P}_n(\vec{\mathbf{\Gamma}})|}\prod_{i=1}^{\infty}(i\cdot b)^{|\mathbf{S}^i_{\to}(\vec{\mathbf{\Gamma}})|}\cdot\alpha^{\ell(\mu(\vec{\mathbf{\Gamma}}))}\,\mathrm{Ch}^{(\alpha)}_{\mu(\vec{\mathbf{\Gamma}})},$$

which, by (4.12), is equal to

$$\sum_{n_1,\dots,n_k\geq 1}\ \sum_{\substack{\ell_1^1,\dots,\ell_1^{n_1}\geq 2,\\ \ell_1^1+\dots+\ell_1^{n_1}=\ell_1}}\cdots\sum_{\substack{\ell_k^1,\dots,\ell_k^{n_k}\geq 2,\\ \ell_k^1+\dots+\ell_k^{n_k}=\ell_k}} B^{(\alpha)}_{\ell_1^1}\cdots B^{(\alpha)}_{\ell_1^{n_1}}\cdots B^{(\alpha)}_{\ell_k^1}\cdots B^{(\alpha)}_{\ell_k^{n_k}}.$$

Relation (4.6) finishes the proof of (4.13).

We now prove (4.12). Using the fact that $J_\lambda^{(\alpha)}\big|_{p_i=\delta_{i,1}} = 1$ (see Eq. (1.10)), we can use Theorem 4.2.6 to write

$$B^{(\alpha)}_{\ell_1}\cdots B^{(\alpha)}_{\ell_k} = \left(PL^{\ell_1-2}P^\dagger\cdots PL^{\ell_k-2}P^\dagger J_\lambda^{(\alpha)}\right)\Big|_{p_i=\delta_{i,1}}.$$

Theorem 4.2.9 allows us to further rewrite

$$B_{\ell_1}^{(\alpha)} \cdots B_{\ell_k}^{(\alpha)} = \sum_{\substack{\vec{\mathbf{\Gamma}} \in \mathbf{R}(\ell_1,\dots,\ell_k),\\ |\mathbf{S}^0(\vec{\mathbf{\Gamma}})|=k}} \prod_{n=1}^{\infty} (\alpha \cdot n)^{|\mathbf{P}_n(\vec{\mathbf{\Gamma}})|} \prod_{i=1}^{\infty} (i \cdot b)^{|\mathbf{S}^i_{\to}(\vec{\mathbf{\Gamma}})|} \cdot \left(\prod_{j=1}^{\infty} p_j^{|\mathbf{S}_j(\vec{\mathbf{\Gamma}})|} \cdot \prod_{m=1}^{\infty} p_{-m}^{|\mathbf{S}_{-m}(\vec{\mathbf{\Gamma}})|} J_\lambda^{(\alpha)} \right)\Bigg|_{p_i=\delta_{i,1}}. \tag{4.14}$$

Fix a ribbon path $\vec{\mathbf{\Gamma}} \in \mathbf{R}(\ell_1, \dots, \ell_k)$. Note that

$$\left(\prod_{j=1}^{\infty} p_j^{|\mathbf{S}_j(\vec{\mathbf{\Gamma}})|} \cdot \prod_{m=1}^{\infty} p_{-m}^{|\mathbf{S}_{-m}(\vec{\mathbf{\Gamma}})|} J_\lambda^{(\alpha)} \right)\Bigg|_{p_i=\delta_{i,1}} = 0 \tag{4.15}$$

whenever there exists $j > 1$ such that $\mathbf{S}_j(\vec{\mathbf{\Gamma}}) \neq \emptyset$. In other terms, if $\vec{\mathbf{\Gamma}}$ is not a Łukasiewicz ribbon path, then its contribution into (4.14) is zero, and the sum in (4.14) actually runs over paths in $\mathbf{L}(\ell_1, \dots, \ell_k)$. For a Łukasiewicz ribbon path, (4.15) simplifies to:

$$\left(p_1^{|\mathbf{S}_1(\vec{\mathbf{\Gamma}})|} \cdot \prod_{m=1}^{\infty} p_{-m}^{|\mathbf{S}_{-m}(\vec{\mathbf{\Gamma}})|} J_\lambda^{(\alpha)} \right)\Bigg|_{p_i=\delta_{i,1}} = \alpha^{\ell(\mu(\vec{\mathbf{\Gamma}}))} \left(p_1^{|\mathbf{S}_1(\vec{\mathbf{\Gamma}})|} \cdot \prod_{j=1}^{\infty} \left(\frac{j\partial}{\partial p_j} \right)^{m_j(\mu(\vec{\mathbf{\Gamma}}))} J_\lambda^{(\alpha)} \right)\Bigg|_{p_i=\delta_{i,1}}.$$

Using Lemma 4.2.11, we get

$$\left(p_1^{|\mathbf{S}_1(\vec{\mathbf{\Gamma}})|} \cdot \prod_{m=1}^{\infty} p_{-m}^{|\mathbf{S}_{-m}(\vec{\mathbf{\Gamma}})|} J_\lambda^{(\alpha)} \right)\Bigg|_{p_i=\delta_{i,1}} = \alpha^{\ell(\mu(\vec{\mathbf{\Gamma}}))} \operatorname{Ch}^{(\alpha)}_{\mu(\vec{\mathbf{\Gamma}})}. \tag{4.16}$$

Plugging this into (4.14) yields precisely the desired identity (4.12), which finishes the proof. □

Consequences

Before we conclude the proof of polynomiality and integrality in Lassalle's conjecture, let us point several applications of Theorem 4.2.10.

In the special case $\alpha = 1$, a problem of positivity between Boolean cumulants and normalized characters of the symmetric group was posed by Rattan and Śniady in [RŚ08], and has been proven very recently by Koshida [Kos23] by use of Khovanov's Heisenberg category. Koshida, however, was not able to find an explicit interpretation of the positivity so that he did not provide the $\alpha = 1$ case of our formula Eq. (4.12) (his proof was a complicated induction) and left it as an open problem. Theorem 4.2.10 solves this problem and provides an explicit combinatorial interpretation for general α; in particular in the special case $\alpha = 1$ it gives an alternative proof to the work of Koshida.

Furthermore, it implies the following theorem.

Theorem 4.2.12. *The following $\mathbb{Z}[\alpha]$-algebras are all equal:*

$$\mathbb{Z}[\alpha][\operatorname{Ch}^{(\alpha)}_\mu \colon \mu \in \mathbb{Y}] = \mathbb{Z}[\alpha, M_2^{(\alpha)}/\alpha, M_3^{(\alpha)}/\alpha, \dots] \\ = \mathbb{Z}[\alpha, R_2^{(\alpha)}/\alpha, R_3^{(\alpha)}/\alpha, \dots] = \mathbb{Z}[\alpha, B_2^{(\alpha)}/\alpha, B_3^{(\alpha)}/\alpha, \dots].$$

This theorem gives the biggest progress so far towards another conjecture of Lassalle from [Las09] that postulates that *cumulants* of Jack characters $\mathrm{Ch}_\mu^{(\alpha)}$ are a polynomials in $b, R_2^{(\alpha)}, R_3^{(\alpha)}, \dots$ with positive integer coefficients (see Conjecture 12 for a precise statement). Rationality of the coefficients of this polynomial (called *Kerov polynomial* for Jack characters) was proven in [DF16], and the top-degree part of the Kerov polynomial was found by Śniady in [Śni19]; these properties have found applications for studying random Young diagrams [DF16, DŚ19, CDM23].

Proof of Theorem 4.2.12. The last two equalities are well-known to the experts and follow from Definition 4.2.3 and the fact that the equations (4.6) and (4.7) are invertible over $\mathbb{Z}$.

In order to prove the first equality, we start by showing the following equation: for any partition λ we have

$$M^{(\alpha)}_{\lambda_1+1}\cdots M^{(\alpha)}_{\lambda_{\ell(\lambda)}+1} = \alpha^{\ell(\lambda)}\,\mathrm{Ch}^{(\alpha)}_\lambda + \sum_{|\rho|<|\lambda|} a^\lambda_\rho(\alpha)\alpha^{\ell(\rho)}\,\mathrm{Ch}^{(\alpha)}_\rho, \tag{4.17}$$

where $a^\mu_\rho(\alpha) \in \mathbb{Z}[\alpha]$. First, (4.13) implies that the contribution of $\alpha^{\ell(\rho)}\,\mathrm{Ch}^{(\alpha)}_\rho$ comes from $\vec{\mathbf{\Gamma}} \in \mathbf{L}(\lambda_1+1,\dots,\lambda_{\ell(\lambda)}+1)$ with $\mu(\vec{\mathbf{\Gamma}}) = \rho$. But from the decomposition of Eq. (4.11), we have

$$\begin{aligned}|\lambda| + \ell(\lambda) &= |\mathbf{S}_\rightarrow(\vec{\mathbf{\Gamma}})| + 2|\mathbf{P}(\vec{\mathbf{\Gamma}})| + |\mathbf{S}_1| + \sum_{n\geq 1}|\mathbf{S}_{-n}|\\ &= |\mathbf{S}_\rightarrow(\vec{\mathbf{\Gamma}})| + 2|\mathbf{P}(\vec{\mathbf{\Gamma}})| + |\mu(\vec{\mathbf{\Gamma}})| + \ell(\mu(\vec{\mathbf{\Gamma}}))\end{aligned}$$

This implies that

$$|\mu(\vec{\mathbf{\Gamma}})| = |\lambda| + \ell(\lambda) - \ell(\mu(\vec{\mathbf{\Gamma}})) - |\mathbf{S}_\rightarrow(\vec{\mathbf{\Gamma}})| - 2|\mathbf{P}(\vec{\mathbf{\Gamma}})|. \tag{4.18}$$

Moreover,

$$\ell(\mu(\vec{\mathbf{\Gamma}})) + |\mathbf{P}(\vec{\mathbf{\Gamma}})| \geq \ell(\lambda). \tag{4.19}$$

Indeed, each one of the ℓ excursions of $\vec{\mathbf{\Gamma}}$ has at least one down step, paired or not. Hence, Eq. (4.18) gives

$$|\mu(\vec{\mathbf{\Gamma}})| \leq |\lambda|$$

with equality if and only if $|\mathbf{S}_\rightarrow(\vec{\mathbf{\Gamma}})| = |\mathbf{P}(\vec{\mathbf{\Gamma}})| = 0$ and $\ell(\mu(\vec{\mathbf{\Gamma}})) = \ell(\lambda)$. There is a unique $\vec{\mathbf{\Gamma}}$ in $\mathbf{L}(\mu_1+1,\dots,\mu_{\ell(\mu)}+1)$ that satisfies these conditions: $\mathbf{P}(\vec{\mathbf{\Gamma}}) = \emptyset$ and $\vec{\Gamma} = (\Gamma_1,\dots,\Gamma_{\ell(\mu)})$, where Γ_i is given by λ_i up steps followed by one down-step of degree λ_i. For this ribbon path, $\mu(\vec{\mathbf{\Gamma}}) = \lambda$. This finishes the proof of Eq. (4.17).

This proves that the expansion of $1/\alpha^{\ell(\lambda)} M^{(\alpha)}_{\lambda_1+1}\cdots M^{(\alpha)}_{\lambda_{\ell(\lambda)}+1}$ in the basis of $\mathrm{Ch}^{(\alpha)}_\rho$ is unitriangular. The coefficients of this expansion are in fact in $\mathbb{Z}[\alpha]$. Indeed, from Eq. (4.13) and Eq. (4.19) one has that for any ρ the normalized coefficient $\alpha^{\ell(\rho)-\ell(\lambda)}a^\lambda_\rho(\alpha)$ is polynomial in α with integer coefficients. This finishes the proof of the theorem. □

We then have the following corollary.

Corollary 4.2.13. *The normalized Jack characters expressed in the Stanley coordinates* $(-1)^{|\mu|}z_\mu\tilde{\theta}^{(\alpha)}_\mu(\boldsymbol{s},\boldsymbol{r})$ *are polynomials in the variables* $b, -s_1, -s_2, \dots, r_1, r_2, \dots$ *with integer coefficients, where* $b := \alpha - 1$.

Proof. From the definition (4.4) of the moments and of the Stanley coordinates one has

$$M_\ell^{(\alpha)}(\boldsymbol{s}^{\boldsymbol{r}}) = [z^{-\ell-1}]\frac{1}{z+r_1+\cdots+r_k}\prod_{i=1}^{k}\frac{z-\big(\alpha\cdot s_i-(r_1+\cdots+r_i)\big)}{z-\big(\alpha\cdot s_i-(r_1+\cdots+r_{i-1})\big)},$$

where we extract the coefficient of $z^{-\ell-1}$ in the above rational function treated as a formal power series in z^{-1}. In particular it is clear that $M_\ell^{(\alpha)}$ is a polynomial in $\alpha, -s_1, -s_2, \ldots, r_1, r_2, \ldots$ with integer coefficients. This is also the case of $M_\ell^{(\alpha)}/\alpha$ since evaluating at $\alpha = 0$ gives $M_\ell^{(0)} = 0$. Then the result follows from the definition of the normalized Jack characters (4.9) and Theorem 4.2.12. □

4.3 The combinatorial model and differential equations

The purpose of this section is to define a family of statistics of non-orientability (see Definition 1.3.8) on layered maps. This construction is a variant of the one given in Section 2.1 which is also extended to layered maps. However, unlike the statistics of Section 2.1, the statistics used here are not "strong".

As in Section 2.1, we start by giving the definition of a measure of non-orientability due to La Croix. For this, we use the classification of edges given in Definition 2.1.1.

Definition 4.3.1 ([La 09, Definition 4.1])**.** *We call measure of non-orientability* (MON) *a function ρ defined on the set of vertex-labelled connected maps (M, e) with a distinguished edge, with values in $\{1, b\}$, satisfying the following conditions:*

- *$\rho(M, e) = b$ if e is a twist.*
- *if e is a handle, then ρ satisfies the condition*

  $$\left\{\boldsymbol{\rho}(M, e), \boldsymbol{\rho}(\widetilde{M}, \tilde{e})\right\} = \{1, b\}.$$

  Moreover, if M is orientable then $\boldsymbol{\rho}(M, e) = 1$.
- *$\boldsymbol{\rho}(M, e) = 1$ otherwise.*

More generally, if M is a vertex-labelled map (not necessarily connected), and e is an edge of M, then we set

$$\boldsymbol{\rho}(M, e) := \boldsymbol{\rho}(M_e, e),$$

where M_e is the connected component of M containing e.

Let M be a vertex-labelled map (connected or not) and let $e_1, e_2, \ldots, e_d$ be d distinct edges of M. For $1 \le i \le d$, we denote M_j the map obtained by deleting the edges e_1, e_2,..., e_{j-1} from M. We define $\rho(M, e_1, e_2, \ldots, e_d)$ as the weight obtained by deleting the edges e_j successively:

$$\rho(M, e_1, e_2, \ldots, e_d) := \prod_{1\le j\le d}\rho(M_j, e_j).$$

One may notice that unlike a SMON defined in Definition 2.1.3, in the definition of a MON we no longer take into account the orientation of the black vertex roots. This explains why we do not distinguish white leaf and black leaf edges, and we always assign the weight 1 to a bridge (an edge connecting two different connected components).

Remark 4.3.2. The fact that we work in this section with MONs rather than SMONs is a technical detail but will be useful to obtain the right normalization to prove positivity Lassalle's conjecture; see Remark 4.3.7.

From now on, all layered maps will be vertex-labelled as defined in Definition 1.3.16, unless stated otherwise.

We now define an order on black vertices of a layered map, which generalizes the one given in Section 2.1.

We fix an integer $k \geq 1$ and a k-layered map M. To each black vertex v of M we associate the triplet of integers $(-i, n, j)$, where i is the layer containing the vertex, n is its degree and j is the number given to v by the labelling of the map. We define then a linear order $\prec_M$ on the black vertices of M given by the lexicographic order on $(-i, n, j)$. In particular, the maximal black vertex with respect to $\prec_M$ is the vertex contained in the layer of the smallest index, and having maximal degree and maximal label. Note that when we delete the maximal vertex from a k-layered map and all the edges incident to it, the map obtained is also k-layered[5].

This order on vertices induces an order $\prec_M$ on edges as explained in Section 2.1. Note that when $k = 1$, layered maps correspond to (simple) maps and the order given here coincides with the one given Section 2.1.

We then obtain statistics of non-orientability from MONs (this is identical to Definition 1.3.12).

Definition 4.3.3 (Statistic of non-orientability on k-layered maps)**.** *Let ρ be a* MON *and let M be a map with n edges. We define the non-orientability weight $\rho(M)$ of M by*

$$\rho(M) := \rho(M, e_n, e_{n-1}, \dots, e_1), \tag{4.20}$$

where e_i denote the edges of M ordered such that $e_1 \prec_M e_2 \prec_M \cdots \prec_M e_n$.

We then define the statistic of non-orientabilty ϑ_ρ on layered maps with non-negative integer values, given for every M by $\rho(M) = b^{\vartheta_\rho(M)}$.

In this section, we fix an integer $k \geq 1$, and k variables $s_1, s_2, \dots, s_k$ and we fix a MON ρ. If M is a k-layered map, then we define its *global weight* $\kappa(M)$ by

$$\kappa(M) := \frac{\rho(M)}{2^{|\mathcal{V}_\bullet(M)| - \mathrm{cc}(M)} \alpha^{\mathrm{cc}(M)}} p_{\lambda^\diamond(M)} \prod_{1 \leq i \leq k} (-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|} \in \mathcal{P}[s_1, \dots, s_k],$$

where $\lambda^\diamond(M)$ is the face-type of M defined in Section 1.1.3.

Note that in addition to the non-orientability weight $\rho(M)$ and the "face-weight" $p_{\lambda^\diamond(M)}$ already used in Section 2.1, the global weight also contains a weight related to the layer structure of the map $\prod_{1 \leq i \leq k} (-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|}$.

[5]In this operation, we also delete white vertices which become isolated, *i.e.* the white vertices that were only incident to the maximal black vertex.

The following proposition is a "multi-layered" version of Proposition 2.2.3 which gives an interpretation of the operators $\mathcal{B}_n$ (see Eq. (1.48) for the definition). The proof is very similar, we refer to [BD23] for a complete proof.

Proposition 4.3.4. *Let M be a k-layered bipartite map and let $n \geq \left(\nu_\bullet^{(1)}(M)\right)_1$, where $\left(\nu_\bullet^{(1)}(M)\right)_1$ is the largest degree of a black vertex in layer 1. Then*

$$\mathcal{B}_n(\boldsymbol{p}, -\alpha s_1) \cdot \kappa(M) = \sum_{M'} \kappa(M'),$$

where the sum is taken over all k-layered maps obtained by adding a black vertex of degree n and of maximal label to the map M in layer 1 (using possibly new white vertices which are necessarily in the layer 1).

Note that the condition $n \geq \left(\nu_\bullet^{(1)}(M)\right)_1$ ensures that the added vertex is maximal, so that it is the first vertex to delete in the decomposition algorithm used to define the weight ρ; see Eq. (4.20).

We define the generating series of k-layered maps by:

$$\begin{aligned} F_k^{(\alpha)}(t, \boldsymbol{p}, s_1, \ldots, s_k) &:= \sum_M (-t)^{|M|} p_{\lambda^\diamond(M)} \frac{\rho(M)}{2^{|\mathcal{V}_\bullet(M)|-\mathrm{cc}(M)} \alpha^{\mathrm{cc}(M)}} \prod_{1 \leq i \leq k} \frac{(-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}} \\ &= \sum_M (-t)^{|M|} \kappa(M) \prod_{1 \leq i \leq k} \frac{1}{z_{\nu_\bullet^{(i)}(M)}}, \end{aligned} \tag{4.21}$$

where the sum runs over k-layered maps. We show that the operators $\mathcal{B}_n$ can be used to build the generating series $F_k^{(\alpha)}$ of k-layered maps. This can be seen as a multilayered extension of the differential construction given in Theorem 2.3.2 for the generating series of (1-layered) maps. The proof is very similar.

Proposition 4.3.5. *The functions $F_k^{(\alpha)}$ satisfy the following induction: $F_0^{(\alpha)} = 1$ and for every $k \geq 1$*

$$F_k^{(\alpha)}(t, \boldsymbol{p}, s_1, \ldots, s_k) = \exp\left(\mathcal{B}_\infty(-t, \boldsymbol{p}, -\alpha s_1)\right) \cdot F_{k-1}^{(\alpha)}(t, \boldsymbol{p}, s_2, \ldots, s_k). \tag{4.22}$$

We recall that

$$\mathcal{B}_\infty(t, \boldsymbol{p}, u) = \sum_{n \geq 1} \frac{t^n}{n} \mathcal{B}_n(\boldsymbol{p}, u). \tag{4.23}$$

This operator maps $\mathcal{P}$ into $\mathcal{P}[u][\![t]\!]_+$, where $\mathcal{P}[\![t]\!]_+$ is the ideal in $\mathcal{P}[\![t]\!]$ generated by t.

Proof of Proposition 4.3.5. For $k = 0$, we know that $F_0^{(\alpha)} = 1$ since the only map with 0 layers is the empty map. Fix now $k \geq 1$. From the definitions

$$F_{k-1}^{(\alpha)}(t, \boldsymbol{p}, s_2 \ldots, s_k) = \sum_M (-t)^{|M|} p_{\lambda^\diamond(M)} \frac{\rho(M)}{2^{|\mathcal{V}_\bullet(M)|-\mathrm{cc}(M)} \alpha^{\mathrm{cc}(M)}} \prod_{1 \leq i \leq k-1} \frac{(-\alpha s_{i+1})^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}},$$

the sum being taken over $k-1$ layered maps. We can rewrite this sum over k-layered maps with empty layer 1, by reindexing the layer j by $j+1$ for $1 \leq j \leq k-1$. Hence,

$$F_{k-1}^{(\alpha)}(t, \boldsymbol{p}, s_2 \dots, s_k) = \sum_M (-t)^{|M|} \kappa(M) \prod_{1 \leq i \leq k-1} \frac{1}{z_{\nu_\bullet^{(i+1)}(M)}},$$

where the sum is taken over all vertex-labelled k-layered maps with empty layer 1. Fix such a map M. To obtain a vertex-labelled k-layered map M' from M, we should add the layer 1 (possibly empty). We proceed as follows.

- We start by fixing a non-negative integer d and a partition μ of length d (this partition is empty if $d = 0$).
- We add successively for each $i = d, \dots, 1$ a black vertex v_i of degree μ_i, using possibly new white vertices, such that all the added vertices are in layer 1 of the map. In particular, the vertices $(v_i)_{1 \leq i \leq d}$ are added in an increasing order of their degrees.
- The edges $e_1, \dots, e_{\mu_i}$ incident to a vertex v_i are added successively in a cyclic order around the vertex. The vertex root[6] is chosen so that if we travel around v_i starting from the root corner we see the edges in the following order $e_{\mu_i}, \dots, e_1$.

Note that one has by definition $\nu_\bullet^{(1)}(M') = \mu$ and $\nu_\bullet^{(j)}(M') = \nu_\bullet^{(j)}(M)$ for $2 \leq j \leq k$. Proposition 4.3.4 implies that the generating series of k-layered maps M' which are obtained from M as described above can be expressed as follows

$$\sum_{M'} (-t)^{|M'|} \kappa(M') \prod_{1 \leq i \leq k} \frac{1}{z_{\nu_\bullet^{(i)}(M')}} = \sum_{\mu \in \mathbb{Y}} \left(\prod_{j \geq 1} \frac{1}{m_j(\mu)!} \right)$$
$$\frac{(-t)^{\mu_1} \mathcal{B}_{\mu_1}(\boldsymbol{p}, -\alpha s_1)}{\mu_1} \cdots \frac{(-t)^{\mu_{\ell(\mu)}} \mathcal{B}_{\mu_{\ell(\mu)}}(\boldsymbol{p}, -\alpha s_1)}{\mu_{\ell(\mu)}} \cdot (-t)^{|M|} \kappa(M) \prod_{1 \leq i \leq k-1} \frac{1}{z_{\nu_\bullet^{(i+1)}(M)}}.$$

Since the operators $\mathcal{B}_{\mu_i}$ commute (see Proposition 2.3.4), and since there are $\ell(\mu)! \prod_{j \geq 1} \frac{1}{m_j(\mu)!}$ reorderings γ of μ, the RHS of the last equation can be rewritten as follows

$$\sum_{\ell \geq 1} \sum_{n_1, \dots, n_\ell \geq 1} \frac{1}{\ell!} \frac{(-t)^{n_\ell} \mathcal{B}_{n_\ell}(\boldsymbol{p}, -\alpha s_1)}{n_\ell} \cdots \frac{(-t)^{n_1} \mathcal{B}_{n_1}(\boldsymbol{p}, -\alpha s_1)}{n_1} (-t)^{|M|} \cdot \kappa(M) \prod_{1 \leq i \leq k-1} \frac{1}{z_{\nu_\bullet^{(i+1)}(M)}}.$$

Hence

$$\sum_{M'} (-t)^{|M'|} \kappa(M') \prod_{1 \leq i \leq k} \frac{1}{z_{\nu_\bullet^{(i)}(M')}} = \exp \left(\sum_{n \geq 1} (-t)^n \frac{\mathcal{B}_n(\boldsymbol{p}, -\alpha s_1)}{n} \right) \cdot \kappa(M) \prod_{1 \leq i \leq k-1} \frac{1}{z_{\nu_\bullet^{(i+1)}(M)}},$$

and we deduce that

$$F_k^{(\alpha)}(t, \boldsymbol{p}, s_1, \dots, s_k) = \exp \left(\sum_{n \geq 1} (-t)^n \frac{\mathcal{B}_n(\boldsymbol{p}, -\alpha s_1)}{n} \right) \cdot F_{k-1}^{(\alpha)}(t, \boldsymbol{p}, s_2, \dots, s_k). \qquad \square$$

[6] We recall that we are working with vertex-labelled maps, so each black vertex has a vertex root.

Remark 4.3.6. Note that unlike the formula given in Section 2.3 for $\tau^{(\alpha)}(\boldsymbol{p},\boldsymbol{q},u)$, we do not control here the degrees of black vertices. Combinatorially, it is always possible to add in the last formula a weight $q_r^{(i)}$ for a black vertex of degree r in the layer i; this corresponds to series satisfying the recursion formula

$$\mathbf{F}_k^{(\alpha)}\left(t,\boldsymbol{p},\boldsymbol{q}^{(1)},\dots,\boldsymbol{q}^{(k)},s_1,\dots,s_k\right) = \exp\left(\sum_{n\geq 1}\frac{q_n^{(1)}\mathcal{B}_n(-t,\boldsymbol{p},-\alpha s_1)}{n}\right)\cdot\mathbf{F}_{k-1}^{(\alpha)}\left(t,\boldsymbol{p},\boldsymbol{q}^{(2)},\dots,\boldsymbol{q}^{(k)},s_2,\dots,s_k\right).$$

However, the algebraic properties discussed in the next sections do not seem to hold for these refined combinatorial series.

Remark 4.3.7. It is actually equivalent to use **strong** statistics of non-orientability in the definition of the series $F_k^{(\alpha)}$ as in Section 2.1 using a slightly different normalization. More precisely, one can show that for any SSON $\boldsymbol{\rho}$ one has

$$F_k^{(\alpha)}\left(t,\boldsymbol{p},s_1,\dots,s_k\right) := \sum_M(-t)^{|M|}p_{\lambda^\diamond(M)}\frac{b^{\boldsymbol{\vartheta}_{\boldsymbol{\rho}}(M)}}{\alpha^{|\mathcal{V}_\bullet(M)|}}\prod_{1\leq i\leq k}\frac{(-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}},$$

where the sum runs over k-layered maps. The advantage of using SON rather than SSON is that the power of α in the denominator is smaller, which will allow us to prove the polynomiality in α in Section 4.7.3.

4.4 The vanishing property

In this section we consider expressions in two different alphabets $\boldsymbol{p} := (p_1,p_2,\dots,)$ and $\boldsymbol{q} := (q_1,q_2,\dots)$. We will use repeatedly without further mention the fact that operators and coefficient extraction which depend on different alphabets trivially commute.

Let λ be a partition, of length $\ell\geq 1$. Then we define

$$F^{(\alpha)}(\lambda) := F_\ell^{(\alpha)}(t,\boldsymbol{p},\lambda_1,\lambda_2,\dots,\lambda_\ell) \tag{4.24}$$

where $F_\ell^{(\alpha)}$ is the generating series of ℓ-layered maps given by Eq. (4.21). The main purpose of this section is to prove the following theorem.

Theorem 4.4.1 (Vanishing property)**.** *Let λ be a partition of size n. Then the function $F^{(\alpha)}(\lambda)$ is a polynomial in t of degree less than or equal to n. In other terms, if $m>n$ then*

$$[t^m]F^{(\alpha)}(\lambda) = 0.$$

Combinatorially, the vanishing property is equivalent to saying that the total contribution of maps with more than $|\lambda|$ edges is zero in the series $F^{(\alpha)}(\lambda)$. In the cases $b=0$ and $b=1$, a combinatorial proof of this property was given in [FŚ11a] and [FŚ11b], respectively. However, such a proof does not seem to work for the general b because of the presence of the non-orientability weight.

4.4.1 The space $\mathcal{P}_{\leq s}$

In this section, we consider operators $\mathcal{B}_m(\boldsymbol{p}, u)$ where $u = -\alpha s$, for some positive integer s. To this purpose, we study their action on the function $\tau^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \underline{-\alpha s})$. We start by some preliminaries.

First, we define for any integer $s \geq 1$, the space $\mathcal{P}_{\leq s} \subset \mathcal{P}$ by

$$\mathcal{P}_{\leq s} := \mathrm{Span}_{\mathbb{Q}(b)} \left\{ J_\xi^{(\alpha)}(\boldsymbol{p}) \right\}_{\xi \in \mathbb{Y}, \xi_1 \leq s}.$$

We prove some useful properties of the action of $\mathcal{B}_m(\boldsymbol{p}, -\alpha s)$ on $\mathcal{P}_{\leq s}$.

Lemma 4.4.2. *Let s be a positive integer and let ξ be a partition. Then $J_\xi^{(\alpha)}(\underline{-\alpha s}) = 0$ if and only if $\xi_1 > s$.*

Proof. From Theorem 1.2.6, we know that

$$J_\xi^{(\alpha)}(\underline{-\alpha s}) = \prod_{\square \in \xi} \left(c_\alpha(\square) - \alpha s \right),$$

where we recall that $c_\alpha(\square) = \alpha(i-1) - (j-1)$ for a cell $\square$ of coordinates (i, j). Hence, $J_\xi^{(\alpha)}(\underline{-\alpha s}) = 0$ if and only if ξ contains the cell $\square_0 = (s+1, 1)$, the only cell satisfying $c_\alpha(\square_0) = \alpha s$. This condition is satisfied if and only if $\xi_1 > s$. □

Remark 4.4.3. We insist on the fact that we think of α here as a formal parameter and of $J_\xi^{(\alpha)}(\underline{-\alpha s}) = 0$ as a polynomial in α. Indeed, the argument of the previous proof does not work when $\alpha = 1$ for example.

The space $\mathcal{P}_{\leq s}$ satisfies a stability property and a vanishing property with respect to the operators $\mathcal{B}_m$, which will play a key role in the proof of Theorem 4.4.1.

Proposition 4.4.4 (Stability property)**.** *For any $s \geq 1$, the space $\mathcal{P}_{\leq s}$ is stable by the operators $\mathcal{B}_m(\boldsymbol{p}, -\alpha s)$ for every $m \geq 1$.*

Proof. It is enough to prove that

$$\mathcal{B}_m(\boldsymbol{p}, -\alpha s) \cdot J_\xi^{(\alpha)}(\boldsymbol{p}) \in \mathcal{P}_{\leq s} \tag{4.25}$$

for every partition ξ such that $\xi_1 \leq s$. Fix such a partition ξ.

From Theorem 2.3.1, we know that

$$t^m \mathcal{B}_m(\boldsymbol{p}, -\alpha s) \cdot \tau^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \underline{-\alpha s}) = \frac{m\partial}{\partial q_m} \tau^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \underline{-\alpha s}).$$

By extracting the coefficient of $t^{|\xi|+m} J_\xi^{(\alpha)}(\boldsymbol{q})$ in the last equation (see also Eq. (1.27)), we get

$$\begin{aligned} \mathcal{B}_m(\boldsymbol{p}, -\alpha s) \cdot \frac{J_\xi^{(\alpha)}(\boldsymbol{p}) J_\xi^{(\alpha)}(\underline{-\alpha s})}{j_\xi^{(\alpha)}} &= \left[t^{|\xi|+m} J_\xi(\boldsymbol{q}) \right] \frac{m\partial}{\partial q_m} \tau^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \underline{-\alpha s}) \\ &= \sum_{\pi \vdash |\xi|+m} \frac{J_\pi^{(\alpha)}(\boldsymbol{p}) J_\pi^{(\alpha)}(\underline{-\alpha s})}{j_\pi^{(\alpha)}} [J_\xi(\boldsymbol{q})] \frac{m\partial}{\partial q_m} J_\pi(\boldsymbol{q}). \end{aligned}$$

Using Lemma 4.4.2, we know that only partitions π such that $\pi_1 \leq s$ contribute the last sum, and as a consequence the right hand-side of the last equation is in $\mathcal{P}_{\leq s}$. This finishes the proof of Eq. (4.25) and hence the proof of the proposition. □

Proposition 4.4.5. *Fix an integer $s \geq 1$ and $\ell > s$. For every ξ such that $\xi_1 \leq s$, we have*

$$[t^\ell]\left(\exp\left(\mathcal{B}_\infty(t,\boldsymbol{p},-\alpha s)\right)\cdot J_\xi^{(\alpha)}(\boldsymbol{p})\right) = 0.$$

Proof. We consider an additional parameter z. Remark 2.3.3 implies that $\exp\left(\mathcal{B}_\infty(t,\boldsymbol{p},-\alpha s)\right)$ is well-defined on $\mathcal{P}$, and consequently it is well-defined on $\mathbb{Q}(\alpha)[\boldsymbol{p},\boldsymbol{q}][\![z]\!]$. In particular, we can investigate the action of $\exp\left(\mathcal{B}_\infty(t,\boldsymbol{p},-\alpha s)\right)$ on $\tau^{(\alpha)}(z,\boldsymbol{p},\boldsymbol{q},\underline{-\alpha s})$. Using Theorem 2.3.2 we have

$$\begin{aligned}[t^\ell]&\left(\exp\left(\mathcal{B}_\infty(t,\boldsymbol{p},-\alpha s)\right)\cdot\tau^{(\alpha)}(z,\boldsymbol{p},\boldsymbol{q},\underline{-\alpha s})\right)\\ &= [t^\ell]\left(\exp\left(\mathcal{B}_\infty(t,\boldsymbol{p},-\alpha s)\right)\exp\left(\sum_{m\geq 1}\frac{z^m q_m}{m}\mathcal{B}_m(\boldsymbol{p},-\alpha s)\right)\cdot 1\right).\end{aligned}$$

But since the operators $\left(\mathcal{B}_m(\boldsymbol{p},-\alpha s)\right)_{m\geq 1}$ commute (see Proposition 2.3.4), the last equation can be rewritten as follows:

$$\begin{aligned}[t^\ell]&\left(\exp\left(\mathcal{B}_\infty(t,\boldsymbol{p},-\alpha s)\right)\cdot\tau^{(\alpha)}(z,\boldsymbol{p},\boldsymbol{q},\underline{-\alpha s})\right)\\ &= \exp\left(\sum_{m\geq 1}\frac{z^m q_m}{m}\mathcal{B}_m(\boldsymbol{p},-\alpha s)\right)[t^\ell]\left(\exp\left(\mathcal{B}_\infty(t,\boldsymbol{p},-\alpha s)\right)\cdot 1\right)\\ &= \exp\left(\sum_{m\geq 1}\frac{z^m q_m}{m}\mathcal{B}_m(\boldsymbol{p},-\alpha s)\right)\cdot[t^\ell]\tau^{(\alpha)}(t,\boldsymbol{p},\underline{1},\underline{-\alpha s}),\end{aligned}$$

where

$$[t^\ell]\tau^{(\alpha)}(t,\boldsymbol{p},\underline{1},\underline{-\alpha s}) = \sum_{\xi\vdash\ell}\frac{J_\xi^{(\alpha)}(\boldsymbol{p})J_\xi^{(\alpha)}(\underline{1})J_\xi^{(\alpha)}(\underline{-\alpha s})}{j_\xi^{(\alpha)}}.$$

We claim that

$$[t^\ell]\tau^{(\alpha)}(t,\boldsymbol{p},\underline{1},\underline{-\alpha s}) = 0. \tag{4.26}$$

Since $\ell > s$, we know that any partition of size ℓ contains at least one of the two cells $(s+1,1)$ and $(1,2)$, of respective α-contents αs and -1. Hence, from Theorem 1.2.6, we know that

$$\frac{J_\xi(\underline{1})J_\xi(\underline{-\alpha s})}{j_\xi^{(\alpha)}} = 0$$

for any partition ξ of size ℓ. This proves Eq. (4.26), and as a consequence,

$$[t^\ell]\left(\exp\left(\mathcal{B}_\infty(t,\boldsymbol{p},-\alpha s)\right)\cdot\tau^{(\alpha)}(z,\boldsymbol{p},\boldsymbol{q},\underline{-\alpha s})\right) = 0.$$

Let ξ be a partition with $\xi_1 \leq s$. We extract the coefficient of $z^{|\xi|}J_\xi^{(\alpha)}(\boldsymbol{q})$ in the last equation, and we use the fact that this extraction commutes with the action of $\exp\left(\mathcal{B}_\infty(t,\boldsymbol{p},-\alpha s)\right)$:

$$[t^\ell]\left(\exp\left(\mathcal{B}_\infty(t,\boldsymbol{p},-\alpha s)\right)\cdot[z^{|\xi|}J_\xi^{(\alpha)}(\boldsymbol{q})]\tau^{(\alpha)}(z,\boldsymbol{p},\boldsymbol{q},\underline{-\alpha s})\right) = 0.$$

Hence

$$[t^\ell]\left(\exp\left(\mathcal{B}_\infty(t,\boldsymbol{p},-\alpha s)\right)\cdot\frac{J^{(\alpha)}_\xi(\boldsymbol{p})J^{(\alpha)}_\xi(\underline{-\alpha s})}{j^{(\alpha)}_\xi}\right)=0.$$

But from Lemma 4.4.2 we know that $J^{(\alpha)}_\xi(\underline{-\alpha s})\neq 0$, which concludes the proof. □

4.4.2 Proof of the vanishing property

We now prove the main theorem of this section.

Proof of Theorem 4.4.1. Fix a partition λ and an integer $m>|\lambda|$, and let us prove that

$$[t^m]F^{(\alpha)}(\lambda)=0.$$

Let ℓ denote the length of λ. Eqs. (4.22) and (4.24) imply that

$$[t^m]F^{(\alpha)}(\lambda)=\sum_{\substack{n_1,\dots,n_\ell\geq 1\\ n_1+\dots+n_\ell=m}}\left([t^{n_1}]\exp\left(\mathcal{B}_\infty(-t,\boldsymbol{p},-\alpha\lambda_1)\right)\right)\cdots\left([t^{n_\ell}]\exp\left(\mathcal{B}_\infty(-t,\boldsymbol{p},-\alpha\lambda_\ell)\right)\right)\cdot 1. \quad (4.27)$$

For each ℓ-tuple $(n_1,\dots,n_\ell)$, let $1\leq i\leq \ell$ be such that $n_i>\lambda_i$ (such an integer exists because $m>|\lambda|$). Since $\lambda_1\geq\cdots\geq\lambda_\ell$, we have a chain of subspaces $1\in\mathcal{P}_{\leq\lambda_\ell}\subset\mathcal{P}_{\leq\lambda_{\ell-1}}\subset\cdots\subset\mathcal{P}_{\leq\lambda_{i+1}}$. Therefore Proposition 4.4.4 implies that

$$\left([t^{n_{i+1}}]\exp\left(\mathcal{B}_\infty(-t,\boldsymbol{p},-\alpha\lambda_{i+1})\right)\right)\cdots\left([t^{n_\ell}]\exp\left(\mathcal{B}_\infty(-t,\boldsymbol{p},-\alpha\lambda_\ell)\right)\right)\cdot 1\in\mathcal{P}_{\leq\lambda_{i+1}}\subset\mathcal{P}_{\leq\lambda_i}.$$

We now apply Proposition 4.4.5 with $s=\lambda_i$ and $\ell=n_i$, and we get

$$\left([t^{n_i}]\exp\left(\mathcal{B}_\infty(-t,\boldsymbol{p},-\alpha\lambda_i)\right)\right)\cdots\left([t^{n_\ell}]\exp\left(\mathcal{B}_\infty(-t,\boldsymbol{p},-\alpha\lambda_\ell)\right)\right)\cdot 1=0. \quad (4.28)$$

As a consequence, the contribution of each term in the RHS of Eq. (4.27) is zero, which concludes the proof of the theorem. □

4.5 Operators $\mathcal{C}_\ell$ and commutation relations

For each $\ell,k\geq 0$, we define the operator $\mathcal{C}_{\ell,k}$ on the space $\mathcal{P}$ by

$$\mathcal{C}_{\ell,k}(\boldsymbol{p}):=[u^\ell]\mathcal{B}_{k+\ell}(\boldsymbol{p},u).$$

With the combinatorial interpretation of the operators $\mathcal{B}_n$ given in Proposition 4.3.4, we get that $\mathcal{C}_{\ell,k}$ acts on the global weight of a bipartite map by adding a black vertex of degree $\ell+k$ with ℓ new white neighbors.

We also define for $\ell\geq 0$ the operators $\mathcal{C}_\ell$ from $\mathcal{P}$ to $\mathcal{P}[\![t]\!]_+$ as the marginal sums

$$\mathcal{C}_\ell(t,\boldsymbol{p}):=\sum_{k\geq 1}\frac{t^{\ell+k}}{\ell+k}\mathcal{C}_{\ell,k}(\boldsymbol{p})+\mathbb{1}_{\ell>0}\frac{t^\ell}{\ell}\mathcal{C}_{\ell,0}(\boldsymbol{p}). \quad (4.29)$$

Hence, we have from Eq. (4.23) that

$$\mathcal{B}_\infty(t,\boldsymbol{p},u) = \sum_{n\geq 1}\frac{t^n}{n}\mathcal{B}_n(\boldsymbol{p},u) = \sum_{\ell\geq 0} u^\ell \mathcal{C}_\ell(t,\boldsymbol{p}) \quad : \mathcal{P}\to\mathcal{P}[u][\![t]\!]_+.$$

The main purpose of this section is to prove the following commutation relation.

Theorem 4.5.1. *Let $m > 0$. Then*

$$[\mathcal{C}_\ell,\mathcal{C}_m] = \begin{cases} 0 & \text{if } \ell > 0,\\ (m+1)\mathcal{C}_{m+1} & \text{if } \ell = 0.\end{cases}$$

The proof of this theorem involves difficult computations but it is independent from the rest of the chapter.

Remark 4.5.2. We recall that the operators $(\mathcal{B}_n(\boldsymbol{p},u))_{n\geq 1}$ commute by Proposition 2.3.4. These commutation relations do not hold when we consider operators in different variables; $[\mathcal{B}_n(\boldsymbol{p},u),\mathcal{B}_m(\boldsymbol{p},v)]\neq 0$. However, Theorem 4.5.1 will allow us to understand commutation relations between $\mathcal{B}_\infty(t,\boldsymbol{p},u)$ and $\mathcal{B}_\infty(t,\boldsymbol{p},v)$ in the next section.

4.5.1 The operators $Y_{\ell,k}$

We consider the following catalytic version of the operators $\mathcal{C}_{\ell,k}$ defined above. If ℓ and k are two integers, then $Y_{\ell,k}$ is defined by

$$Y_{\ell,k} := \begin{cases} [u^\ell]\left(\Gamma_Y + uY_+\right)^{\ell+k} & \text{if } \ell,k\geq 0,\\ 0 & \text{otherwise.}\end{cases}$$

on $\mathcal{P}_Y$. In practice, we think of $Y_{\ell,k}$ as the sum of all successions of operators Y_+ and Γ_Y in which Y_+ appears ℓ times and Γ_Y appears k times. $Y_{\ell,k}$ and $\mathcal{C}_{\ell,k}$ are related by

$$\mathcal{C}_{\ell,k} = \Theta_Y Y_{\ell,k}\frac{y_0}{1+b}. \tag{4.30}$$

These operators satisfy the following recursive relations.

Lemma 4.5.3. *Fix a pair of integers (ℓ,k). One has*

$$Y_{\ell,k} = Y_{\ell,k-1}\Gamma_Y + Y_{\ell-1,k}Y_+ + \delta_{\ell,0}\delta_{k,0}, \tag{4.31}$$

where δ denotes the Kronecker delta. Moreover, if $1\leq m\leq \ell$ then

$$Y_{\ell,k} = \sum_{0\leq i\leq k} Y_{m-1,i}Y_+Y_{\ell-m,k-i}, \tag{4.32}$$

and if $1\leq m\leq \ell+k$, then

$$Y_{\ell,k} = \sum_{0\leq j\leq m} Y_{j,m-j}Y_{\ell-j,k-m+j}. \tag{4.33}$$

Proof. If $\ell\leq 0$ or $k\leq 0$ then the equations are immediate from the definition. Let us suppose that $\ell>0$ and $k>0$. In order to obtain Eq. (4.31), we expand $Y_{\ell,k}$ according to the rightmost operator; the sum of terms ending with Γ_Y (reps. Y_+) give $Y_{\ell,k-1}\Gamma_Y$ (resp. $Y_{\ell-1,k}Y_+$).

Similarly, we obtain Eq. (4.32) by expanding $Y_{\ell,k}$ according to the position of the m-th occurrence of the operator Y_+, and we obtain Eq. (4.33) by expanding on the number of occurrences of Y_+ in the m left operators. □

4.5.2 Catalytic operators in $\widetilde{Y}$ and $\widetilde{Z}$

We consider three new alphabets

$$Y' := \{y'_0, y'_1 \dots\}, \qquad Z := \{z_0, z_1, \dots\}, \quad \text{and} \quad Z' := \{z'_0, z'_1, \dots\}.$$

We also denote

$$\widetilde{Y} := Y \cup Y', \quad \text{and} \quad \widetilde{Z} := Z \cup Z'.$$

Let $\mathcal{P}_{\widetilde{Y},\widetilde{Z}}$ be the space

$$\mathcal{P}_{\widetilde{Y},\widetilde{Z}} = \operatorname{Span}_{\mathbb{Q}(b)} \left\{ y_i z_j p_\lambda, y'_i z'_j p_\lambda \right\}_{i,j \in \mathbb{N}, \lambda \in \mathbb{Y}}.$$

In this section, we use several differential operators acting on the space $\mathcal{P}_{\widetilde{Y},\widetilde{Z}}$ which have been introduced in [CD22].

We define the operators Y'_+ and $\Gamma_{Y'}$ by replacing y_i by y'_i in Eq. (1.46) and Eq. (1.47), respectively. Similarly, we define Z_+, Z'_+, Γ_Z and $\Gamma_{Z'}$. We also consider the catalytic operators in the two variables $\widetilde{Y}$ and $\widetilde{Z}$.

$$\Gamma^{Y,Y'}_{Z,Z'} = (1+b) \cdot \sum_{i,j,k \geq 1} \frac{y'_{i+j-1} z'_k \partial^2}{\partial y_{i+k-1} \partial z_{j-1}} + \sum_{i,j,k \geq 1} \frac{y_{i+j-1} z_k \partial^2}{\partial y'_{i+k-1} \partial z'_{j-1}} + b \cdot \sum_{i,j,k \geq 1} \frac{y'_{i+j-1} z'_k \partial^2}{\partial y'_{i+k-1} \partial z'_{j-1}},$$

$$\Gamma_{\widetilde{Z}} = \Gamma_Z + \Gamma_{Z'} + \Gamma^{Y,Y'}_{Z,Z'}, \qquad \text{and} \qquad \widetilde{Z}_+ = Z_+ + Z'_+.$$

We also consider the following operator

$$\Theta_{\widetilde{Z}} = \sum_{i \geq 0} p_i \frac{\partial}{\partial z_i} + \sum_{i,j \geq 0} y_{i+j} \frac{\partial^2}{\partial y'_i \partial z'_j} : \mathcal{P}_{\widetilde{Y},\widetilde{Z}} \to \mathcal{P}_Y.$$

Similarly, the operators $\Gamma^{Z,Z'}_{Y,Y'}$, $\Gamma_{\widetilde{Y}}$, $\widetilde{Y}_+$, $\Theta_{\widetilde{Y}}$ are defined by exchanging $z_i \leftrightarrow y_i$ and $z'_i \leftrightarrow y'_i$ in the previous definitions. Moreover, let Δ be the operator

$$\Delta := (1+b) \cdot \sum_{i,j \geq 0} \frac{y'_j z'_i \partial^2}{\partial y_i \partial z_j} + \sum_{i,j \geq 0} \frac{y_j z_i \partial^2}{\partial y'_i \partial z'_j} + b \cdot \sum_{i,j \geq 0} \frac{y'_j z'_i \partial^2}{\partial y'_i \partial z'_j}.$$

We now consider a two catalytic variables version of $Y_{\ell,k}$, defined as the operators on $\mathcal{P}_{\widetilde{Y},\widetilde{Z}}$ given by

$$\widetilde{Y}_{\ell,k} := \begin{cases} [u^\ell] \left(\Gamma_{\widetilde{Y}} + u \widetilde{Y}_+ \right)^{\ell+k} & \text{if } \ell, k \geq 0, \\ 0 & \text{otherwise} \end{cases}, \quad \widetilde{Z}_{\ell,k} := \begin{cases} [u^\ell] \left(\Gamma_{\widetilde{Z}} + u \widetilde{Z}_+ \right)^{\ell+k} & \text{if } \ell, k \geq 0, \\ 0 & \text{otherwise} \end{cases},$$

Finally, we define the operator on $\mathcal{P}_Y$:

$$\mathcal{C}^Y_{\ell,k} := \Theta_{\widetilde{Z}} \widetilde{Z}_{\ell,k} \frac{z_0}{1+b}. \tag{4.34}$$

Eventhough the combinatorics of operators on the space $\mathcal{P}_{\widetilde{Y},\widetilde{Z}}$ will not be used in the proof of the commutation relations, we briefly explain the combinatorial background behind introducing these operators.

Computing the commutator $[\mathcal{C}_\ell, \mathcal{C}_m]$ corresponds to understanding the "difference" between the two ways of adding two black vertices with ℓ and m new white vertices, respectively. In the intermediate steps of such computations, we want to add some edges of the first vertex, add the second vertex, and then complete the remaining edges of the first vertex.

In such operations, we work with maps having two roots. The alphabets $\widetilde{Y}$ and $\widetilde{Z}$ are then used to control the degrees of the two root faces. Roughly, a face weight $y_i z_j p_\lambda$ corresponds to the case where the two root faces are distinct while $y'_i z'_j p_\lambda$ corresponds to the case where the two roots splits the same face into parts of degrees i and j. Hence, the operator $\widetilde{Z}_+$ as an example acts on the weight of a map by increasing the degree of the $\widetilde{Z}$-root of a map by 1. Similar interpretation can be given to the other operators.

4.5.3 Preliminary commutation relations

In this section, we prove some commutation relations satisfied by these operators. We will use some identities from the work of Chapuy and Dołęga [CD22], which are stated for the operators $\Lambda_{\tilde{Y}}, \Lambda_{\tilde{Z}}$ that are related to our $\Gamma_{\tilde{Y}}, \Gamma_{\tilde{Z}}$ by

$$\Gamma_{\tilde{Y}} = \widetilde{Y}_+ \Lambda_{\tilde{Y}}, \quad \Gamma_{\tilde{Z}} = \widetilde{Z}_+ \Lambda_{\tilde{Z}}. \tag{4.35}$$

Lemma 4.5.4 ([CD22, Lemma 4.12]). *We have the following equalities between operators on $\mathcal{P}_{\widetilde{Y},\widetilde{Z}}$*

$$\Delta\Gamma_{\tilde{Y}} = \Gamma_{\tilde{Z}}\Delta,\ \Delta\Gamma_{\tilde{Z}} = \Gamma_{\tilde{Y}}\Delta, \qquad \textit{and} \qquad \Delta\widetilde{Y}_+ = \widetilde{Z}_+\Delta,\ \Delta\widetilde{Z}_+ = \widetilde{Y}_+\Delta.$$

As a consequence, for any $\ell, k \geq 0$, we have

$$\Delta\widetilde{Y}_{\ell,k} = \widetilde{Z}_{\ell,k}\Delta,\ \Delta\widetilde{Z}_{\ell,k} = \widetilde{Y}_{\ell,k}\Delta. \tag{4.36}$$

Proof. Note that each identity comes in pair with an identity obtained by exchanging the alphabets $y_i \leftrightarrow z_i$, $y'_i \leftrightarrow z'_i$. Therefore it is enough to prove only the first identity for each pair (this argument will appear all over this section, so we will always prove only one of the identities that appear in such pairs). The second identity is direct from the definitions, and the first one follows from the second one, Eq. (4.35), and the identity $\Lambda_{\tilde{Z}}\Delta = \Delta\Lambda_{\tilde{Y}}$ from [CD22, Eq. (29a)]. Eq. (4.36) follows immediately. □

We have the following commutation relations between $\widetilde{Y}$ and $\widetilde{Z}$ operators.

Lemma 4.5.5 ([CD22]). *We have the following commutation relations on $\mathcal{P}_{\widetilde{Y},\widetilde{Z}}$,*

$$\left[\widetilde{Z}_+, \widetilde{Y}_+\right] = 0, \qquad \left[\Gamma_{\tilde{Z}}, \Gamma_{\tilde{Y}}\right] = 0, \tag{4.37}$$

$$\left[\Gamma_{\tilde{Z}}, \widetilde{Y}_+\right] = -\left[\widetilde{Z}_+, \Gamma_{\tilde{Y}}\right] = \widetilde{Y}_+\Delta\widetilde{Y}_+. \tag{4.38}$$

Proof. [CD22, Eq. (30)] says that for $m, n \geq 0$ the following equation holds

$$[\tilde{Z}_+\Lambda_{\tilde{Z}}^{m+1}, \tilde{Y}_+\Lambda_{\tilde{Y}}^{n}] + [\tilde{Z}_+\Lambda_{\tilde{Z}}^{n+1}, \tilde{Y}_+\Lambda_{\tilde{Y}}^{m}] = \tilde{Y}_+\Lambda_{\tilde{Y}}^{n}\tilde{Z}_+\Lambda_{\tilde{Z}}^{m}\Delta + \tilde{Y}_+\Lambda_{\tilde{Y}}^{m}\tilde{Z}_+\Lambda_{\tilde{Z}}^{n}\Delta.$$

Special cases of this equation ($m = n = 0$, $m = n = 1$ and $(m, n) = (1, 0)$ resp.), Eq. (4.35), and Lemma 4.5.4 finish the proof. □

Lemma 4.5.6. *The following equalities between operators hold:*

$$\Theta_{\widetilde{Z}}\widetilde{Y}_{i,j} = Y_{i,j}\Theta_{\widetilde{Z}} \colon \mathcal{P}_{\widetilde{Y},\widetilde{Z}} \to \mathcal{P}_Y, \quad \Theta_{\widetilde{Y}}\widetilde{Z}_{i,j} = Z_{i,j}\Theta_{\widetilde{Y}} \colon \mathcal{P}_{\widetilde{Y},\widetilde{Z}} \to \mathcal{P}_Z \textit{ for } i,j \geq 0, \tag{4.39a}$$

$$\Theta_Y\Theta_{\widetilde{Z}} = \Theta_Z\Theta_{\widetilde{Y}} \colon \mathcal{P}_{\widetilde{Y},\widetilde{Z}} \to \mathcal{P}, \tag{4.39b}$$

$$\Theta_{\widetilde{Y}}z_i = z_i\Theta_Y \colon \mathcal{P}_Y \to \mathcal{P}_Z, \quad \Theta_{\widetilde{Z}}y_i = y_i\Theta_z \colon \mathcal{P}_Z \to \mathcal{P}_Y \textit{ for } i \geq 0, \tag{4.39c}$$

$$\Theta_{\widetilde{Z}}\Delta\frac{z_0}{1+b} = 1 \colon \mathcal{P}_Y \to \mathcal{P}_Y, \quad \Theta_{\widetilde{Y}}\Delta\frac{y_0}{1+b} = 1 \colon \mathcal{P}_Z \to \mathcal{P}_Z, \tag{4.39d}$$

$$\widetilde{Y}_{i,j}z_0 = z_0Y_{i,j} \colon \mathcal{P}_Y \to \mathcal{P}_{\widetilde{Y},\widetilde{Z}}, \quad \widetilde{Z}_{i,j}y_0 = y_0Z_{i,j} \colon \mathcal{P}_Z \to \mathcal{P}_{\widetilde{Y},\widetilde{Z}}, \textit{ for } i,j \geq 0, \tag{4.39e}$$

$$\mathcal{C}^Y_{i,j}\frac{y_0}{1+b} = \frac{y_0}{1+b}\mathcal{C}_{i,j} \textit{ for } i,j \geq 0, \textit{ as operators from } \mathcal{P} \textit{ to } \mathcal{P}_Y. \tag{4.39f}$$

Proof. Eq. (4.39a) is a consequence of Eq. (4.35) and the identities

$$\Theta_{\tilde{Z}}\Lambda_{\tilde{Y}} = \Lambda_Y\Theta_{\tilde{Z}},\ \Theta_{\tilde{Y}}\Lambda_{\tilde{Z}} = \Lambda_Z\Theta_{\tilde{Y}}, \quad \Theta_{\tilde{Z}}\tilde{Y}_+ = Y_+\Theta_{\tilde{Z}},\ \Theta_{\tilde{Y}}\tilde{Z}_+ = Z_+\Theta_{\tilde{Y}}$$

proved in [CD22, Eqs. (29e) and (29f)]. Eq. (4.39b)—(4.39e) are direct from the definitions. Eq. (4.39e) gives

$$\widetilde{Z}_{i,j}\frac{y_0}{1+b} = \frac{y_0}{1+b}Z_{i,j} \colon \mathcal{P}_Z \to \mathcal{P}_{\widetilde{Y},\widetilde{Z}}.$$

Eq. (4.34) implies that by applying $\Theta_{\widetilde{Z}}$ on the left and $\frac{z_0}{1+b}$ on the right we get

$$\mathcal{C}^Y_{i,j}\frac{y_0}{1+b} = \Theta_{\widetilde{Z}}\frac{y_0}{1+b}Z_{i,j}\frac{z_0}{1+b}.$$

We deduce Eq. (4.39f) from Eq. (4.39c), and definition of $\mathcal{C}_{i,j}$ (see Eq. (4.30)). ☐

We conclude this section with the following lemma.

Lemma 4.5.7. *Let* $\ell, k \geq 0$. *Then,*

$$\Theta_Y\mathcal{C}^Y_{\ell,k} = \mathcal{C}_{\ell,k}\Theta_Y, \textit{ as operators from } \mathcal{P}_Y \textit{ to } \mathcal{P}.$$

Proof. Applying Eqs. (4.39b), (4.39a) and (4.39c) successively, we get that

$$\begin{aligned}\Theta_Y\mathcal{C}^Y_{\ell,k} &= \Theta_Y\Theta_{\widetilde{Z}}\widetilde{Z}_{\ell,k}\frac{z_0}{1+b}\\ &= \Theta_Z\Theta_{\widetilde{Y}}\widetilde{Z}_{\ell,k}\frac{z_0}{1+b}\\ &= \Theta_Z Z_{\ell,k}\Theta_{\widetilde{Y}}\frac{z_0}{1+b}\\ &= \Theta_Z Z_{\ell,k}\frac{z_0}{1+b}\Theta_Y\\ &= \mathcal{C}_{\ell,k}\Theta_Y.\end{aligned}$$

☐

4.5.4 Proof of Theorem 4.5.1

In this subsection, we prove Theorem 4.5.1.

The idea of the proof is to start from Lemma 4.5.5 which computes the commutator of a linear monomial in $\widetilde{Y}_+, \Gamma_{\widetilde{Y}}$ with a linear monomial in $\widetilde{Z}_+, \Gamma_{\widetilde{Z}}$, and use inductions to obtain the commutators of such monomials of arbitrary degrees.

The proof is organized as follows: we start from Lemma 4.5.5 which gives an expression for the commutator $[\widetilde{Z}_{\ell,k}, \widetilde{Y}_+]$ and $[\widetilde{Z}_{\ell,k}, \Gamma_{\widetilde{Z}}]$ when $\ell + k = 1$. By induction we obtain in Lemma 4.5.8 an expression for these commutators for any ℓ and k. By "forgetting" the first catalytic operator $\widetilde{Z}$, we deduce in Corollary 4.5.9 the commutators $[\mathcal{C}^Y_{\ell,k}, \widetilde{Y}_+]$. We then use induction to obtain an expression for $\sum_{0\le i\le k} \left[\frac{1}{\ell+i}\mathcal{C}^Y_{\ell,i}, Y_{m,k-i}\right]$. Finally, we deduce Theorem 4.5.1 by forgetting the catalytic variable $\widetilde{Y}$.

Lemma 4.5.8. *For any integers $\ell, k \ge -1$, we have the following equalities between operators on $\mathcal{P}_{\widetilde{Y},\widetilde{Z}}$.*

$$\left[\widetilde{Z}_{\ell,k}, \widetilde{Y}_+\right] = \sum_{i=0}^{\ell}\sum_{j=0}^{k-1} \widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-i,k-1-j}\widetilde{Y}_+, \tag{4.40}$$

and

$$\left[\widetilde{Z}_{\ell,k}, \Gamma_{\widetilde{Y}}\right] = -\sum_{i=0}^{\ell-1}\sum_{j=0}^{k} \widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-1-i,k-j}\widetilde{Y}_+. \tag{4.41}$$

In particular, by exchanging the variables $\widetilde{Y}$ and $\widetilde{Z}$, we get that

$$\left[\widetilde{Y}_{\ell-1,k}, \widetilde{Z}_+\right] + \left[\widetilde{Y}_{\ell,k-1}, \Gamma_{\widetilde{Z}}\right] = 0, \tag{4.42}$$

Proof. We prove simultaneously the three equations by induction on $\ell + k$. If $\ell = -1$ or $k = -1$ or $\ell = k = 0$ the result is immediate from the definitions. For $(\ell = 0, k = 1)$ and $(\ell = 1, k = 0)$ it corresponds to Lemma 4.5.5.

We now fix $\ell, k \ge 0$ such that $(\ell, k) \ne (0, 0)$ and we suppose that Eqs. (4.40) and (4.41) hold for all (i, j) such that $i + j < \ell + k$. First, since $(\ell, k) \ne (0, 0)$, we have an analogue of Eq. (4.31):

$$\widetilde{Z}_{\ell,k} = \widetilde{Z}_{\ell,k-1}\Gamma_{\widetilde{Z}} + \widetilde{Z}_{\ell-1,k}\widetilde{Z}_+ \tag{4.43}$$

Applying Eq. (4.40) with the pairs $(\ell, k-1)$ and $(\ell-1, k)$, and using Lemma 4.5.5 we get that

$$\begin{aligned}\left[\widetilde{Z}_{\ell,k}, \widetilde{Y}_+\right] &= \left[\widetilde{Z}_{\ell,k-1}\Gamma_{\widetilde{Z}}, \widetilde{Y}_+\right] + \left[\widetilde{Z}_{\ell-1,k}\widetilde{Z}_+, \widetilde{Y}_+\right] \\ &= \sum_{i=0}^{\ell}\sum_{j=0}^{k-2} \widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-i,k-2-j}\widetilde{Y}_+\Gamma_{\widetilde{Z}} + \widetilde{Z}_{\ell,k-1}\widetilde{Y}_+\Delta\widetilde{Y}_+ \\ &\quad + \sum_{i=0}^{\ell-1}\sum_{j=0}^{k-1} \widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-1-i,k-1-j}\widetilde{Y}_+\widetilde{Z}_+\end{aligned}$$

$$= \sum_{i=0}^{\ell}\sum_{j=0}^{k-2} \widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-i,k-2-j}\Gamma_{\widetilde{Z}}\widetilde{Y}_+ - \sum_{i=0}^{\ell}\sum_{j=0}^{k-2} \widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-i,k-2-j}\widetilde{Y}_+\Delta\widetilde{Y}_+ \\ + \widetilde{Z}_{\ell,k-1}\widetilde{Y}_+\Delta\widetilde{Y}_+ + \sum_{i=0}^{\ell-1}\sum_{j=0}^{k-1} \widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-1-i,k-1-j}\widetilde{Z}_+\widetilde{Y}_+. \tag{4.44}$$

Fix $0 \leq i \leq \ell$ and $0 \leq j \leq k-1$. Eq. (4.42), and Lemma 4.5.4 show that the operator $\widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-i,k-2-j}\Gamma_{\widetilde{Z}}\widetilde{Y}_+$ is equal to:

$$\widetilde{Y}_+\widetilde{Y}_{i,j}\Gamma_{\widetilde{Y}}\Delta\widetilde{Y}_{\ell-i,k-2-j}\widetilde{Y}_+ - \widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-i-1,k-1-j}\widetilde{Z}_+\widetilde{Y}_+ + \widetilde{Y}_{i,j}\widetilde{Y}_+\Delta\widetilde{Y}_{\ell-i-1,k-1-j}\widetilde{Y}_+.$$

Therefore, we get that the sum of the first and the fourth item in the RHS of Eq. (4.44) is equal to

$$\sum_{i=0}^{\ell}\sum_{j=0}^{k-2} \widetilde{Y}_+\widetilde{Y}_{i,j}\Gamma_{\widetilde{Y}}\Delta\widetilde{Y}_{\ell-i,k-2-j}\widetilde{Y}_+ + \sum_{i=0}^{\ell-1}\sum_{j=0}^{k-1} \widetilde{Y}_+\widetilde{Y}_{i,j}\widetilde{Y}_+\Delta\widetilde{Y}_{\ell-1-i,k-1-j}\widetilde{Y}_+.$$

On the other hand, applying Eq. (4.40) with the pair $(\ell, k-1)$, we obtain that

$$\widetilde{Z}_{\ell,k-1}\widetilde{Y}_+\Delta\widetilde{Y}_+ = \widetilde{Y}_+\Delta\widetilde{Y}_{\ell,k-1}\widetilde{Y}_+ + \sum_{i=0}^{\ell}\sum_{j=0}^{k-2} \widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-i,k-2-j}\widetilde{Y}_+\Delta\widetilde{Y}_+.$$

Hence

$$\left[\widetilde{Z}_{\ell,k}, \widetilde{Y}_+\right] = \sum_{i=0}^{\ell}\sum_{j=0}^{k-2} \widetilde{Y}_+\widetilde{Y}_{i,j}\Gamma_{\widetilde{Y}}\Delta\widetilde{Y}_{\ell-i,k-2-j}\widetilde{Y}_+ + \widetilde{Y}_+\Delta\widetilde{Y}_{\ell,k-1}\widetilde{Y}_+ \\ + \sum_{i=0}^{\ell-1}\sum_{j=0}^{k-1} \widetilde{Y}_+\widetilde{Y}_{i,j}\widetilde{Y}_+\Delta\widetilde{Y}_{\ell-1-i,k-1-j}\widetilde{Y}_+.$$

Shifting the summation indices we get

$$\left[\widetilde{Z}_{\ell,k}, \widetilde{Y}_+\right] = \sum_{i=0}^{\ell}\sum_{j=1}^{k-1} \widetilde{Y}_+\widetilde{Y}_{i,j-1}\Gamma_{\widetilde{Y}}\Delta\widetilde{Y}_{\ell-i,k-1-j}\widetilde{Y}_+ + \widetilde{Y}_+\Delta\widetilde{Y}_{\ell,k-1}\widetilde{Y}_+ \\ + \sum_{i=1}^{\ell}\sum_{j=0}^{k-1} \widetilde{Y}_+\widetilde{Y}_{i-1,j}\widetilde{Y}_+\Delta\widetilde{Y}_{\ell-i,k-1-j}\widetilde{Y}_+ \\ = \sum_{i=0}^{\ell}\sum_{j=0}^{k-1} \widetilde{Y}_+\widetilde{Y}_{i,j-1}\Gamma_{\widetilde{Y}}\Delta\widetilde{Y}_{\ell-i,k-1-j}\widetilde{Y}_+ + \widetilde{Y}_+\Delta\widetilde{Y}_{\ell,k-1}\widetilde{Y}_+ \\ + \sum_{i=0}^{\ell}\sum_{j=0}^{k-1} \widetilde{Y}_+\widetilde{Y}_{i-1,j}\widetilde{Y}_+\Delta\widetilde{Y}_{\ell-i,k-1-j}\widetilde{Y}_+,$$

where the second equality follows from the fact that $\widetilde{Y}_{i,j} = 0$ if $i < 0$ or $j < 0$. For each couple of indices $(i, j) \neq (0, 0)$, we regroup the terms in the two sums of the last equation by applying Eq. (4.43). On the other hand, note that the second term in the last equation can be written $\widetilde{Y}_+\widetilde{Y}_{0,0}\Delta\widetilde{Y}_{\ell,k-1}\widetilde{Y}_+$. We deduce that

$$\left[\widetilde{Z}_{\ell,k},\widetilde{Y}_+\right]=\sum_{i=0}^{\ell}\sum_{j=0}^{k-1}\widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-i,k-1-j}\widetilde{Y}_+.$$

We prove Eq. (4.41) in a similar way. Using Eq. (4.43) and the induction hypothesis, we have

$$\begin{aligned}\left[\widetilde{Z}_{\ell,k},\Gamma_{\widetilde{Y}}\right]&=\left[\widetilde{Z}_{\ell,k-1}\Gamma_{\widetilde{Z}},\Gamma_{\widetilde{Y}}\right]+\left[\widetilde{Z}_{\ell-1,k}\widetilde{Z}_+,\Gamma_{\widetilde{Y}}\right]\\&=-\sum_{i=0}^{\ell-1}\sum_{j=0}^{k-1}\widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-1-i,k-1-j}\widetilde{Y}_+\Gamma_{\widetilde{Z}}-\sum_{i=0}^{\ell-2}\sum_{j=0}^{k}\widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-2-i,k-j}\widetilde{Y}_+\widetilde{Z}_+\\&-\widetilde{Z}_{\ell-1,k}\widetilde{Y}_+\Delta\widetilde{Y}_+.\end{aligned}$$

From Lemma 4.5.5, and (4.40) we have

$$\begin{aligned}\left[\widetilde{Z}_{\ell,k},\Gamma_{\widetilde{Y}}\right]=&-\sum_{i=0}^{\ell-1}\sum_{j=0}^{k-1}\widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-1-i,k-1-j}\Gamma_{\widetilde{Z}}\widetilde{Y}_++\sum_{i=0}^{\ell-1}\sum_{j=0}^{k-1}\widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-1-i,k-1-j}\widetilde{Y}_+\Delta\widetilde{Y}_+\\&-\sum_{i=0}^{\ell-2}\sum_{j=0}^{k}\widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-2-i,k-j}\widetilde{Z}_+\widetilde{Y}_+-\widetilde{Y}_+\Delta\widetilde{Y}_{\ell-1,k}\widetilde{Y}_+\\&-\sum_{i=0}^{\ell-1}\sum_{j=0}^{k-1}\widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-1-i,k-1-j}\widetilde{Y}_+\Delta\widetilde{Y}_+.\end{aligned}$$

Applying Eq. (4.42) with $(\ell-i,k-1-j)$ for $0\leq i\leq\ell$ and $0\leq j\leq k-1$, we get that

$$\begin{aligned}\left[\widetilde{Z}_{\ell,k},\Gamma_{\widetilde{Y}}\right]&=-\sum_{i=0}^{\ell-1}\sum_{j=0}^{k-1}\widetilde{Y}_+\widetilde{Y}_{i,j}\Gamma_{\widetilde{Y}}\Delta\widetilde{Y}_{\ell-1-i,k-1-j}\widetilde{Y}_+-\sum_{i=0}^{\ell-2}\sum_{j=0}^{k}\widetilde{Y}_+\widetilde{Y}_{i,j}\widetilde{Y}_+\Delta\widetilde{Y}_{\ell-2-i,k-j}\widetilde{Y}_+\\&-\widetilde{Y}_+\Delta\widetilde{Y}_{\ell-1,k}\widetilde{Y}_+\\&=-\sum_{i=0}^{\ell-1}\sum_{j=0}^{k}\widetilde{Y}_+\widetilde{Y}_{i,j}\Delta\widetilde{Y}_{\ell-1-i,k-j}\widetilde{Y}_+\end{aligned}$$

□

We deduce the following corollary.

Corollary 4.5.9. *Let $\ell,k\geq 0$. As operators on $\mathcal{P}_Y$,*

$$\left[\mathcal{C}^Y_{\ell,k},Y_+\right]=(\ell+k)Y_+Y_{\ell,k-1}Y_+,\qquad\left[\mathcal{C}^Y_{\ell,k},\Gamma_Y\right]=-(\ell+k)Y_+Y_{\ell-1,k}Y_+.$$

Proof. We start by multiplying Eq. (4.40) by $\Theta_{\widetilde{Z}}$ on the left and $\frac{z_0}{1+b}$ on the right, and we use

Equations (4.39a), (4.39e) and (4.39d) to obtain:

$$\begin{aligned}\left[\mathcal{C}^Y_{\ell,k}, Y_+\right] &= \Theta_{\widetilde{Z}} \sum_{i=0}^{\ell} \sum_{j=0}^{k-1} \widetilde{Y}_+ \widetilde{Y}_{i,j} \Delta \widetilde{Y}_{\ell-i,k-1-j} \widetilde{Y}_+ \frac{z_0}{1+b} \\ &= \sum_{i=0}^{\ell} \sum_{j=0}^{k-1} Y_+ Y_{i,j} \Theta_{\widetilde{Z}} \Delta \frac{z_0}{1+b} Y_{\ell-i,k-1-j} Y_+ \\ &= \sum_{i=0}^{\ell} \sum_{j=0}^{k-1} Y_+ Y_{i,j} Y_{\ell-i,k-1-j} Y_+. \\ &= \sum_{m=0}^{\ell+k-1} \sum_{i=0}^{\ell} Y_+ Y_{i,m-i} Y_{\ell-i,k-1-m+i} Y_+.\end{aligned}$$

From Eq. (4.33), we know that in the last line, for each m, the second sum is equal to $Y_+ Y_{\ell,k-1} Y_+$, which concludes the proof of the first equation. Similarly, we obtain the second equation of the corollary from Eq. (4.41). □

We deduce the following proposition.

Proposition 4.5.10. *Fix $m \geq 0$ and $k \geq -1$. If $\ell > 0$ then*

$$\sum_{0 \leq i \leq k} \left[\frac{1}{\ell+i} \mathcal{C}^Y_{\ell,i}, Y_{m,k-i}\right] = -Y_{\ell+m+1,k-1}. \tag{4.45}$$

Moreover, if $\ell = 0$ then

$$\sum_{1 \leq i \leq k} \left[\frac{1}{i} \mathcal{C}^Y_{0,i}, Y_{m,k-i}\right] = m Y_{m+1,k-1}. \tag{4.46}$$

Proof. We proceed by induction on $k+m$. For $k=-1$ the two equations are immediate from the definitions.

Let us start by proving Eq. (4.45). Fix (k,m) with $k \geq 0$, such that Eq. (4.45) is satisfied for every (j,s) such that $s+j < k+m$. Fix $0 \leq i \leq k$. We rewrite Eq. (4.31) as follows.

$$Y_{m,k-i} = \mathbb{1}_{m>0} Y_{m-1,k-i} Y_+ + Y_{m,k-i-1} \Gamma_Y + \delta_{m,0} \delta_{k,i}.$$

Hence, using Corollary 4.5.9, we get for each $0 \leq i \leq k$

$$\begin{aligned}\left[\frac{1}{\ell+i} \mathcal{C}^Y_{\ell,i}, Y_{m,k-i}\right] = \mathbb{1}_{m>0} \left[\frac{1}{\ell+i} \mathcal{C}^Y_{\ell,i}, Y_{m-1,k-i}\right] Y_+ + \mathbb{1}_{m>0} Y_{m-1,k-i} Y_+ Y_{\ell,i-1} Y_+ \\ + \left[\frac{1}{\ell+i} \mathcal{C}^Y_{\ell,i}, Y_{m,k-i-1}\right] \Gamma_Y - Y_{m,k-1-i} Y_+ Y_{\ell-1,i} Y_+.\end{aligned}$$

Applying the induction hypothesis on the pairs $(m-1,k)$ and $(m,k-1)$, we get that

$$\begin{aligned}\sum_{0 \leq i \leq k} \left[\frac{1}{\ell+i} \mathcal{C}^Y_{\ell,i}, Y_{m,k-i}\right] = -\mathbb{1}_{m>0} Y_{\ell+m,k-1} Y_+ + \mathbb{1}_{m>0} \sum_{0 \leq i \leq k} Y_{m-1,k-i} Y_+ Y_{\ell,i-1} Y_+ \\ - Y_{\ell+m+1,k-2} \Gamma_Y - \sum_{0 \leq i \leq k} Y_{m,k-1-i} Y_+ Y_{\ell-1,i} Y_+.\end{aligned}$$

Using Eq. (4.32), we know that the two sums in the right-hand side of the last equality are both equal to $Y_{\ell+m,k-1}Y_+$. On the other hand, from Eq. (4.31), we know that

$$Y_{\ell+m+1,k-2}\Gamma_Y + Y_{\ell+m,k-1}Y_+ = Y_{\ell+m+1,k-1},$$

which concludes the proof of Eq. (4.45).

We now prove Eq. (4.46) in a similar way. Let (k,m) be two non-negative integers such that Eq. (4.46) is satisfied for every (j,s) such that $s+j<k+m$. For each $1\le i\le k$, one has

$$\left[\frac{1}{i}\mathcal{C}^Y_{0,i}, Y_{m,k-i}\right] = \mathbb{1}_{m>0}\left[\frac{1}{i}\mathcal{C}^Y_{0,i}, Y_{m-1,k-i}\right]Y_+ + \mathbb{1}_{m>0}Y_{m-1,k-i}Y_+Y_{0,i-1}Y_+ + \left[\frac{1}{i}\mathcal{C}^Y_{0,i}, Y_{m,k-1-i}\right]\Gamma_Y.$$

Applying the induction hypothesis on the pairs $(m-1,k)$ and $(m,k-1)$, we get that

$$\sum_{1\le i\le k}\left[\frac{1}{i}\mathcal{C}^Y_{0,i}, Y_{m,k-i}\right] = (m-1)\mathbb{1}_{m>0}Y_{m,k-1}Y_+ + \mathbb{1}_{m>0}\sum_{0\le i\le k}Y_{m-1,k-i}Y_+Y_{0,i-1}Y_+ + mY_{m+1,k-2}\Gamma_Y.$$

But from Eq. (4.32), we know that the sum in the right hand is equal to $Y_{m,k-1}Y_+$. Hence

$$\begin{aligned}\sum_{1\le i\le k}\left[\frac{1}{i}\mathcal{C}^Y_{0,i}, Y_{m,k-i}\right] &= m\mathbb{1}_{m>0}Y_{m,k-1}Y_+ + mY_{m+1,k-2}\Gamma_Y\\ &= mY_{m,k-1}Y_+ + mY_{m+1,k-2}\Gamma_Y\\ &= mY_{m+1,k-1},\end{aligned}$$

where we use Eq. (4.31) to obtain the last equality. □

We deduce the following corollary by forgetting the catalytic variable Y.

Corollary 4.5.11. *Fix $m,k\ge 0$. We have the following equalities between operators on $\mathcal{P}$:*

$$\sum_{0\le i\le k}\left[\frac{1}{\ell+i}\mathcal{C}_{\ell,i}, \mathcal{C}_{m,k-i}\right] = -\mathcal{C}_{\ell+m+1,k-1},\ \text{if } \ell>0, \tag{4.47}$$

and

$$\sum_{1\le i\le k}\left[\frac{1}{i}\mathcal{C}_{0,i}, \mathcal{C}_{m,k-i}\right] = m\mathcal{C}_{m+1,k-1}. \tag{4.48}$$

Proof. Let us prove Eq. (4.47). Starting from Eq. (4.45), and applying Θ_Y on the left and $\frac{y_0}{1+b}$ on the right, we get

$$\sum_{0\le i\le k}\frac{1}{\ell+i}\left(\Theta_Y\mathcal{C}^Y_{\ell,i}Y_{m,k-i}\frac{y_0}{1+b} - \Theta_Y Y_{m,k-i}\mathcal{C}^Y_{\ell,i}\frac{y_0}{1+b}\right) = -\Theta_Y Y_{\ell+m+1,k-1}\frac{y_0}{1+b}.$$

Using Lemma 4.5.7 and Eq. (4.39f) we obtain

$$\sum_{0\leq i\leq k}\frac{1}{\ell+i}\left(\mathcal{C}_{\ell,i}\Theta_Y Y_{m,k-i}\frac{y_0}{1+b}-\Theta_Y Y_{m,k-i}\frac{y_0}{1+b}\mathcal{C}_{\ell,i}\right)=-\Theta_Y Y_{\ell+m+1,k-1}\frac{y_0}{1+b}.$$

We deduce Eq. (4.47) using Eq. (4.30). We obtain in a similar way Eq. (4.48) from Eq. (4.46). □

We now prove the main result of this section.

Proof of Theorem 4.5.1. Let $\ell, m > 0$. Recall the relation between $\mathcal{C}_\ell$ and $\mathcal{C}_{\ell,m}$ given by (4.29). It gives

$$\begin{aligned}[\mathcal{C}_\ell,\mathcal{C}_m]&=\sum_{k\geq 0}t^{k+\ell+m}\sum_{0\leq i\leq k}\frac{[\mathcal{C}_{\ell,i},\mathcal{C}_{m,k-i}]}{(i+\ell)(k-i+m)}\\&=\sum_{k\geq 0}\frac{t^{k+\ell+m}}{k+\ell+m}\left(\sum_{0\leq i\leq k}\left[\frac{\mathcal{C}_{\ell,i}}{i+\ell},\mathcal{C}_{m,k-i}\right]+\sum_{0\leq i\leq k}\left[\mathcal{C}_{\ell,i},\frac{\mathcal{C}_{m,k-i}}{k-i+m}\right]\right).\end{aligned}$$

Eq. (4.47) implies that it is equal to 0. Fix now $m > 0$. We have

$$\begin{aligned}[\mathcal{C}_0,\mathcal{C}_m]&=\sum_{k\geq 0}t^{k+m}\sum_{1\leq i\leq k}\frac{[\mathcal{C}_{0,i},\mathcal{C}_{m,k-i}]}{i(k-i+m)}\\&=\sum_{k\geq 0}\frac{t^{k+m}}{k+m}\left(\sum_{1\leq i\leq k}\left[\frac{\mathcal{C}_{0,i}}{i},\mathcal{C}_{m,k-i}\right]+\sum_{1\leq i\leq k}\left[\mathcal{C}_{0,i},\frac{\mathcal{C}_{m,k-i}}{k-i+m}\right]\right).\end{aligned}$$

Moreover, $\mathcal{C}_{0,0} = 0$ by definition. We then write

$$\begin{aligned}[\mathcal{C}_0,\mathcal{C}_m]&=\sum_{k\geq 0}\frac{t^{k+m}}{k+m}\left(\sum_{1\leq i\leq k}\left[\frac{\mathcal{C}_{0,i}}{i},\mathcal{C}_{m,k-i}\right]+\sum_{0\leq i\leq k}\left[\mathcal{C}_{0,i},\frac{\mathcal{C}_{m,k-i}}{k-i+m}\right]\right)\\&=\sum_{k\geq 0}\frac{t^{k+m}}{k+m}\left(m\mathcal{C}_{m+1,k-1}+\mathcal{C}_{m+1,k-1}\right)\\&=(m+1)\mathcal{C}_{m+1}.\end{aligned}$$

□

4.6 The shifted symmetry property

It will be convenient in this section to separate the constant part of the operator $\mathcal{B}_\infty$ in the variable u. Namely, we consider

$$\mathcal{B}_\infty^{>}(t,\boldsymbol{p},u):=\sum_{\ell\geq 1}u^\ell\mathcal{C}_\ell(t,\boldsymbol{p}). \tag{4.49}$$

In the following, when there is no ambiguity, we will simply denote

$$\mathcal{B}_\infty(u)\equiv\mathcal{B}_\infty(t,\boldsymbol{p},u),\quad\text{and}\quad\mathcal{B}_\infty^{>}(u)\equiv\mathcal{B}_\infty^{>}(t,\boldsymbol{p},u).$$

We recall that $F_k^{(\alpha)}(t,\boldsymbol{p},s_1,\dots,s_k)$ is the generating series of layered maps defined in Eq. (4.21). The purpose of this section is to prove the following theorem.

Theorem 4.6.1 (Shifted symmetry property)**.** *Fix $k \geq 1$. The function $F_k^{(\alpha)}(t, \boldsymbol{p}, s_1, \dots, s_k)$ is α-shifted symmetric into the variables $s_1, s_2, \dots, s_k$. Moreover,*

$$\begin{aligned} &F_k^{(\alpha)}(-t, \boldsymbol{p}, s_1, s_2 \dots, s_k) \\ &= \exp\big(\mathcal{B}_\infty(k-1)\big) \cdots \exp\big(\mathcal{B}_\infty(1)\big) \exp\big(\mathcal{C}_0 + \mathcal{B}_\infty^{>}(-\alpha s_1' - k) + \cdots + \mathcal{B}_\infty^{>}(-\alpha s_k' - k)\big) \cdot 1, \end{aligned} \tag{4.50}$$

where $s_i' := s_i - i/\alpha$.

We start by proving some general commutation relations which will be useful in the proof of Theorem 4.6.1.

4.6.1 Preliminaries

Let X be a vector space and let $\mathcal{O}(X)$ denote the set of linear operators on X. We also denote $\mathcal{O}(X)[\![z]\!]_+$ the ideal of $\mathcal{O}(X)[\![z]\!]$ generated by z. Whenever $A \in \mathcal{O}(X)[\![z]\!]_+$, then $\exp(A) := 1 + \sum_{n\geq 1} \frac{A^n}{n!}$ is a well-defined element of $\mathcal{O}(X)[\![z]\!]$. Note that $\mathcal{O}(X)[\![z]\!]$ is a Lie algebra with the standard commutator

$$[A, B] := AB - BA.$$

For $C \in \mathcal{O}(X)[\![z]\!]$, we consider the adjoint action of C, denoted by ad_C, as the linear map defined on the space $\mathcal{O}(X)[\![z]\!]$ by

$$\mathrm{ad}_C(A) := [C, A].$$

We then have for any integer $m \geq 0$

$$\mathrm{ad}_C^m(A) = \underbrace{[C, [C, \dots, [C, A] \dots]]}_{m \text{ commutators}}.$$

We will also use the convention

$$\mathrm{ad}_C^0(A) = A.$$

By a direct induction we obtain the following lemma.

Lemma 4.6.2. *Let A_1, A_2 and C be three operators such that*

$$[A_2, \mathrm{ad}_C^m(A_1)] = 0, \text{ for every } m \geq 0.$$

Then

$$\mathrm{ad}_{A_2+C}^m(A_1) = \mathrm{ad}_C^m(A_1), \text{ for } m \geq 0.$$

We have the following classical identities between the elements of the Lie algebra $\mathcal{O}(X)[\![z]\!]$ (see e.g [Hal15, Section 5.4] for a more general context):

$$e^C A e^{-C} = e^{\mathrm{ad}_C}(A) \text{ for } C \in \mathcal{O}(X)[\![z]\!]_+, \tag{4.51}$$

$$\frac{d}{dt} e^{tA+C} = e^{tA+C} \frac{e^{\mathrm{ad}_{-C-tA}} - 1}{\mathrm{ad}_{-C-tA}}(A) \quad \text{for } A \in \mathcal{O}(X)[\![z]\!], C \in \mathcal{O}(X)[\![z]\!]_+, \tag{4.52}$$

where the last equality holds in $\mathcal{O}(X)[\![t,z]\!]$. Moreover,

$$\frac{e^{\mathrm{ad}_{-C-tA}}-1}{\mathrm{ad}_{-C-tA}} := \sum_{k\geq 0} \frac{\mathrm{ad}^k_{-C-tA}}{(k+1)!}.$$

The following lemma will be quite useful for proving Theorem 4.6.1, and might be interpreted as a special case of the Baker–Campbell–Hausdorff formula.

Lemma 4.6.3. *Let $A, C \in \mathcal{O}(X)[\![z]\!]_+$ be two operators such that $[A, \mathrm{ad}^m_C(A)] = 0$ for every $m \geq 0$. Then*

$$e^{A+C}e^{-C} = \exp\left(\frac{e^{\mathrm{ad}_C}-1}{\mathrm{ad}_C}(A)\right).$$

Proof. We consider the following function in t

$$\Phi(t) := e^C e^{-tA-C} \exp\left(\frac{e^{\mathrm{ad}_C}-1}{\mathrm{ad}_C}(tA)\right).$$

We want to prove that $\Phi(1) = 1$. Since $\Phi(0) = 1$, it is enough then to prove that $\frac{d}{dt}\Phi(t) = 0$. But

$$\frac{d}{dt}\Phi(t) = e^C\left(\frac{d}{dt}e^{-tA-C}\right)\exp\left(\frac{e^{\mathrm{ad}_C}-1}{\mathrm{ad}_C}(tA)\right) + e^C e^{-tA-C}\frac{d}{dt}\exp\left(\frac{e^{\mathrm{ad}_C}-1}{\mathrm{ad}_C}(tA)\right).$$

On the one hand, using Eq. (4.52) and Lemma 4.6.2 we have that

$$\frac{d}{dt}e^{-tA-C} = e^{tA-C}\cdot\frac{e^{\mathrm{ad}_C}-1}{\mathrm{ad}_C}(-A) = -e^{-tA-C}\cdot\frac{e^{\mathrm{ad}_C}-1}{\mathrm{ad}_C}(A).$$

On the other hand,

$$\frac{d}{dt}\exp\left(\frac{e^{\mathrm{ad}_C}-1}{\mathrm{ad}_C}(tA)\right) = \frac{e^{\mathrm{ad}_C}-1}{\mathrm{ad}_C}(A)\exp\left(\frac{e^{\mathrm{ad}_C}-1}{\mathrm{ad}_C}(tA)\right).$$

This finishes the proof of the lemma. □

We deduce the main result of this section.

Proposition 4.6.4. *Let $A_1, A_2, C \in \mathcal{O}(X)[\![z]\!]_+$ be three operators such that*

$$\left[\mathrm{ad}^\ell_C A_i, \mathrm{ad}^m_C A_j\right] = 0, \quad \textit{for } 1 \leq i,j \leq 2 \textit{ and } m,\ell \geq 0.$$

Then

$$e^{A_1+C}e^{-C}e^{A_2+C} = e^{A_1+A_2+C}.$$

Proof. We prove that

$$e^{A_1+C}e^{-C} = \exp\left(\frac{e^{\mathrm{ad}_C}-1}{\mathrm{ad}_C}(A_1)\right) = e^{A_1+A_2+C}e^{-A_2-C}.$$

The left equality in the line above is guaranteed by Lemma 4.6.3. Let us prove the right equality. Since the operators A_1, A_2 and C satisfy the conditions of Lemma 4.6.2, then

$$\mathrm{ad}^m_{A_2+C}(A_1) = \mathrm{ad}^m_C(A_1) \quad \text{for } m \geq 0. \tag{4.53}$$

Hence,

$$[A_1, \mathrm{ad}^m_{A_2+C}(A_1)] = [A_1, \mathrm{ad}^m_C(A_1)] = 0.$$

Consequently, the operators A_1 and $A_2 + C$ fulfill then the conditions of Lemma 4.6.3, and we obtain that

$$e^{A_1+A_2+C}e^{-A_2-C} = \exp\left(\frac{e^{\mathrm{ad}_{A_2+C}}-1}{\mathrm{ad}_{A_2+C}}(A_1)\right) = \exp\left(\frac{e^{\mathrm{ad}_C}-1}{\mathrm{ad}_C}(A_1)\right),$$

where the last equality follows from Eq. (4.53) and Lemma 4.6.2. This finishes the proof. □

4.6.2 Proof of Theorem 4.6.1

From Proposition 4.3.5 we know that

$$F_k^{(\alpha)}(-t, \boldsymbol{p}, s_1, \ldots, s_k) = \exp\left(\mathcal{B}_\infty(-\alpha s_1)\right)\cdots\exp\left(\mathcal{B}_\infty(-\alpha s_k)\right)\cdot 1. \tag{4.54}$$

We recall that the purpose of this section is to prove that this function is symmetric in the shifted variables $s_i' = s_i - i/\alpha$ and to give a symmetric expression of it (Theorem 4.6.1).

We know from Theorem 4.5.1 that

$$[\mathcal{B}^>_\infty(u), \mathcal{B}^>_\infty(v)] = 0. \tag{4.55}$$

The following lemma establishes a relation between the operators $\mathcal{B}_\infty(u+1)$ and $\mathcal{B}^>_\infty(u)$.

Proposition 4.6.5. *For any variable u, we have*

$$\mathcal{B}_\infty(u+1) = \mathcal{B}_\infty(1) + e^{\mathcal{C}_0}\mathcal{B}^>_\infty(u)e^{-\mathcal{C}_0}.$$

Proof. Note that $\mathcal{B}^>_\infty(u), \mathcal{B}_\infty(u), \mathcal{C}_0 \in \mathcal{O}(\mathcal{P}[u])[\![t]\!]_+$. In particular $\mathcal{B}_\infty(1) \in \mathcal{O}(\mathcal{P})[\![t]\!]_+$ is well-defined and we can use formulas from Section 4.6.1. From the definition, we have

$$\begin{aligned}
\mathcal{B}_\infty(u+1) &= \sum_{\ell\geq 0}(u+1)^\ell\mathcal{C}_\ell \\
&= \sum_{\ell\geq 0}\sum_{0\leq k\leq \ell}\binom{\ell}{k}u^k\mathcal{C}_\ell \\
&= \mathcal{B}_\infty(1) + \sum_{k\geq 1}\sum_{\ell\geq k}\binom{\ell}{k}u^k\mathcal{C}_\ell.
\end{aligned}$$

Moreover, have for any $1 \leq k \leq \ell$ that

$$\frac{\ell!}{k!}\mathcal{C}_\ell = \mathrm{ad}^{\ell-k}_{\mathcal{C}_0}\mathcal{C}_k. \tag{4.56}$$

This is obtained by applying Theorem 4.5.1 inductively on $\ell - k$. Hence,

$$\begin{aligned}\mathcal{B}_\infty(u+1) &= \mathcal{B}_\infty(1) + \sum_{k\geq 1} u^k \left(\sum_{i\geq 0} \frac{(\mathrm{ad}_{\mathcal{C}_0})^i}{i!}\right) \mathcal{C}_k \\ &= \mathcal{B}_\infty(1) + \sum_{k\geq 1} u^k e^{\mathcal{C}_0} \mathcal{C}_k e^{-\mathcal{C}_0} \\ &= \mathcal{B}_\infty(1) + e^{\mathcal{C}_0} \mathcal{B}_\infty^{>}(u) e^{-\mathcal{C}_0},\end{aligned}$$

where the second equality follows from Eq. (4.51). □

We now prove that the operators $\mathcal{B}_\infty^{>}(u)$, $\mathcal{B}_\infty^{>}(v)$ and $\mathcal{C}_0$ satisfy the conditions of Proposition 4.6.4.

Lemma 4.6.6. *Let u and v be two variables, and let m and ℓ be two non negative integers. Then*

$$\left[\mathrm{ad}^{\ell}_{\mathcal{C}_0} \mathcal{B}_\infty^{>}(u), \mathrm{ad}^{m}_{\mathcal{C}_0} \mathcal{B}_\infty^{>}(v)\right] = 0.$$

Proof. Eq. (4.49) implies that it is enough to prove that $\left[\mathrm{ad}^{\ell}_{\mathcal{C}_0} \mathcal{C}_i, \mathrm{ad}^{m}_{\mathcal{C}_0} \mathcal{C}_j\right] = 0$ for all $i, j \geq 1$. The latter follows from Eq. (4.56) and Theorem 4.5.1. □

Lemma 4.6.7. *For any variable u, we have*

$$\exp(\mathcal{B}_\infty(u+1)) = \exp\big(\mathcal{B}_\infty(1)\big) \cdot \exp\big(\mathcal{B}_\infty(u)\big) \cdot \exp(-\mathcal{C}_0).$$

Proof. We know from Proposition 4.6.5 that

$$\begin{aligned}\exp(\mathcal{B}_\infty(u+1)) &= \exp\left(\mathcal{B}_\infty(1) + e^{\mathcal{C}_0} \mathcal{B}_\infty^{>}(u) e^{-\mathcal{C}_0}\right) \\ &= \exp\left(\mathcal{C}_0 + \mathcal{B}_\infty^{>}(1) + e^{\mathcal{C}_0} \mathcal{B}_\infty^{>}(u) e^{-\mathcal{C}_0}\right).\end{aligned}$$

But from Lemma 4.6.6 we know that the three operators $\mathcal{B}_\infty^{>}(1)$, $\mathcal{B}_\infty^{>}(u)$ and $\mathcal{C}_0$ satisfy the conditions of Proposition 4.6.4. Hence, this is also the case for the three operators $\mathcal{B}_\infty^{>}(1)$, $e^{\mathcal{C}_0} \mathcal{B}_\infty^{>}(u) e^{-\mathcal{C}_0}$ and $\mathcal{C}_0$, since $e^{\mathcal{C}_0} \mathcal{B}_\infty^{>}(u) e^{-\mathcal{C}_0} = e^{\mathrm{ad}_{\mathcal{C}_0}} \mathcal{B}_\infty^{>}(u)$. As a consequence, we get

$$\begin{aligned}\exp(\mathcal{B}_\infty(u+1)) &= \exp\left(\mathcal{C}_0 + \mathcal{B}_\infty^{>}(1)\right) \cdot \exp(-\mathcal{C}_0) \cdot \exp(\mathcal{C}_0 + e^{\mathcal{C}_0} \mathcal{B}_\infty^{>}(u) e^{-\mathcal{C}_0}) \\ &= \exp\left(\mathcal{C}_0 + \mathcal{B}_\infty^{>}(1)\right) \cdot \exp(-\mathcal{C}_0) \cdot \exp\left(e^{\mathcal{C}_0}\left(\mathcal{C}_0 + \mathcal{B}_\infty^{>}(u)\right) e^{-\mathcal{C}_0}\right) \\ &= \exp\left(\mathcal{C}_0 + \mathcal{B}_\infty^{>}(1)\right) \cdot \exp(-\mathcal{C}_0) \cdot \exp\left(e^{\mathcal{C}_0} \mathcal{B}_\infty(u) e^{-\mathcal{C}_0}\right) \\ &= \exp\left(\mathcal{B}_\infty(1)\right) \cdot \exp(-\mathcal{C}_0) \cdot \exp\left(e^{\mathcal{C}_0} \mathcal{B}_\infty(u) e^{-\mathcal{C}_0}\right)\end{aligned}$$

Finally, by expanding the last exponential, and by observing that for each $\ell \geq 0$

$$\exp(-\mathcal{C}_0) \cdot \left(e^{\mathcal{C}_0} \mathcal{B}_\infty(u) e^{-\mathcal{C}_0}\right)^{\ell} = \mathcal{B}_\infty(u)^{\ell} \cdot \exp(-\mathcal{C}_0),$$

we deduce that

$$\exp(\mathcal{B}_\infty(u+1)) = \exp\big(\mathcal{B}_\infty(1)\big) \cdot \exp\big(\mathcal{B}_\infty(u)\big) \cdot \exp(-\mathcal{C}_0).$$ □

Remark 4.6.8. Fix now two variables u and v. By applying Lemma 4.6.7 and Proposition 4.6.4 with $\mathcal{B}^{>}_{\infty}(u)$, $\mathcal{B}^{>}_{\infty}(v)$ and $\mathcal{C}_0$, we get that

$$\begin{aligned}
&\exp(\mathcal{B}_\infty(u+1))\exp(\mathcal{B}_\infty(v)) \qquad (4.57)\\
&\quad= \exp\big(\mathcal{B}_\infty(1)\big)\cdot\exp\big(\mathcal{C}_0+\mathcal{B}^{>}_{\infty}(u)\big)\cdot\exp(-\mathcal{C}_0)\cdot\exp(\mathcal{B}_\infty(v))\\
&\quad= \exp\big(\mathcal{B}_\infty(1)\big)\cdot\exp\big(\mathcal{C}_0+\mathcal{B}^{>}_{\infty}(u)+\mathcal{B}^{>}_{\infty}(v)\big).
\end{aligned}$$

In particular, we deduce that

$$\exp\big(\mathcal{B}_\infty(u+1)\big)\exp\big(\mathcal{B}_\infty(v)\big)=\exp\big(\mathcal{B}_\infty(v+1)\big)\exp\big(\mathcal{B}_\infty(u)\big). \qquad (4.58)$$

We now prove the main theorem of this section.

Proof of Theorem 4.6.1. We prove it by induction on k. For $k=1$ the result corresponds to Proposition 4.3.5. Fix now $k\geq 1$ and suppose that the theorem holds for $F^{(\alpha)}_k$. The induction hypothesis and Eq. (4.22) imply that

$$\begin{aligned}
&F^{(\alpha)}_{k+1}(-t,\boldsymbol{p},s_1,s_2,\dots,s_{k+1})\\
&\quad= \exp\big(\mathcal{B}_\infty(-\alpha s_1'-1)\big)\cdot\exp\big(\mathcal{B}_\infty(k-1)\big)\cdots\exp\big(\mathcal{B}_\infty(1)\big)\cdot\\
&\qquad \exp\Big(\mathcal{C}_0+\mathcal{B}^{>}_{\infty}(-\alpha s_2'-k-1)+\dots+\mathcal{B}^{>}_{\infty}(-\alpha s_{k+1}'-k-1)\Big)\cdot 1.
\end{aligned}$$

Using $k-1$ times Eq. (4.58), we obtain

$$\begin{aligned}
&F^{(\alpha)}_{k+1}(-t,\boldsymbol{p},s_1,s_2,\dots,s_{k+1})\\
&\quad= \exp(\mathcal{B}_\infty(k))\cdots\exp(\mathcal{B}_\infty(2))\cdot\exp\big(\mathcal{B}_\infty(-\alpha s_1'-k)\big)\cdot\\
&\qquad \exp\Big(\mathcal{C}_0+\mathcal{B}^{>}_{\infty}(-\alpha s_2'-k-1)+\dots+\mathcal{B}^{>}_{\infty}(-\alpha s_{k+1}'-k-1)\Big)\cdot 1.
\end{aligned}$$

Using Lemma 4.6.7, this can be rewritten as follows

$$\begin{aligned}
&F^{(\alpha)}_{k+1}(-t,\boldsymbol{p},s_1,s_2,\dots,s_{k+1})\\
&\quad= \exp(\mathcal{B}_\infty(k))\cdots\exp(\mathcal{B}_\infty(1))\cdot\exp(\mathcal{B}_\infty(-\alpha s_1'-k-1))\cdot\exp(-\mathcal{C}_0)\cdot\\
&\qquad \exp\Big(\mathcal{C}_0+\mathcal{B}^{>}_{\infty}(-\alpha s_2'-k-1)+\dots+\mathcal{B}^{>}_{\infty}(-\alpha s_{k+1}'-k-1)\Big)\cdot 1.
\end{aligned}$$

Finally, Lemma 4.6.6 allows us to apply Proposition 4.6.4 with the operators $\mathcal{B}^{>}_{\infty}(-\alpha s_1'-k-1)$, $\mathcal{B}^{>}_{\infty}(-\alpha s_2'-k-1)+\dots+\mathcal{B}^{>}_{\infty}(-\alpha s_{k+1}'-k-1)$ and $\mathcal{C}_0$ in order to reassemble the last three exponentials, which concludes the proof of the theorem. □

4.7 Proof of the main results

4.7.1 End of proof of Theorem 1.5.3

In this section, we finish the proof of Theorem 1.5.3 which gives a differential construction for Jack characters. We start by taking the limit of the generating series of k-layered maps. We recall that a map is said layered if it is k-layered for some $k\geq 1$.

Lemma 4.7.1. *The functions $F_{k+1}^{(\alpha)}$ satisfy the condition of Eq. (4.1):*

$$F_{k+1}^{(\alpha)}(t, \boldsymbol{p}, s_1, \dots, s_k, 0) = F_k^{(\alpha)}(t, \boldsymbol{p}, s_1, \dots, s_k). \tag{4.59}$$

As a consequence, the projective limit $F_\infty^{(\alpha)} := \varprojlim F_k^{(\alpha)}$ is well defined in $\mathcal{S}_\alpha^*[\![t, p_1, p_2, \dots]\!]$, and

$$F_\infty^{(\alpha)}(t, \boldsymbol{p}, s_1, s_2, \dots) = \sum_M (-t)^{|M|} p_{\lambda^\diamond(M)} \frac{b^{\vartheta_\rho(M)}}{2^{|\mathcal{V}_\bullet(M)| - \mathrm{cc}(M)} \alpha^{\mathrm{cc}(M)}} \prod_{i \geq 1} \frac{(-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}, \tag{4.60}$$

where the sum is taken over all layered maps.

Proof. From the combinatorial definition of the series $F_{k+1}^{(\alpha)}$ (see Eq. (4.21)), we know that $F_{k+1}^{(\alpha)}(t, \boldsymbol{p}, s_1, \dots, s_k, 0)$ corresponds to the series of $k+1$-layered maps with no white vertices in the layer $k+1$. But by definition such a map does not have black vertices in layer $k+1$ either and is then a k-layered map. As a consequence, $F_{k+1}^{(\alpha)}(t, \boldsymbol{p}, s_1, \dots, s_k, 0)$ is the generating series of k-layered maps. which gives Eq. (4.59).

The second part of the lemma is a consequence of Theorem 4.6.1 and Eq. (4.21). □

We now finish the proof of the first main result of the chapter.

Proof of Theorem 1.5.3. We want to prove that for any partition μ,

$$\theta_\mu^{(\alpha)}(s_1, s_2, \dots) = [t^{|\mu|} p_\mu] F_\infty^{(\alpha)}(t, \boldsymbol{p}, s_1, s_2, \dots). \tag{4.61}$$

This implies the first part of Theorem 1.5.3 by Eq. (4.54) and its second part by Eq. (4.60).

Fix a partition μ. In order to obtain Eq. (4.61), it is enough to prove that the coefficient of $t^{|\mu|} p_\mu$ in $F_\infty^{(\alpha)}(t, \boldsymbol{p}, \lambda)$ satisfies the conditions of Theorem 4.1.11. The fact that this coefficient vanishes on partitions λ of size $|\lambda| < |\mu|$ is given by Theorem 4.4.1, and we know that it is α-shifted symmetric from Lemma 4.7.1.

Let us now prove that the top homogeneous part in $[t^{|\mu|} p_\mu] F_k^{(\alpha)}(t, \boldsymbol{p}, \lambda)$ in the variables λ_i is equal to $\frac{\alpha^{|\mu| - \ell(\mu)}}{z_\mu} p_\mu(\lambda_1, \lambda_2, \dots, \lambda_k)$ for any $k \geq 1$. This part corresponds to vertex-labelled k-layered maps of face-type μ and with maximal number of white vertices, *i.e.* maps with $|\mu|$ white vertices which are all of degree 1. Note that adding a black vertex connected to n white vertices in a layer i of a map M corresponds to multiplying its global weight $\kappa(M)$ by $(-\alpha\lambda_i)^n p_n/\alpha$. Thus, in order to obtain the top homogeneous part in $F_k^{(\alpha)}$ we replace $\mathcal{B}_n(\boldsymbol{p}, \lambda_i)$ by $(-\alpha\lambda_i)^n p_n$ in Eq. (4.22). As a consequence, the top homogeneous part in $[t^{|\mu|} p_\mu] F_k^{(\alpha)}$ is given by

$$\begin{aligned}
[t^{|\mu|} p_\mu] \exp\left(\sum_{n \geq 1} \frac{(-t)^n \cdot (-\alpha\lambda_1)^n p_n}{\alpha n}\right) \cdots \exp\left(\sum_{n \geq 1} \frac{(-t)^n \cdot (-\alpha\lambda_k)^n p_n}{\alpha n}\right)& \\
= [t^{|\mu|} p_\mu] \exp\left(\sum_{n \geq 1} \frac{t^n \alpha^{n-1} \cdot p_n(\lambda_1, \dots, \lambda_k) p_n}{n}\right)& \\
= \frac{\alpha^{|\mu| - \ell(\mu)}}{z_\mu} p_\mu(\lambda_1, \dots, \lambda_k).&
\end{aligned}$$

□

4.7.2 Proof of Theorem 1.5.5

In this section, we give a direct way to construct Jack polynomials $J_\lambda^{(\alpha)}$ by adding the rows of λ in increasing order of their size. In some sense, it is a simplified version of the one given in Eq. (1.50).

Theorem 4.7.2. *Fix a partition λ. Let $\mu := \lambda \backslash \lambda_1$ be the partition obtained from λ by removing the largest part. Then*

$$\begin{aligned} J_\lambda^{(\alpha)} &= [t^{\lambda_1}] \exp\left(\mathcal{B}_\infty(-t, \boldsymbol{p}, -\alpha\lambda_1)\right) \cdot J_\mu^{(\alpha)} \\ &= \sum_{\ell \geq 1} \sum_{\substack{n_1, \dots, n_\ell \geq 1 \\ n_1 + \cdots + n_\ell = \lambda_1}} \frac{1}{\ell!} \frac{\mathcal{B}_{n_1}(\boldsymbol{p}, -\alpha\lambda_1)}{n_1} \cdots \frac{\mathcal{B}_{n_\ell}(\boldsymbol{p}, -\alpha\lambda_1)}{n_\ell} \cdot J_\mu^{(\alpha)}. \end{aligned}$$

Proof. From the definition of the Jack characters and Eq. (1.50) (proved in the previous subsection), we have

$$\begin{aligned} J_\lambda^{(\alpha)} &= [t^{|\lambda|}] \exp\left(\mathcal{B}_\infty(-t, \boldsymbol{p}, -\alpha\lambda_1)\right) \cdots \exp\left(\mathcal{B}_\infty(-t, \boldsymbol{p}, -\alpha\lambda_{\ell(\lambda)})\right) \cdot 1 \\ &= \sum_{\substack{n_1, \dots, n_{\ell(\lambda)} \geq 1 \\ n_1 + \cdots + n_{\ell(\lambda)} = |\lambda|}} \left([t^{n_1}] \exp\left(\mathcal{B}_\infty(-t, \boldsymbol{p}, -\alpha\lambda_1)\right)\right) \cdots \left([t^{n_{\ell(\lambda)}}] \exp\left(\mathcal{B}_\infty(-t, \boldsymbol{p}, -\alpha\lambda_{\ell(\lambda)})\right)\right) \cdot 1. \end{aligned}$$

But from the proof of Theorem 4.4.1, the only choice for $(n_i)_{1 \leq i \leq \ell(\lambda)}$ that does not give a null term is $n_i = \lambda_i$ for all $1 \leq i \leq \ell(\lambda)$. Indeed, otherwise there exists i such that $n_i > \lambda_i$, and from Eq. (4.28), we have

$$\left([t^{n_i}] \exp\left(\mathcal{B}_\infty(-t, \boldsymbol{p}, -\alpha\lambda_i)\right)\right) \cdots \left([t^{n_{\ell(\lambda)}}] \exp\left(\mathcal{B}_\infty(-t, \boldsymbol{p}, -\alpha\lambda_{\ell(\lambda)})\right)\right) \cdot 1 = 0.$$

As a consequence, we obtain

$$J_\lambda^{(\alpha)} = \left([t^{\lambda_1}] \exp\left(\mathcal{B}_\infty(-t, \boldsymbol{p}, -\alpha\lambda_1)\right)\right) \cdots \left([t^{\lambda_{\ell(\lambda)}}] \exp\left(\mathcal{B}_\infty(-t, \boldsymbol{p}, -\alpha\lambda_{\ell(\lambda)})\right)\right) \cdot 1. \quad (4.62)$$

Comparing this expression for the partitions λ and μ we obtain the first part of the equation of the theorem. To obtain the second part of the equation we expand the exponential. □

4.7.3 Positivity in Lassalle's conjecture

We now prove positivity in Lassalle's conjecture. This finishes the proof of Lassalle's conjecture Theorem 4.1.7, since integrality has been proved in Section 4.2.

We consider two sequences of variables $s_1, \dots, s_k$ and $r_1, \dots, r_k$. We define the generating series $\widetilde{F}_k^{(\alpha)}\left(t, \boldsymbol{p}, \begin{matrix} s_1 & \dots & s_k \\ r_1 & \dots & r_k \end{matrix}\right)$ inductively by $F_0^{(\alpha)} = 1$ and for every $k \geq 0$

$$\widetilde{F}_{k+1}^{(\alpha)}\left(t, \boldsymbol{p}, \begin{matrix} s_1 & \dots & s_{k+1} \\ r_1 & \dots & r_{k+1} \end{matrix}\right) = \exp\left(r_1 \mathcal{B}_\infty(-t, \boldsymbol{p}, -\alpha s_1)\right) \cdot \widetilde{F}_k^{(\alpha)}\left(t, \boldsymbol{p}, \begin{matrix} s_2 & \dots & s_{k+1} \\ r_2 & \dots & r_{k+1} \end{matrix}\right).$$

Hence the functions $F_k^{(\alpha)}$ defined in Eq. (4.22) are obtained as specializations of $\widetilde{F}_k^{(\alpha)}$:

$$F_k^{(\alpha)}(t, \boldsymbol{p}, s_1, \dots, s_k) = \widetilde{F}_k^{(\alpha)}\left(t, \boldsymbol{p}, \begin{matrix} s_1 & \dots & s_k \\ 1 & \dots & 1 \end{matrix}\right).$$

Using the combinatorial interpretation for the operator $\mathcal{B}_\infty$ from Section 2.2, the argument that allowed us to interpret the function $F_k^{(\alpha)}$ also gives the following interpretation for $\widetilde{F}_k^{(\alpha)}$:

$$\widetilde{F}_k^{(\alpha)}\left(t, \boldsymbol{p}, \begin{array}{ccc} s_1 & \dots & s_k \\ r_1 & \dots & r_k \end{array}\right) = \sum_M \frac{(-t)^{|M|} p_{\lambda^\circ(M)} b^{\vartheta_\rho(M)}}{2^{|\mathcal{V}_\bullet(M)|-\mathrm{cc}(M)} \alpha^{\mathrm{cc}(M)}} \prod_{1 \le i \le k} \frac{r_i^{|\mathcal{V}_\bullet^{(i)}(M)|} (-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}, \tag{4.63}$$

where the sum runs over k-layered maps.

The two following properties follow from the definition of the functions $\widetilde{F}_k^{(\alpha)}$:

(i) For every $1 \le i \le k-1$,

$$\widetilde{F}_k^{(\alpha)}\left(t, \boldsymbol{p}, \begin{array}{ccc} s_1 & \dots & s_k \\ r_1 & \dots & r_k \end{array}\right)\Bigg|_{s_i = s_{i+1} = s} = \widetilde{F}_{k-1}^{(\alpha)}\left(t, \boldsymbol{p}, \begin{array}{cccccc} s_1 & \dots & s & s_{i+2} & \dots & s_k \\ r_1 & \dots & r_i + r_{i+1} & r_{i+2} & \dots & r_k \end{array}\right).$$

(ii) For every $1 \le i \le k$,

$$\widetilde{F}_k^{(\alpha)}\left(t, \boldsymbol{p}, \begin{array}{ccc} s_1 & \dots & s_k \\ r_1 & \dots & r_k \end{array}\right)\Bigg|_{r_i = 0} = \widetilde{F}_{k-1}^{(\alpha)}\left(t, \boldsymbol{p}, \begin{array}{cccccc} s_1 & \dots & s_{i-1} & s_{i+1} & \dots & s_k \\ r_1 & \dots & r_{i-1} & r_{i+1} & \dots & r_k \end{array}\right)$$

We deduce the following proposition.

Proposition 4.7.3. *Let λ be a partition, and let $\left(\begin{array}{ccc} s_1 & \dots & s_k \\ r_1 & \dots & r_k \end{array}\right)$ be multirectangular coordinates of λ, i.e. $\lambda = [s_1^{r_1}, \dots, s_k^{r_k}]$. Then*

$$\widetilde{F}_k^{(\alpha)}\left(t, \boldsymbol{p}, \begin{array}{ccc} s_1 & \dots & s_k \\ r_1 & \dots & r_k \end{array}\right) = F_{\ell(\lambda)}^{(\alpha)}(t, \boldsymbol{p}, \lambda_1, \dots, \lambda_{\ell(\lambda)}).$$

In particular, the quantity $\widetilde{F}_k^{(\alpha)}\left(t, \boldsymbol{p}, \begin{array}{ccc} s_1 & \dots & s_k \\ r_1 & \dots & r_k \end{array}\right)$ does not depend on the multirectangular coordinates of λ chosen.

Proof. We start by removing all the pairs (s_i, r_i) for which $r_i = 0$, using property (ii) above. Then, we use property (i) in order to decrease the remaining coordinates r_i until they are all equal to 1. More precisely, we apply multiple times the following equation

$$\widetilde{F}_k^{(\alpha)}\left(t, \boldsymbol{p}, \begin{array}{ccc} s_1 & \dots & s_k \\ r_1 & \dots & r_k \end{array}\right) = \widetilde{F}_{k+1}^{(\alpha)}\left(t, \boldsymbol{p}, \begin{array}{cccccc} s_1 & \dots & s_i & s_i & \dots & s_k \\ r_1 & \dots & r_i - 1 & 1 & \dots & r_k \end{array}\right).$$

Note that in each one of these operations the new coordinates obtained are also multirectangular coordinates for λ. Hence when $r_i = 1$ for every i, we have $k = \ell(\lambda)$ and $s_i = \lambda_i$ for each $1 \le i \le \ell(\lambda)$. $\square$

As a consequence of Theorem 1.5.3 and Proposition 4.7.3, we obtain that for any partition μ

$$\tilde{\theta}_\mu^{(\alpha)}(\boldsymbol{s}, \boldsymbol{r}) = [t^{|\mu|} p_\mu](-1)^{|\mu|} \widetilde{F}_k^{(\alpha)}\left(t, \boldsymbol{p}, \begin{array}{ccc} s_1 & \dots & s_k \\ r_1 & \dots & r_k \end{array}\right), \tag{4.64}$$

where $\boldsymbol{s} = (s_1 \dots, s_k, 0, \dots)$ and $\boldsymbol{r} = (r_1, \dots, r_k, 0, \dots)$. We now prove the positivity in Lassalle's conjecture.

Proof of positivity in Theorem 1.5.6. Comparing Eq. (4.63) and Eq. (4.64) and taking the limit as $k \to \infty$ we get

$$\tilde{\theta}_{\mu}^{(\alpha)}(\boldsymbol{s}, \boldsymbol{r}) = \sum_{M} \frac{b^{\vartheta_\rho(M)}}{2^{|\mathcal{V}_\bullet(M)|-\mathrm{cc}(M)}\alpha^{\mathrm{cc}(M)}} \prod_{i\geq 1} \frac{r_i^{|\mathcal{V}_\bullet^{(i)}(M)|}(-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}, \tag{4.65}$$

where the sum is taken over layered maps M of face-type μ. Since every connected component of a bipartite map contains at least one white vertex, the α-term which appears in the denominator is compensated. This concludes the proof of the theorem. □

4.7.4 A new formula of Lassalle's isomorphism

We conclude this chapter by a new formula for Lassalle's isomorphism (see Eq. (4.2)) between $\mathcal{S}_\alpha$ and $\mathcal{S}_\alpha^*$, which maps Jack polynomials to shifted Jack polynomials.

Theorem 4.7.4. *Let $f \in \mathcal{S}_\alpha$ and fix $k \geq 1$. Then,*

$$f^*(s_1, \dots, s_k) = \langle f, \exp\left(\mathcal{B}_\infty(-1/\alpha, \boldsymbol{p}, -\alpha s_1)\right) \dots \exp\left(\mathcal{B}_\infty(-1/\alpha, \boldsymbol{p}, -\alpha s_k)\right) \cdot 1\rangle. \tag{4.66}$$

Proof. We recall that for any partition μ we have

$$\theta_\mu^{(\alpha)} = \frac{\alpha^{|\mu|-\ell(\mu)}}{z_\mu} p_\mu^*,$$

see Lemma 4.1.9. Combining this equation with Eq. (1.50), we get

$$\alpha^{|\mu|} p_\mu^*(s_1, \dots, s_k) = \langle p_\mu, [t^{|\mu|}] \exp\left(\mathcal{B}_\infty(-t, \boldsymbol{p}, -\alpha s_1)\right) \dots \exp\left(\mathcal{B}_\infty(-t, \boldsymbol{p}, -\alpha s_k)\right) \cdot 1\rangle.$$

We deduce that

$$p_\mu^*(s_1, \dots, s_k) = \langle p_\mu, \exp\left(\mathcal{B}_\infty(-1/\alpha, \boldsymbol{p}, -\alpha s_1)\right) \dots \exp\left(\mathcal{B}_\infty(-1/\alpha, \boldsymbol{p}, -\alpha s_k)\right) \cdot 1\rangle,$$

which gives the claimed formula for $f = p_\mu$. We conclude using the fact that power-sum symmetric functions form a basis of $\mathcal{S}_\alpha$ and that both sides of Eq. (4.66) are linear in f. □

This theorem can be thought of as a generalization of Theorem 4.1.7, since it gives a formula for f^* as a shifted symmetric function and not only its evaluation at Young diagrams.

Chapter 5

Differential equations for the generating series of maps with controlled profile

This chapter is based on [Ben24].

A very classical problem in map enumeration consists in establishing differential equations for the generating series of maps. Such equations can be used to obtain recursion formulas for the number of maps satisfying given properties. We are interested here in the generating series of maps with control of the three partitions of the profile in the orientable and the non-orientable case, and more generally for their deformation given by the function $\tau^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r})$. The specializations of these series consisting in keeping two alphabets $\boldsymbol{p}$ and $\boldsymbol{q}$ and replacing the alphabet $\boldsymbol{r}$ by a single variable u as in Eq. (1.27) have been well studied and are known to satisfy differential equations related to the integrable hierarchies of KP and 2-Toda in the orientable case, and BKP in the non-orientable one, see *e.g.* [AvM01]. Moreover, Theorem 2.3.1 of Chapuy–Dołęga gives a family of decomposition equations satisfied by these specialized series. However, we are not aware of any differential equations satisfied by the full generating series of hypermaps, *i.e.* without any specialization of the variables.

In order to understand the recursive structure of a family of coefficients indexed by three partitions of the same size, it is sometimes convenient to start by considering a generalized family of coefficients indexed by partitions of arbitrary size (see [AF17] for an example of application of this idea). It turns out that one way to make such a generalization in the case of the coefficients $c^{\pi}_{\mu,\nu}$ is to consider the structure coefficients $g^{\pi}_{\mu,\nu}$ of the Jack characters $\theta^{(\alpha)}_{\mu}$ defined by:

$$\theta^{(\alpha)}_{\mu}\theta^{(\alpha)}_{\nu} = \sum_{\pi} g^{\pi}_{\mu,\nu}(\alpha)\theta^{(\alpha)}_{\pi}. \tag{5.1}$$

When $|\pi| = |\mu| = |\nu|$, we recover the coefficients $c^{\pi}_{\mu,\nu}$ by Proposition 1.3.21.

We recall from Eq. (1.53), that the associated series $G^{(\alpha)}$ is given by

$$G^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) := \sum_{\pi,\mu,\nu} \frac{g^{\pi}_{\mu,\nu}(\alpha)}{z_{\pi}\alpha^{\ell(\pi)}} t^{|\mu|+|\nu|-|\pi|} p_{\pi} q_{\mu} r_{\nu}. \tag{5.2}$$

The main result of this chapter is to establish a family of differential equations for the series $G^{(\alpha)}$, using the operators $\mathcal{B}_{\infty}$; see Theorem 5.1.6 below. We prove that these equations characterize the series and we use it to obtain a recursive formula for the coefficients $g^{\pi}_{\mu,\nu}$ in Eq. (5.40).

The obtained family of equations is new even in the cases $\alpha = 1$ and $\alpha = 2$, for which $G^{(\alpha)}$ has an interpretation in terms of bipartite maps with controlled profile and with some marked vertices of degree 1; see Proposition 5.1.5.

The proof is based on the differential construction of Jack characters given in Eq. (1.50). We also provide a combinatorial proof in the case $\alpha = 1$ using a family of maps with both vertices and faces colored.

The approach used here to prove the main theorem applies also to the case of constellations and allows us to extend the differential equation to series with k alphabets; see Theorem 5.3.5. In other words, we obtain an equation for the generating function of Hurwitz numbers (and their α-deformations) with control of *full ramification profiles* above an *arbitrary* number of points.

As a byproduct of our approach, we prove integrality in Śniady's conjecture (Conjecture 9) from integrality in the Matching-Jack conjecture; see Corollary 5.2.2. We also prove the positivity of the top degree terms in this conjecture; see Corollary 5.1.13.

In this chapter, we consider generating series of hypermaps not necessarily connected. Nevertheless, it is possible to obtain the generating series of connected hypermaps by taking a logarithm as explained in Section 1.3.1. More precisely, the series

$$\widehat{G}^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) = \alpha \cdot \log(G^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}))$$

is an α-deformation of the generating series of connected maps. The homogeneous part of this series coincides with the series $\widehat{\tau}^{(\alpha)}$ defined in Eq. (1.23) and which is the object of the b-conjecture. In Theorem 5.6.5, we derive from the main theorem a differential equation for $\widehat{G}^{(\alpha)}$.

5.1 Definitions and main results

Before stating the main result of this chapter, we start by giving a combinatorial interpretation of the series $G^{(\alpha)}$ in terms of maps when $\alpha \in \{1, 2\}$.

Actually, the combinatorial interpretation of $c^{\pi}_{\mu,\nu}(\alpha)$ when $\alpha \in \{1, 2\}$ in terms of bipartite maps (see Eqs. (1.30) and (1.31)), can be extended to $g^{\pi}_{\mu,\nu}(\alpha)$ which also count bipartite maps with some marked vertices of degree 1. It will be convenient to give this interpretation using hypermaps rather than bipartite maps, the two of them being related by duality. The proof is postponed to Section 5.2.3.

5.1.1 Generating series of hypermaps

A *hypermap* is a map whose faces are colored in two colors $(+)$ and $(-)$, and such that each edge is incident to two faces of different colors. Usually the faces of one color are called *hyperedges*, and the faces of the other color are the *faces* of the hypermap.

Hypermaps were first introduced by Cori in [Cor75] and are in bijection with bipartite maps by duality [Wal75]. We refer to [Lou20, Section 1.1] or [CD22, Definition 2.4] for more details.

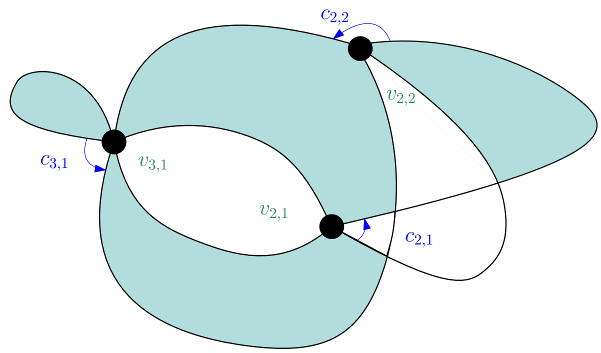


Figure 5.1: An example of an oriented vertex-labelled hypermap of profile $([3,2,2],[5,2],[6,1])$. Faces of color $(-)$ are represented in blue, the root of the vertex $v_{d,i}$ is denoted by $c_{d,i}$.

We start by defining the notions of labelling and profiles for hypermaps in a similar way to that for bipartite maps.

Definition 5.1.1. *We say that a hypermap (orientable or not) is vertex-labelled if:*

1. *for each $d \geq 1$, vertices of same degree*[1] *$2d$ are labelled $v_{d,1}, v_{d,2}, \dots$,*
2. *each vertex has a marked oriented corner in a face colored $(+)$. This corner is called the vertex root.*

A vertex-labelled hypermap is ***oriented*** *if the map is orientable and the orientation endowed by the vertex roots are consistent (see Fig. 1.6). If M is hypermap, then we associate to it three integer partitions of size $|M|$:*

- *its vertex-type, denoted $\lambda^\bullet(M)$, is the partition obtained by reordering the vertex degrees divided by 2.*
- *its $(+)$ type, denoted $\lambda^+(M)$, is the partition obtained by reordering the degrees of the $(+)$ faces.*
- *its $(-)$ type, denoted $\lambda^-(M)$, is the partition obtained by reordering the degrees of the $(-)$ faces.*

The profile of M, is then the tuple of partitions $(\lambda^\bullet(M), \lambda^+(M), \lambda^-(M))$.

We give in Fig. 5.1 an example of an oriented vertex-labelled hypermap.

Remark 5.1.2. The duality mentioned above between bipartite maps and hypermaps preserves the profiles; it sends a bipartite map of profile (π, μ, ν) to a hypermap of profile (μ, π, ν) (we use here the convention of Section 1.1.3 for the profile of a bipartite map).

Understanding hypermaps with controlled profile is a hard combinatorial problem. Indeed, usual techniques to enumerate maps cannot be used to control the three partitions of the

[1] When we turn around a vertex in a hypermap, colors $(+)$ and $(-)$ alternate. By consequence, each vertex has necessarily even degree.

profile and the only known results in this direction are obtained using representation theory tools as explained in Section 1.3.1. We reformulate here these results in terms of hypermaps.

First, we consider the two generating series

$$H^{(1)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) = \sum_{M} \frac{t^{|M|}}{z_{\lambda^\bullet(M)}} p_{\lambda^\bullet(M)} q_{\lambda^+(M)} r_{\lambda^-(M)}, \tag{5.3}$$

$$H^{(2)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) = \sum_{M} \frac{t^{|M|}}{2^{\ell(\lambda^\bullet(M))} z_{\lambda^\bullet(M)}} p_{\lambda^\bullet(M)} q_{\lambda^+(M)} r_{\lambda^-(M)}, \tag{5.4}$$

where the sum runs over vertex-labelled oriented (resp. oriented or not) hypermaps in Eq. (5.3) (resp. Eq. (5.4)).

The formulas given in Section 1.3.1 can be reformulated as follows.

Theorem 5.1.3 ([GJ96b]). *Fix three partitions π, μ and ν of the same size. For $\alpha = 1$ (resp. $\alpha = 2$), the coefficient $c^{\pi}_{\mu,\nu}(1)$ (resp. $c^{\pi}_{\mu,\nu}(2)$) counts the number of oriented (resp. oriented or not) vertex-labelled hypermaps of profile (π,μ,ν).*

Equivalently, for $\alpha \in \{1,2\}$ we have

$$\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) = H^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}).$$

Proof. We say that a bipartite map is face-labelled if faces of the same degree $d \geq 1$ are numbered by 1,2,..., and such that each face has a marked oriented black corner.

Using the argument of Lemma 1.3.4, we obtain the following variant of Eqs. (1.30) and (1.31):

$$c^{\pi}_{\mu,\nu}(1) = \left|\{\text{Oriented face-labelled bipartite maps of profile } (\mu,\pi,\nu)\}\right|, \tag{5.5}$$

$$c^{\pi}_{\mu,\nu}(2) = \left|\{\text{Face-labelled bipartite maps of profile } (\mu,\pi,\nu),\ \text{oriented or not}\}\right|. \tag{5.6}$$

We now apply the duality explained in Remark 5.1.2. Since this duality sends faces of bipartite maps into vertices of hypermaps, the image of a face-labelled map of profile (μ,π,ν) is a vertex-labelled hypermap of profile (π,μ,ν). Hence, Eqs. (5.5) and (5.6) become

$$c^{\pi}_{\mu,\nu}(1) = \left|\{\text{Oriented vertex-labelled hypermaps of profile } (\pi,\mu,\nu)\}\right|,$$

$$c^{\pi}_{\mu,\nu}(2) = \left|\{\text{Vertex-labelled hypermaps of profile } (\pi,\mu,\nu),\ \text{oriented or not}\}\right|$$

as claimed. □

In order to have a similar interpretation for the series $G^{(\alpha)}$, we introduce a family of hypermaps with marked faces.

Definition 5.1.4. *Let π, μ and ν be three partitions. We denote by $\mathcal{OH}^{\pi}_{\mu,\nu}$ (resp. $\mathcal{H}^{\pi}_{\mu,\nu}$) the set of vertex-labelled oriented hypermaps (resp. oriented hypermaps or not) of profile $(\pi, \mu\cup 1^{|\pi|-|\mu|}, \nu\cup 1^{|\pi|-|\nu|})$, with $|\pi|-|\mu|$ marked $(+)$ faces of degree 1, $|\pi|-|\nu|$ marked $(-)$ faces of degree 1, with the condition that in an isolated loop[2], we cannot have both the $(+)$ face and the $(-)$ face marked. By definition, $\mathcal{OH}^{\pi}_{\mu,\nu}$ and $\mathcal{H}^{\pi}_{\mu,\nu}$ are empty when $|\pi| < \max(|\mu|,|\nu|)$.*

[2]An isolated loop is a connected component of a hypermap with exactly one edge and one vertex. It has necessarily two faces, one colored $(+)$ and one colored $(-)$.

We then define the generating series of hypermaps with marked faces:

$$\widetilde{H}^{(1)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) = \sum_{\pi,\mu,\nu} \frac{|\mathcal{OH}^{\pi}_{\mu,\nu}|}{z_\pi} t^{|\mu|+|\nu|-|\pi|} p_\pi q_\mu r_\nu,$$

$$\widetilde{H}^{(2)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) = \sum_{\pi,\mu,\nu} \frac{|\mathcal{H}^{\pi}_{\mu,\nu}|}{2^{\ell(\pi)} z_\pi} t^{|\mu|+|\nu|-|\pi|} p_\pi q_\mu r_\nu.$$

The integer $|\mu|+|\nu|-|\pi|$ corresponds to the number of edges of the hypermap which are not incident to a marked face. It is straightforward from the definitions that the series $H^{(1)}$ and $H^{(2)}$ are respectively the homogeneous parts of $\widetilde{H}^{(1)}$ and $\widetilde{H}^{(2)}$. Conversely, $\widetilde{H}^{(1)}$ and $\widetilde{H}^{(2)}$ can be obtained from $H^{(1)}$ and $H^{(2)}$ by simple operations (see Eq. (5.16)).

The series $G^{(\alpha)}$ of structure coefficients is actually an α-deformation of the generating series of hypermaps with marked faces (see Section 5.2.3 for a proof).

Proposition 5.1.5. *For $\alpha \in \{1,2\}$, we have*

$$G^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) = \widetilde{H}^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}).$$

5.1.2 The main theorem

We recall that the operator $\mathcal{B}_\infty(t,\boldsymbol{p},u)$ is defined by

$$\mathcal{B}_\infty(t,\boldsymbol{p},u) := \sum_{n\geq 1} \frac{t^n}{n} \mathcal{B}_n(\boldsymbol{p},u),$$

where $\mathcal{B}_n$ is the operator defined in Eq. (1.48). Moreover, $\mathcal{B}_n^\perp(\boldsymbol{p},u)$ is the adjoint operator of $\mathcal{B}_n(\boldsymbol{p},u)$ with respect to the scalar product of $\mathcal{S}_\alpha$. From Example 1.4.1, we have

$$\mathcal{B}_1(\boldsymbol{p},u) = \frac{up_1}{\alpha} + \sum_{i\geq 1} p_{i+1} \frac{i\partial}{\partial p_i},$$

implying

$$\mathcal{B}_1^\perp(\boldsymbol{p},u) = \frac{u\partial}{\partial p_1} + \sum_{i\geq 1} p_i \frac{(i+1)\partial}{\partial p_{i+1}}.$$

We use here the duality relations of Lemma 1.2.4. Furthermore, one gets by linearity that

$$\mathcal{B}_\infty^\perp(t,\boldsymbol{p},u) = \sum_{n\geq 1} \frac{t^n}{n} \mathcal{B}_n^\perp(\boldsymbol{p},u).$$

In Section 5.6.2, we derive a catalytic expression of $\mathcal{B}_n^\perp$ from the one defining $\mathcal{B}_n$. We now state the main theorem of this chapter.

Theorem 5.1.6. *The function $G^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})$ satisfies the following equation*

$$\left(\mathcal{B}_\infty(-t,\boldsymbol{q},u) + \mathcal{B}_\infty(-t,\boldsymbol{r},u)\right) \cdot G^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) = \mathcal{B}_\infty^\perp(-t,\boldsymbol{p},u) \cdot G^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}). \quad (5.7)$$

By extracting coefficients in the variable u, Eq. (5.7) can be alternatively written as a family of equations (independent of u) which are indexed by non-negative integers; see Section 5.3.4.

The fact that the action of the operators $\mathcal{B}_\infty^\perp(-t,\boldsymbol{p},u)$, $\mathcal{B}_\infty(-t,\boldsymbol{q},u)$ and $\mathcal{B}_\infty(-t,\boldsymbol{r},u)$ on $G^{(\alpha)}$ is well defined is not obvious and is related to some properties of this series that we now explain. We start by proving the following lemma.

Lemma 5.1.7. *The coefficient $g^\pi_{\mu,\nu}$ is zero unless* $\max(|\mu|,|\nu|) \le |\pi| \le |\mu|+|\nu|$.

Proof. Since $g^\pi_{\mu,\nu}$ are the structure coefficients of Jack characters (see Eq. (5.1)), the upper bound is a direct consequence of the fact that $\theta^{(\alpha)}_\mu$ is a shifted symmetric function of degree $|\mu|$ and the fact that $(\theta^{(\alpha)}_\pi)_{|\pi|\le d}$ is a basis of shifted symmetric functions of degree less or equal than d; see Theorem 4.1.11. In order to obtain the lower bound we use the vanishing properties of $\theta^{(\alpha)}_\mu$. Fix two partitions μ and ν. Set $m := \max(|\mu|,|\nu|)$ and

$$F := \theta^{(\alpha)}_\mu \theta^{(\alpha)}_\nu - \sum_{m\le|\pi|\le|\mu|+|\nu|} g^\pi_{\mu,\nu}\theta^{(\alpha)}_\pi \tag{5.8}$$

$$= \sum_{|\pi|<m} g^\pi_{\mu,\nu}\theta^{(\alpha)}_\pi. \tag{5.9}$$

From Eq. (5.9), the function F is shifted symmetric with degree at most $m-1$. Moreover, using Eq. (5.8) and the definition of Jack characters (Eq. (1.40)), we get that $F(\lambda)=0$ for any $|\lambda|<m$. Applying Theorem 4.1.3 and using the fact that $\theta^{(\alpha)}$ are linearly independent in $\mathcal{S}_\alpha$, we deduce that $F=0$ and that $g^\pi_{\mu,\nu}=0$ for any $|\pi|<m$. □

We get from Lemma 5.1.7 that $G^{(\alpha)}$ belongs to the algebra $\mathcal{A} := \mathbb{Q}(\alpha)[t,\boldsymbol{p}][\![\boldsymbol{q},\boldsymbol{r}]\!]$. This is the space of formal power series in the variables q_i and r_i, whose coefficients are polynomial in t and p_i. Similarly, the series $G^{(\alpha)}$ is well defined in $\mathbb{Q}(\alpha)[t,\boldsymbol{q},\boldsymbol{r}][\![\boldsymbol{p}]\!]$.

Moreover, we have the following lemma.

Lemma 5.1.8. *The three operators $\mathcal{B}_\infty(-t,\boldsymbol{q},u)$, $\mathcal{B}_\infty(-t,\boldsymbol{r},u)$ and $\mathcal{B}^\perp_\infty(-t,\boldsymbol{p},u)$ are well defined as operators from $\mathcal{A}$ to $\mathcal{A}[\![u]\!]$.*

Proof. First, it follows from the definition of the operator $\mathcal{B}_n(\boldsymbol{p},u)$ (see Eq. (1.48)) that it is a homogeneous operator of degree n. More precisely, for any λ, we have that $\mathcal{B}_n(\boldsymbol{p},u)\cdot p_\lambda$ is a linear combination of $u^\ell p_\mu$ for μ of size $|\lambda|+n$ and $\ell\le n$. Hence, if $n\le|\lambda|$, then $\mathcal{B}^\perp_n(\boldsymbol{p},u)\cdot p_\lambda$ is a linear combination of $u^\ell p_\mu$ for μ of size $|\lambda|-n$ and $\ell\le n$. Moreover, if $|\lambda|<n$ then $\mathcal{B}^\perp_n(\boldsymbol{p},u)\cdot p_\lambda=0$.

We deduce that $\mathcal{B}_\infty(t,\boldsymbol{p},u)\cdot p_\lambda$ is a combination of $t^{|\mu|-|\lambda|}u^\ell p_\mu$ for $|\mu|\ge|\lambda|$ and $\ell\ge 0$. Moreover, $\mathcal{B}^\perp_\infty(t,\boldsymbol{p},u)\cdot p_\lambda$ is a linear combination of $t^{|\lambda|-|\mu|}p_\mu u^\ell$ for $|\mu|\le|\lambda|$ and $\ell\ge0$. Consequently, operators

$$\mathcal{B}_\infty(t,\boldsymbol{p},u) : \mathbb{Q}(\alpha)[t][\![\boldsymbol{p}]\!] \longrightarrow \mathbb{Q}(\alpha)[t][\![\boldsymbol{p},u]\!]$$

$$\mathcal{B}^\perp_\infty(t,\boldsymbol{p},u) : \mathbb{Q}(\alpha)[t,\boldsymbol{p}] \longrightarrow \mathbb{Q}(\alpha)[t,\boldsymbol{p}][\![u]\!]$$

are well defined. We deduce that operators $\mathcal{B}_\infty(t,\boldsymbol{q},u), \mathcal{B}_\infty(t,\boldsymbol{r},u)$ and $\mathcal{B}^\perp_\infty(t,\boldsymbol{p},u)$ are well defined from $\mathcal{A}$ to $\mathcal{A}[\![u]\!]$. □

Combining Lemma 5.1.7 and Lemma 5.1.8 we deduce that both sides of Eq. (5.7) are well defined in $\mathcal{A}[\![u]\!]$.

In Theorem 5.5.1, we solve the differential equation given in Theorem 5.1.6 to obtain an explicit expression of the coefficients $g^{\pi}_{\mu,\nu}$ in terms of a family of coefficients a^{λ}_{μ} obtained using the operator $\mathcal{B}_\infty$, which have combinatorial interpretation in terms of maps.

5.1.3 The operator $\mathcal{G}$ and a reformulation of the main result

It may be more convenient to think of the differential equation Eq. (5.7) as a commutation relation that we now explain. Let $\mathcal{G}^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})$ be the operator defined by

$$\begin{array}{rccc} \mathcal{G}^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}): & \mathbb{Q}(\alpha)[\boldsymbol{p}] & \longrightarrow & \mathbb{Q}(\alpha)[\boldsymbol{q},\boldsymbol{r}][t] \\ & p_\pi & \longmapsto & \displaystyle\sum_{\mu,\nu} t^{|\mu|+|\nu|-|\pi|} g^{\pi}_{\mu,\nu}(\alpha) q_\mu r_\nu. \end{array} \tag{5.10}$$

As for operators $\mathcal{B}_n$, we write $\mathcal{G} \equiv \mathcal{G}^{(\alpha)}$ unless we are considering specializations in α as in Section 5.4.

The following is a variant of the main theorem; see Section 5.3.2 for the proof.

Theorem 5.1.9. *We have the following relation*

$$(\mathcal{B}_\infty(-t,\boldsymbol{q},u)+\mathcal{B}_\infty(-t,\boldsymbol{r},u))\cdot\mathcal{G}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) = \mathcal{G}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\cdot\mathcal{B}_\infty(-t,\boldsymbol{p},u) \tag{5.11}$$

between operators from $\mathbb{Q}(\alpha)[\boldsymbol{p}]$ *to* $\mathbb{Q}(\alpha)[\boldsymbol{q},\boldsymbol{r},u][\![t]\!]$.

It turns out that for $\alpha = 1$, $\mathcal{G}$ is a hypermap construction operator, which acts on a map by adding edges and coloring faces. In Section 5.4, we use this interpretation to give a combinatorial proof of Theorem 5.1.9 for $\alpha = 1$.

5.1.4 Integrality and top degree terms in Śniady's conjecture

We conclude this section by giving some consequences of the main theorem of the chapter. The proofs will be given later.

We recall that Śniady's conjecture (Conjecture 9) suggests that the coefficients $g^{\pi}_{\mu,\nu}$ are non-negative integer polynomials in the parameter b.

In Corollary 5.2.2, we deduce the integrality part in Conjecture 9 from the integrality in the Matching-Jack conjecture proved in Chapter 3, by expressing the coefficients $g^{\pi}_{\mu,\nu}$ in terms of the coefficients $c^{\pi}_{\mu,\nu}$.

Unfortunately, we have not been able to use the formula for the coefficients $g^{\pi}_{\mu,\nu}$ provided in Section 5.5 to prove the positivity in b, the remaining part in Śniady's conjecture. It is however possible to use Theorem 5.1.9 to prove the top degree cases in the conjecture. Let us start by an example.

Example 5.1.10. We have the following formula for the top structure coefficients; if $|\pi| = |\mu| + |\nu|$, then

$$g^{\pi}_{\mu,\nu} = \begin{cases} \frac{z_{\mu\cup\nu}}{z_\mu . z_\nu} = \prod_{i\geq 1}\binom{m_i(\mu)+m_i(\nu))}{m_i(\mu)} & \text{if } \pi = \mu\cup\nu \\ 0 & \text{otherwise.} \end{cases}$$

To obtain this formula, we use the fact that the top homogeneous part of $\theta_\mu^{(\alpha)}$ as a shifted symmetric function is $\alpha^{|\mu|-\ell(\mu)}p_\mu/z_\mu$ (see Section 4.1). The formulas given in Theorem 5.1.11 below allow one to compute these coefficients when $|\pi| \geq |\mu| + |\nu| - 2$.

We first consider the homogeneous parts of the operators $\mathcal{G}$ defined for any $k \geq 0$ by

$$\mathcal{G}_k \cdot p_\pi = \sum_{|\mu|+|\nu|=|\pi|+k} g^\pi_{\mu,\nu}(\alpha) q_\mu r_\nu.$$

From Lemma 3.2.6 we know that $g^\pi_{\mu,\nu} = 0$ if $|\pi| > |\mu| + |\nu|$. Hence,

$$\mathcal{G}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) = \sum_{k\geq 0} t^k \mathcal{G}_k(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}).$$

In Section 5.4.3, we prove that for $\alpha = 1$, the operator $\mathcal{G}_k^{(1)}$ is an operator which acts on a map by adding k edges satisfying some conditions.

In Section 5.5, we use a variant of the main theorem to give a differential expression for the operators $\mathcal{G}_0, \mathcal{G}_1$ and $\mathcal{G}_2$. First, we introduce the operator

$$\begin{array}{rcl} \Psi(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) : \mathbb{Q}(\alpha)[\boldsymbol{p}] & \longrightarrow & \mathbb{Q}(\alpha)[\boldsymbol{q}, \boldsymbol{r}] \\ p_\pi & \longmapsto & \displaystyle\prod_{1\leq i\leq \ell(\pi)} (q_{\pi_i} + r_{\pi_i}). \end{array}$$

Combinatorially, when we think of p_i (resp. q_i, r_i) as the weight of an uncolored face (reps $(+)$ face, $(-)$ face) of degree i in a map, Ψ is the operator that chooses a color $(+)$ or $(-)$ for each face.

Theorem 5.1.11. *We have the following differential expressions for $\mathcal{G}_0, \mathcal{G}_1$ and $\mathcal{G}_2$.*

$$\mathcal{G}_0 = \Psi, \tag{5.12}$$

$$\mathcal{G}_1 = \sum_{m\geq 1} \sum_{\substack{m_1+m_2=m+1 \\ m_1,m_2\geq 1}} q_{m_1} r_{m_2} \cdot \Psi \cdot \frac{m\partial}{\partial p_m}, \tag{5.13}$$

and

$$\begin{aligned} \mathcal{G}_2 =& \frac{1}{2} \sum_{m\geq 1} \sum_{\substack{m_1+m_2=m+2 \\ m_1,m_2\geq 1}} b(m_1-1)(m_2-1) q_{m_1} r_{m_2} \Psi \frac{m\partial}{\partial p_m} \\ &+ \frac{1}{2} \sum_{m\geq 1} \sum_{\substack{m_1+m_2+m_3=m+2 \\ m_1,m_2,m_3\geq 1}} (m_1-1)(q_{m_1} r_{m_2} r_{m_3} + r_{m_1} q_{m_2} q_{m_3}) \Psi \frac{m\partial}{\partial p_m} \\ &+ \frac{1}{2} \sum_{k,m\geq 1} \sum_{\substack{i_1+i_2=k+m+2 \\ i_1,i_2\geq 1}} \alpha \min(m, k, i_1-1, i_2-1) q_{i_1} r_{i_2} \Psi \frac{m\partial}{\partial p_m} \frac{k\partial}{\partial p_k} \\ &+ \frac{1}{2} \sum_{m,k\geq 1} \sum_{\substack{m_1+m_2=m+1 \\ m_1,m_2\geq 1}} \sum_{\substack{k_1+k_2=k+1 \\ k_1,k_2\geq 1}} q_{m_1} q_{k_1} r_{m_2} r_{k_2} \Psi \frac{m\partial}{\partial p_m} \frac{k\partial}{\partial p_k}. \end{aligned} \tag{5.14}$$

Remark 5.1.12. One may notice that Eq. (5.12) is equivalent to Example 5.1.10.

Trying to give a combinatorial proof for Theorem 5.1.11 seems to be quite challenging. However, it can be obtained from the characterizing differential equation of Theorem 5.1.6 by routine computations.

Theorem 5.1.11 yields the following partial result towards Conjecture 9.

Corollary 5.1.13. *Fix three π, μ and ν partitions such that $|\pi| \geq |\mu| + |\nu| - 2$. Then $2g^{\pi}_{\mu,\nu}$ is a polynomial in $b := \alpha - 1$ with non-negative integer coefficients.*

Since the integrality in Conjecture 9 will be proved using a different method (see Corollary 5.2.2), the factor 2 in Corollary 5.1.13 can be omitted. The interest of Corollary 5.1.13 lies then in the positivity part.

We hope that a better understanding of the differential structure of the operator $\mathcal{B}_\infty$ could allow one to generalize Theorem 5.1.11 in order to obtain a differential formula of $\mathcal{G}_k$ for any k. This would eventually give the missing positivity part in Conjecture 9 and in Goulden–Jackson's Matching-Jack conjecture.

Structure of the remaining sections of the chapter

In Section 5.2, we establish some useful properties of structure coefficients $g^{\pi}_{\mu,\nu}$ and we prove Corollary 5.1.13 and Proposition 5.1.5. Section 5.3 is dedicated to the proof of the main theorem as well as its generalized version Theorem 5.3.5 related to Hurwitz numbers. In Section 5.4, we give a combinatorial proof of Theorem 5.1.9 for $\alpha = 1$ (this section is quite independent from the rest of the chapter). We use the main theorem in Section 5.5 to give an explicit expression for coefficients $g^{\pi}_{\mu,\nu}$ and to prove Theorem 5.1.11. Finally, we prove Theorem 5.6.5 in Section 5.6.

5.2 Preliminary results

The purpose of this section to establish a relation between the coefficients $g^{\pi}_{\mu,\nu}$ and the coefficients $c^{\pi}_{\mu,\nu}$ which we use to prove integrality in Śniady's conjecture (Conjecture 9) as well as Proposition 5.1.5.

5.2.1 Relation between coefficients $g^{\pi}_{\mu,\nu}$ and $c^{\pi}_{\mu,\nu}$

We recall that the coefficients $g^{\pi}_{\mu,\nu}$ coincide with $c^{\pi}_{\mu,\nu}$ when indexed by three partitions of the same size; see Proposition 1.3.21. In this section, we establish another connection between the two families of coefficients which allows us to express the coefficients $g^{\pi}_{\mu,\nu}$ in terms of $c^{\pi}_{\mu,\nu}$. If π is a partition, we denote by $\widetilde{\pi} := \pi \backslash 1^{m_1(\pi)}$ the partition obtained by deleting all parts of size 1. The following proposition is a generalization of [DF16, Equation (19)]. The proof is quite the same.

Proposition 5.2.1. *For any partitions π, μ and ν such that $\pi \vdash n \geq |\mu|, |\nu|$, we have*

$$\sum_{i=0}^{m_1(\pi)} \binom{m_1(\pi)}{i} g^{\tilde{\pi}\cup 1^i}_{\mu,\nu} = \binom{m_1(\mu) + n - |\mu|}{m_1(\mu)} \binom{m_1(\nu) + n - |\nu|}{m_1(\nu)} c^{\pi}_{\mu\cup 1^{n-|\mu|}, \nu\cup 1^{n-|\nu|}}, \quad (5.15)$$

Equivalently,

$$\exp\left(\frac{p_1}{t\alpha}\right) G^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) = \exp\left(\frac{\partial}{t\partial q_1}+\frac{\partial}{t\partial r_1}\right)\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}), \tag{5.16}$$

where the last equality holds in $\mathbb{Q}(\alpha)[t,1/t,\boldsymbol{q},\boldsymbol{r}][\![\boldsymbol{p}]\!]$.

Proof. To simplify expressions, we denote $\bar{\mu} := \mu \cup 1^{n-|\mu|}$ and $\bar{\nu} := \nu \cup 1^{n-|\nu|}$. We get from the definition of Jack characters and Lemma 5.1.7 that for any partition λ of size n,

$$\begin{aligned}\theta^{(\alpha)}_{\mu}(\lambda)\theta^{(\alpha)}_{\nu}(\lambda) &= \binom{n-|\mu|+m_1(\mu)}{m_1(\mu)}\binom{n-|\nu|+m_1(\nu)}{m_1(\nu)}\theta^{(\alpha)}_{\bar{\mu}}(\lambda)\theta^{(\alpha)}_{\bar{\nu}}(\lambda)\\ &= \binom{n-|\mu|+m_1(\mu)}{m_1(\mu)}\binom{n-|\nu|+m_1(\nu)}{m_1(\nu)}\sum_{n\le|\rho|\le 2n} g^{\rho}_{\bar{\mu},\bar{\nu}}\theta^{(\alpha)}_{\rho}(\lambda).\end{aligned}$$

Terms corresponding to $|\rho| > n$ vanish since characters are evaluated at a partition of size n. Hence

$$\theta^{(\alpha)}_{\mu}(\lambda)\theta^{(\alpha)}_{\nu}(\lambda) = \binom{n-|\mu|+m_1(\mu)}{m_1(\mu)}\binom{n-|\nu|+m_1(\nu)}{m_1(\nu)}\sum_{\rho\vdash n} g^{\rho}_{\bar{\mu},\bar{\nu}}\theta^{(\alpha)}_{\rho}(\lambda).$$

Using Proposition 1.3.21 we get that,

$$\theta^{(\alpha)}_{\mu}(\lambda)\theta^{(\alpha)}_{\nu}(\lambda) = \binom{n-|\mu|+m_1(\mu)}{m_1(\mu)}\binom{n-|\nu|+m_1(\nu)}{m_1(\nu)}\sum_{\rho\vdash n} c^{\rho}_{\bar{\mu},\bar{\nu}}\theta^{(\alpha)}_{\rho}(\lambda). \tag{5.17}$$

Moreover, for any $\lambda \vdash n$,

$$\begin{aligned}\theta^{(\alpha)}_{\mu}(\lambda)\theta^{(\alpha)}_{\nu}(\lambda) &= \sum_{|\kappa|\le n} g^{\kappa}_{\mu,\nu}\theta^{(\alpha)}_{\kappa}(\lambda)\\ &= \sum_{|\kappa|\le n} g^{\kappa}_{\mu,\nu}\binom{m_1(\kappa)+n-|\kappa|}{m_1(\kappa)}\theta^{(\alpha)}_{\kappa\cup 1^{n-|\kappa|}}(\lambda)\\ &= \sum_{\rho\vdash n}\left(\theta^{(\alpha)}_{\rho}(\lambda)\sum_{i=0}^{m_1(\rho)} g^{\widetilde{\rho}\cup 1^i}_{\mu,\nu}\binom{m_1(\rho)}{i}\right).\end{aligned} \tag{5.18}$$

The last equation is obtained by regrouping terms with respect to $\rho := \kappa \cup 1^{n-|\kappa|}$. We obtain Eq. (5.15) by comparing the coefficient of $\theta^{(\alpha)}_{\pi}$ in Eqs. (5.17) and (5.18).

Let us now prove Eq. (5.16). Let π, μ and ν be three partitions. We want to prove that the coefficient of $t^{|\mu|+|\nu|-|\pi|}p_\pi q_\mu r_\nu/(z_\pi \alpha^{\ell(\pi)})$ is the same in both sides of Eq. (5.16). It is easy to check that this is given by Eq. (5.15) if $|\pi| \ge \max(|\mu|,|\nu|)$. Otherwise, each one of these coefficients is 0; this is a consequence of Lemma 5.1.7 and the fact that $\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})$ is homogeneous in the three alphabets $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$. $\square$

5.2.2 Integrality in Śniady's conjecture

We deduce the following corollary from Proposition 5.2.1.

Corollary 5.2.2 (Integrality in Conjecture 9)**.** *The coefficients $g^{\pi}_{\mu,\nu}$ are polynomials in b with integer coefficients.*

Proof. Inverting Eq. (5.16), we get

$$G^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) = \exp\left(-\frac{p_1}{t\alpha}\right)\exp\left(\frac{\partial}{t\partial q_1}+\frac{\partial}{t\partial r_1}\right)\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}). \tag{5.19}$$

By extracting the coefficient of $t^{|\mu|+|\nu|-|\pi|}p_\pi q_\mu r_\nu/(z_\pi\alpha^{\ell(\pi)})$, we get

$$g^{\pi}_{\mu,\nu} = \sum_{e\leq i\leq m_1(\pi)} (-1)^{m_1(\pi)-i}\binom{m_1(\pi)}{i}\binom{|\widetilde{\pi}|+i-|\mu|+m_1(\mu)}{m_1(\mu)} \binom{|\widetilde{\pi}|+i-|\nu|+m_1(\nu)}{m_1(\nu)} c^{\widetilde{\pi}\cup 1^i}_{\mu\cup 1^{|\widetilde{\pi}|+i-|\mu|},\nu\cup 1^{|\widetilde{\pi}|+i-|\nu|}},$$

where $\widetilde{\pi} := \pi\backslash 1^{m_1(\pi)}$ and $e := \max(0,|\mu|-|\widetilde{\pi}|,|\nu|-|\widetilde{\pi}|)$. We conclude using the fact that $c^{\pi}_{\mu,\nu}$ are integer polynomials; see Theorem 1.5.2. □

5.2.3 Combinatorial interpretation of $g^{\pi}_{\mu,\nu}$ for $\alpha\in\{1,2\}$

Proof of Proposition 5.1.5. We prove the proposition for $\alpha = 1$. The proof is exactly the same for $\alpha = 2$. Fix three partitions π,μ and ν. We want to prove that

$$g^{\pi}_{\mu,\nu}(1) = \left|\mathcal{OH}^{\pi}_{\mu,\nu}\right|, \tag{5.20}$$

where $\mathcal{OH}^{\pi}_{\mu,\nu}$ is the set of hypermaps defined in Definition 5.1.4. Since Eq. (5.15) fully characterizes the coefficients $g^{\pi}_{\mu,\nu}$, it is enough to prove that

$$\sum_{i=0}^{m_1(\pi)}\binom{m_1(\pi)}{i}\left|\mathcal{OH}^{\tilde{\pi}\cup 1^i}_{\mu,\nu}\right| = \binom{m_1(\mu)+n-|\mu|}{m_1(\mu)}\binom{m_1(\nu)+n-|\nu|}{m_1(\nu)} c^{\pi}_{\mu\cup 1^{n-|\mu|},\nu\cup 1^{n-|\nu|}}(1),$$

holds for any partitions π, μ and ν, with $n = |\pi|$. Using Theorem 5.1.3, we know that the right-hand side of the last equation counts vertex-labelled oriented hypermaps of profile $(\pi,\mu\cup 1^{|\pi|-|\mu|},\nu\cup 1^{|\pi|-|\nu|})$, with $|\pi|-|\mu|$ marked $(+)$ faces and $|\pi|-|\nu|$ marked $(-)$ faces of degree 1 (unlike in Definition 5.1.4, it is possible here to have both faces of isolated loops marked).

Moreover, the set of such maps M with a fixed number j of isolated loops with both $(+)$ and $(-)$ faces marked, can be obtained as follows:

- choose the labels of black vertices of degree 2, which form the isolated loops with two marked faces; there are $\binom{m_1(\pi)}{j}$ such possible choices,
- choose a hypermap in $\mathcal{OH}^{\widetilde{\pi}\cup 1^{|\pi|-j}}_{\mu,\nu}$ and associate to black vertices of degree 2 the labels not chosen in the first step.

Summing over all $i := m_1(\pi)-j$ between 0 and $m_1(\pi)$, we obtain the left hand-side. This finishes the proof of the proposition. □

5.3 Proof of the main theorem

The purpose of this section is to prove the main result, Theorem 5.1.6, as well as its generalization to series with arbitrary many alphabets of variables, Theorem 5.3.5.

5.3.1 Skew Jack characters

Before proving Theorem 5.1.6, we introduce a skew[3] version $\theta^{(\alpha)}_{\mu/\nu}$ of Jack characters.

Definition 5.3.1. *We consider a sequence of variables $u_1, u_2 \dots$. For any partitions μ and ν satisfying $|\mu| \geq |\nu|$, we define the coefficient $\theta^{(\alpha)}_{\mu/\nu}$ which depends on one variable v, by*

$$\theta^{(\alpha)}_{\mu}(v, u_1, u_2 \dots) = \sum_{\nu} \theta^{(\alpha)}_{\mu/\nu}(v)\theta^{(\alpha)}_{\nu}(u_1, u_2 \dots).$$

This expansion is well defined, since $\theta^{(\alpha)}_{\mu}(v, u_1, u_2 \dots)$ is a shifted symmetric function in $u_1, u_2 \dots$, and $(\theta^{(\alpha)}_{\nu})_{\nu \in \mathbb{Y}}$ is a basis of $\mathcal{S}^*_\alpha$.

The following lemma relates the structure coefficients to skew characters.

Lemma 5.3.2. *For any partitions μ, ν, π one has*

$$\sum_{\kappa} g^{\kappa}_{\mu,\nu}\theta^{(\alpha)}_{\kappa/\pi}(v) = \sum_{\rho,\xi} g^{\pi}_{\rho,\xi}\theta^{(\alpha)}_{\mu/\rho}(v)\theta^{(\alpha)}_{\nu/\xi}(v).$$

Proof. We have for any partitions μ and ν

$$\begin{aligned}\sum_{\pi}\theta^{(\alpha)}_{\pi}(u_1, u_2 \dots)\left(\sum_{\kappa} g^{\kappa}_{\mu,\nu}\theta^{(\alpha)}_{\kappa/\pi}(v)\right) &= \sum_{\kappa} g^{\kappa}_{\mu,\nu}\theta^{(\alpha)}_{\kappa}(v, u_1, u_2 \dots)\\ &= \theta^{(\alpha)}_{\mu}(v, u_1, u_2 \dots)\theta^{(\alpha)}_{\nu}(v, u_1, u_2 \dots)\\ &= \sum_{\rho,\xi}\theta^{(\alpha)}_{\mu/\rho}(v)\theta^{(\alpha)}_{\rho}(u_1, u_2 \dots)\theta^{(\alpha)}_{\nu/\xi}(v)\theta^{(\alpha)}_{\xi}(u_1, u_2 \dots)\\ &= \sum_{\pi}\theta^{(\alpha)}_{\pi}(u_1, u_2 \dots)\left(\sum_{\rho,\xi} g^{\pi}_{\rho,\xi}\theta^{(\alpha)}_{\mu/\rho}(v)\theta^{(\alpha)}_{\nu/\xi}(v)\right).\end{aligned}$$

We conclude by extracting the coefficient of $\theta^{(\alpha)}_{\pi}(u_1, u_2, \dots)$. □

The following proposition gives a differential construction for skew characters.

Proposition 5.3.3. *For any partitions μ and ν one has*

$$\theta^{(\alpha)}_{\mu/\nu}(v) = [t^{|\mu|-|\nu|}p_\mu]\exp\left(\mathcal{B}_\infty(-t, \boldsymbol{p}, -\alpha v)\right) \cdot p_\nu, \tag{5.21}$$

and

$$\frac{\alpha^{\ell(\mu)} z_\mu}{\alpha^{\ell(\nu)} z_\nu}\theta^{(\alpha)}_{\mu/\nu}(v) = [t^{|\mu|-|\nu|}p_\nu]\exp\left(\mathcal{B}^{\perp}_\infty(-t, \boldsymbol{p}, -\alpha v)\right) \cdot p_\mu. \tag{5.22}$$

[3]This is not the usual definition of skew characters in which we consider skew diagrams in the argument of the character; $\theta^{(\alpha)}_{\mu}(\lambda/\rho)$.

Remark 5.3.4. Using Eq. (5.21) and the combinatorial description of the operators $\mathcal{B}_n$ (Proposition 4.3.4), one can see that $\theta^{(\alpha)}_{\mu/\nu}$ corresponds to the weights collected while adding one layer to a map of face-type ν to obtain a map of face-type μ.

Proof. Fix $k \geq 0$. From Eq. (1.50), we have

$$\sum_{\nu} t^{|\nu|}\theta^{(\alpha)}_{\nu}(u_1,\dots,u_k)p_\nu = \exp\left(\mathcal{B}_\infty(-t,\boldsymbol{p},-\alpha u_1)\right)\dots\exp\left(\mathcal{B}_\infty\left(-t,\boldsymbol{p},-\alpha u_k\right)\right)\cdot 1.$$

By applying $\exp\left(\mathcal{B}_\infty(-t,\boldsymbol{p},-\alpha v)\right)$ and using Eq. (1.50) for $k+1$, we get

$$\sum_{\nu}\left(\exp\left(\mathcal{B}_\infty(-t,\boldsymbol{p},-\alpha v)\right)\cdot t^{|\nu|}p_\nu\right)\theta^{(\alpha)}_{\nu}(u_1,\dots,u_k) = \sum_{\mu} t^{|\mu|}\theta^{(\alpha)}_{\mu}(v,u_1,\dots,u_k)p_\mu.$$

Taking the limit over k and extracting the coefficient of p_μ, we get

$$[p_\mu]\sum_{\nu}\left(\exp\left(\mathcal{B}_\infty(-t,\boldsymbol{p},-\alpha v)\right)\cdot t^{|\nu|}p_\nu\right)\theta^{(\alpha)}_{\nu}(u_1,u_2,\dots) = t^{|\mu|}\theta^{(\alpha)}_{\mu}(v,u_1,u_2,\dots).$$

We obtain Eq. (5.21) by extracting the coefficient of $\theta^{(\alpha)}_{\nu}(u_1,u_2\dots)$. For Eq. (5.22), we write, using Eq. (5.21)

$$\begin{aligned} t^{|\mu|-|\nu|}\theta^{(\alpha)}_{\mu/\nu}(v) &= \left\langle \exp\left(\mathcal{B}_\infty(-t,\boldsymbol{p},-\alpha v)\right)\cdot p_\nu, \frac{p_\mu}{z_\mu\alpha^{\ell(\mu)}}\right\rangle \\ &= \left\langle p_\nu, \exp\left(\mathcal{B}^\perp_\infty(-t,\boldsymbol{p},-\alpha v)\right)\cdot \frac{p_\mu}{z_\mu\alpha^{\ell(\mu)}}\right\rangle \\ &= [p_\nu]\exp\left(\mathcal{B}^\perp_\infty(-t,\boldsymbol{p},-\alpha v)\right)\cdot \frac{z_\nu\alpha^{\ell(\nu)}}{z_\mu\alpha^{\ell(\mu)}}p_\mu. \end{aligned}$$

□

5.3.2 Proof of the main theorem

Proof of Theorem 5.1.6. We have from Eq. (5.21)

$$\begin{aligned} &\exp\left(\mathcal{B}_\infty(-t,\boldsymbol{q},-\alpha v)\right)\exp\left(\mathcal{B}_\infty(-t,\boldsymbol{r},-\alpha v)\right)\cdot G^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) \\ &\qquad = \sum_{\pi,\mu,\nu}\left(\sum_{\rho,\xi}\frac{g^{\pi}_{\rho,\xi}(\alpha)}{z_\pi\alpha^{\ell(\pi)}}\theta^{(\alpha)}_{\mu/\rho}(v)\theta^{(\alpha)}_{\nu/\xi}(v)\right)t^{|\mu|+|\nu|-|\pi|}p_\pi q_\mu r_\nu. \end{aligned}$$

Moreover, Eq. (5.22) gives

$$\exp\left(\mathcal{B}^\perp_\infty(-t,\boldsymbol{p},-\alpha v)\right)\cdot G^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) = \sum_{\pi,\mu,\nu}\left(\sum_{\kappa}\frac{g^{\kappa}_{\mu,\nu}}{z_\kappa\alpha^{\ell(\kappa)}}\theta^{(\alpha)}_{\kappa/\pi}(v)\right)t^{|\mu|+|\nu|-|\pi|}p_\pi q_\mu r_\nu.$$

Combining these two equations with Lemma 5.3.2, we deduce that

$$\begin{aligned} &\exp\left(\mathcal{B}_\infty(-t,\boldsymbol{q},-\alpha v)+\mathcal{B}_\infty(-t,\boldsymbol{r},-\alpha v)\right)\cdot G^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) \\ &\qquad = \exp\left(\mathcal{B}^\perp_\infty(-t,\boldsymbol{p},-\alpha v)\right)\cdot G^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}). \end{aligned} \tag{5.23}$$

Since $\mathcal{B}_\infty^\perp(-t,\boldsymbol{p},-\alpha v)$ commutes with each one of the operators $\mathcal{B}_\infty(-t,\boldsymbol{q},-\alpha v)$ and $\mathcal{B}_\infty(-t,\boldsymbol{r},-\alpha v)$ as operators in $\mathcal{O}(\mathcal{A})[\![v]\!]$, we obtain by induction on $\ell \geq 1$

$$\exp\left(\mathcal{B}_\infty(-t,\boldsymbol{q},-\alpha v)+\mathcal{B}_\infty(-t,\boldsymbol{r},-\alpha v)\right)^\ell \cdot G^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) \\ = \exp\left(\mathcal{B}_\infty^\perp(-t,\boldsymbol{p},-\alpha v)\right)^\ell \cdot G^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}).$$

This allows us to take the logarithm of the operators in Eq. (5.23). We get that

$$\left(\mathcal{B}_\infty(-t,\boldsymbol{q},-\alpha v)+\mathcal{B}_\infty(-t,\boldsymbol{r},-\alpha v)\right)\cdot G^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) = \mathcal{B}_\infty^\perp(-t,\boldsymbol{p},-\alpha v)\cdot G^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}).$$

We conclude by replacing v by $-u/\alpha$. ☐

Let us now prove Theorem 5.1.9.

Proof of Theorem 5.1.9. Using Eq. (5.21), Lemma 5.3.2 can be rewritten as the following commutation relation

$$\exp\left(\mathcal{B}_\infty(-t,\boldsymbol{q},-\alpha v)+\mathcal{B}_\infty(-t,\boldsymbol{r},-\alpha v)\right)\cdot \mathcal{G}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) \\ = \mathcal{G}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\cdot \exp\left(\mathcal{B}_\infty(-t,\boldsymbol{p},-\alpha v)\right).$$

Finally, we "take the logarithm" of operators as in the proof of Theorem 5.1.6. ☐

5.3.3 Generalization to constellations

The purpose of this subsection is to briefly explain how to generalize the differential equation of the main theorem to series with finitely many alphabets, and how these are related to constellations.

Fix an integer $k \geq 1$. We introduce the following generalization of the series $G^{(\alpha)}$;

$$G_k^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q}^{(0)},\dots,\boldsymbol{q}^{(k)}) := \sum_{\pi,\mu^{(0)},\dots,\mu^{(k)}} \frac{g^\pi_{\mu^{(0)},\dots,\mu^{(k)}}(\alpha)}{z_\pi \alpha^{\ell(\pi)}} t^{|\mu^{(0)}|+\dots+|\mu^{(k)}|-|\pi|} p_\pi q^{(0)}_{\mu^{(0)}}\dots q^{(k)}_{\mu^{(k)}},$$

where $g^\pi_{\mu^{(0)},\dots,\mu^{(k)}}$ are defined as the structure coefficients;

$$\theta^{(\alpha)}_{\mu^{(0)}}\dots\theta^{(\alpha)}_{\mu^{(k)}} = \sum_\pi g^\pi_{\mu^{(0)},\dots,\mu^{(k)}}(\alpha)\theta^{(\alpha)}_\pi.$$

We recall that the series $G_k^{(\alpha)}$ (or more precisely its homogeneous part $\tau_k^{(\alpha)}$) gives access to all Hurwitz numbers (and their b-deformation in the sense of [CD22]); see Remark 1.3.15.

Using the same arguments as in the case of three alphabets, we obtain the following theorem.

Theorem 5.3.5. *For any $k \geq 1$, we have,*

$$\left(\mathcal{B}_\infty(-t,\boldsymbol{q}^{(0)},u)+\dots+\mathcal{B}_\infty(-t,\boldsymbol{q}^{(k)},u)\right)\cdot G_k^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q}^{(0)},\dots,\boldsymbol{q}^{(k)}) \\ = \mathcal{B}_\infty^\perp(-t,\boldsymbol{p},u)\cdot G_k^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q}^{(0)},\dots,\boldsymbol{q}^{(k)}).$$

As mentioned before, such an equation seems to be new even in the classical case $\alpha = 1$. We now explain the connection of $G_k^{(\alpha)}$ to the series of k-constellations. We recall that $\tau_k^{(\alpha)}$ is a generalized version of $\tau^{(\alpha)}$ with $k+2$ alphabets which has been introduced in Section 1.3.1, and that when $\alpha \in \{1,2\}$, it corresponds to series of constellations with control of the full profile.

Extending the proof of Proposition 5.2.1 to $k+2$ alphabets, we get

$$\exp\left(\frac{p_1}{t\alpha}\right) G_k^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}^{(0)}, \dots, \boldsymbol{q}^{(k)}) = \exp\left(\frac{\partial}{t\partial q_1^{(0)}} + \dots + \frac{\partial}{t\partial q_1^{(k)}}\right) \tau_k^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}^{(0)}, \dots, \boldsymbol{q}^{(k)}).$$

As in the case $k = 1$, the series $G_k^{(\alpha)}$ has an interpretation when $\alpha \in \{1,2\}$ in terms of k-constellations with some marked vertices of degrees 1.

5.3.4 Reformulation of the main theorem with the operators $\mathcal{C}_\ell$

We recall that the operators $\mathcal{C}_\ell$ introduced in Section 4.5 are defined by

$$\mathcal{B}_\infty(t, \boldsymbol{p}, u) = \sum_{\ell \geq 0} u^\ell \mathcal{C}_\ell(t, \boldsymbol{p}) \colon \mathcal{P} \to \mathcal{P}[u][\![t]\!]_+. \tag{5.24}$$

The differential equation Eq. (5.7) of the main theorem can then be reformulated as follows

$$(\mathcal{C}_\ell(-t, \boldsymbol{q}) + \mathcal{C}_\ell(-t, \boldsymbol{r})) \cdot G^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) = \mathcal{C}_\ell^{\perp}(-t, \boldsymbol{p}) \cdot G^{(\alpha)}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}), \quad \text{for } \ell \geq 0.$$

Similarly, Theorem 5.1.9 is equivalent to

$$(\mathcal{C}_\ell(-t, \boldsymbol{q}) + \mathcal{C}_\ell(-t, \boldsymbol{r})) \cdot \mathcal{G}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) = \mathcal{G}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) \cdot \mathcal{C}_\ell(-t, \boldsymbol{p}) \;\text{ for } \ell \geq 0. \tag{5.25}$$

We deduce the following corollary which will be useful to solve the differential equation of the main theorem in Section 5.5.1.

Corollary 5.3.6. *For any partition $\lambda = [\lambda_1, \dots, \lambda_k]$, we have*

$$\prod_{1 \leq i \leq k} (\mathcal{C}_{\lambda_i}(-t, \boldsymbol{q}) + \mathcal{C}_{\lambda_i}(-t, \boldsymbol{r})) \cdot \mathcal{G}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) = \mathcal{G}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) \cdot \prod_{1 \leq i \leq k} \mathcal{C}_{\lambda_i}(-t, \boldsymbol{p}). \tag{5.26}$$

Actually, the products in Eq. (5.26) can be taken in any order, since the operators $(\mathcal{C}_\ell)_{\ell \geq 1}$ commute by Theorem 4.5.1.

5.4 Combinatorial proof of the differential equation for $\alpha = 1$

The purpose of this section is to give a combinatorial proof when $\alpha = 1$ for the commutation relation Eq. (5.25) (equivalently the differential equation Eq. (5.7)). To this purpose, we start by recalling the combinatorial interpretation of the operators $\mathcal{C}_\ell^{(1)}$ given in [BD23], see Proposition 5.4.2. We then use the combinatorial interpretation of $G^{(1)}$ given by Proposition 5.1.5 to obtain a combinatorial interpretation of the operator $\mathcal{G}^{(1)}$, see Corollary 5.4.13.

We believe that the combinatorial constructions of Subsections 5.4.2 and 5.4.3 are of independent interest and might be useful to shed some light on the combinatorics of hypermaps with controlled profile.

All maps considered in this section are oriented. Instead of orienting each one of the vertex roots as in Section 1.1.3, we will use the following convention; each one of the connected surfaces of a map has an orientation which will be called the *direct orientation of the surface* and all corners will be (implicitly) oriented with respect to it.

5.4.1 Interpretation of the operator $\mathcal{C}_\ell$ for $\alpha = 1$

It will be more convenient at some steps of the proof to work with maps with labelled edges rather than labelled vertices.

Definition 5.4.1. *We say that a map M is labelled if its edges are numbered $1, 2, \dots, |M|$. We say that a map is bipartite if its vertices are colored in white and black and each edge connects two vertices of different colors. If M is a bipartite map of size n, then its face-type is the partition of n obtained by reordering the face degrees divided by 2.*

The following proposition is a variant of Proposition 2.2.3.

Proposition 5.4.2. *Fix $\ell \geq 0$ and let N be a labelled orientable bipartite map. Then,*

$$\mathcal{C}_\ell^{(1)}(-t, \boldsymbol{p}) \cdot \frac{p_{\lambda^\diamond(N)}}{|N|!} = \sum_{n \geq \ell} (-t)^n \sum_{M} \frac{p_{\lambda^\diamond(M)}}{|M|!},$$

where the second sum is taken over labelled orientable maps M obtained from N as follows:

- *we add a black vertex v and ℓ new white vertices,*
- *we add n edges all connecting v to some white vertex, and so that each new white vertex is connected to v by at least one edge,*
- *we relabel the edges of M in any way.*

An example of such maps N and M is given in Fig. 5.2.

Proof. We recall that the operators $\mathcal{C}_\ell^{(1)}$ are related to the operators $\mathcal{B}_n^{(1)}$ by

$$\mathcal{C}_\ell^{(1)}(-t, \boldsymbol{p}) = [u^\ell] \sum_{n \geq 1} (-t)^n \frac{\mathcal{B}_n^{(1)}(\boldsymbol{p}, u)}{n}. \tag{5.27}$$

But we know from Proposition 2.2.3 that for a bipartite map N we have

$$\mathcal{B}_n^{(1)}(\boldsymbol{p}, u) \cdot p_{\lambda^\diamond(N)} u^{|\mathcal{V}_\circ(N)|} = \sum_{M} p_{\lambda^\diamond(M)} u^{|\mathcal{V}_\circ(M)|},$$

where $|\mathcal{V}_\circ(.)|$ denotes the number of white vertices and where the sum is taken over maps M obtained from N by adding a black vertex v of degree n with a rooted corner. If N and M are now labelled then the previous equation becomes

$$\frac{\mathcal{B}_n^{(1)}(\boldsymbol{p}, u)}{n} \cdot \frac{p_{\lambda^\diamond(N)}}{|N|!} u^{|\mathcal{V}_\circ(N)|} = \sum_{M} \frac{p_{\lambda^\diamond(M)}}{|M|!} u^{|\mathcal{V}_\circ(M)|}.$$

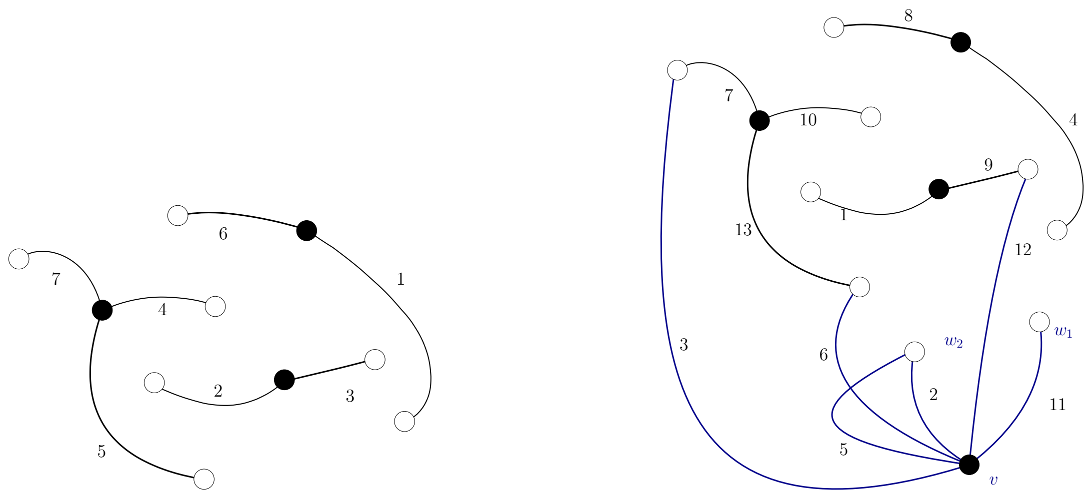


Figure 5.2: Example of the action of $\mathcal{C}_2^{(1)}$ on a map N. On the left the map N, and on the right a map M obtained by adding one black vertex v, two white vertices w_1 and w_2 and 6 edges (represented in blue).

Indeed, we have $|M|!$ ways to choose new labels for the edges of M, and then we divide by $|N|!$ to "forget" the old labels in N, and by n to forget the root of the added black vertex v. Combining this equation with Eq. (5.27) concludes the proof of the proposition. □

5.4.2 BFC maps and pre-hypermaps

We start by introducing a family of maps which allows us to encode the hypermaps with marked faces defined in Definition 5.1.4.

Definition 5.4.3. *We say that a map is* bipartite face-colored (BFC *map*) *if its vertices are colored in black and white, its faces are colored in two colors* $(+)$ *and* $(-)$*, and such that each edge connects two vertices of different colors (but it does not necessarily separate two faces of different colors).*

Moreover, a BFC *map will be called a pre-hypermap if it satisfies the following additional conditions:*

1. *white vertices have degree at most 2.*

2. *all white vertices of degree 2 are incident to two faces of different colors.*

Remark 5.4.4. Notice that a hypermap can be seen as a pre-hypermap; we color all the vertices of the hypermap in black and add in the middle of each edge a white vertex of degree 2. Hence, hypermaps are pre-hypermaps maps with only white vertices of degree 2. Conversely, if we delete all white vertices of degree 1 in a pre-hypermap we obtain a hypermap.

We distinguish two types of edges in a BFC map.

Definition 5.4.5. *An edge is said to be* bicolor *if it is incident to two faces of different colors. We have two types of bicolor edges in a* BFC *map* (*see Fig. 5.3*)*:*

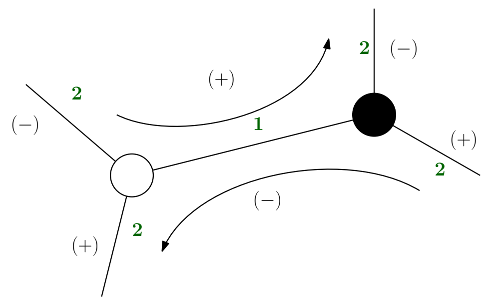


Figure 5.3: Types of bicolor edges in a BFC map.

- *Type 1: on the* $(+)$ *side-face, we see the white vertex and then the black one when we travel along the edge-side in the direct orientation.*
- *Type 2: on the* $(+)$ *side-face, we see the black vertex and then the white one when we travel along the edge-side in the direct orientation.*

By definition, all non bicolor edges will be considered of type 1.

Remark 5.4.6. Note that the types can be equivalently defined by conditions around vertices. For instance, when we turn around a black vertex in the direct orientation, then edges of type 2 are exactly those that separate faces $(-)/(+)$ in this order. Consequently, a vertex which is not monochromatic (*i.e.* incident at least to one $(+)$ and one $(-)$ face) is necessarily incident to a type 2 edge.

It is easy to check that each black vertex of a pre-hypermap has an even number of white neighbors of degree 2 (see *e.g* Remark 5.4.4). This allows us to define the degree of a black vertex v in a pre-hypermap as follows

$$\deg(v) = \frac{1}{2}|\{\text{neighbors of } v \text{ of degree } 2\}| + |\{\text{neighbors of } v \text{ of degree } 1\}| \tag{5.28}$$

$$= |\{\text{incident edges to } v \text{ of type } 1\}|. \tag{5.29}$$

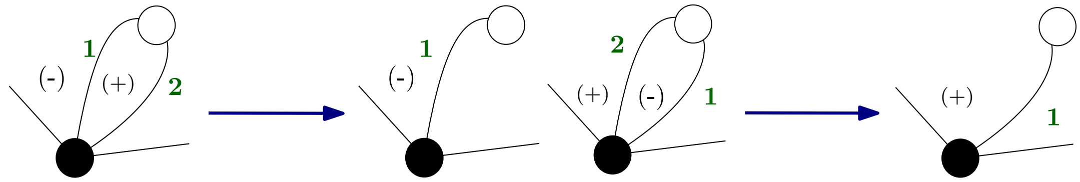
We now extend the notion of profile to pre-hypermaps.

Definition 5.4.7. *Let* M *be a pre-hypermap. We denote by* $\lambda^\bullet(M)$ *the partition given by black vertices degrees* (*as defined in Eq.* (5.28)). *We also denote by* $\lambda^+(M)$ (*resp.* $\lambda^-(M)$ *the partition given by the* $(+)$ *face* (*resp. the* $(-)$ *face*) *degrees divided by 2. We call the profile of* M *the triple of partitions* $(\lambda^\bullet(M), \lambda^+(M), \lambda^-(M))$.

One can check that the profile of a hypermap as defined in Definition 5.1.1 coincides with its pre-hypermap profile as defined in Definition 5.4.7.

Definition 5.4.8. *We say that a pre-hypermap is vertex-labelled if:*

- *for each* $d \geq 1$, *vertices of same degree* d *are numbered 1,2,....*
- *each black vertex has a marked corner, oriented in the direct orientation and followed by an edge of type 1.*



(a) The marked face is colored $(+)$ (b) The marked face is colored $(-)$

Figure 5.4: Deleting edges of a hypermap with marked faces to obtain a pre-hypermap.

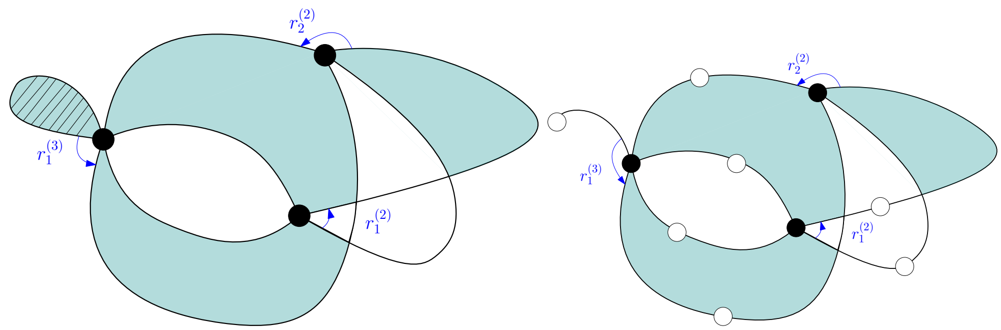


Figure 5.5: On the left a labelled hypermap with one marked face; $(-)$ faces are represented in blue and the marked face is crossed. On the right the associated pre-hypermap.

Fix three partitions π, μ and ν. We define $\mathcal{OPH}^{\pi}_{\mu,\nu}$ as the set of vertex-labelled oriented pre-hypermaps of profile (π, μ, ν).

The following lemma connects pre-hypermaps to hypermaps with marked faces.

Lemma 5.4.9. *Fix three partitions π, μ and ν. There is a bijection between $\mathcal{OH}^{\pi}_{\mu,\nu}$ (defined in Definition 5.1.4) and $\mathcal{OPH}^{\pi}_{\mu,\nu}$.*

Proof. Let $M \in \mathcal{OH}^{\pi}_{\mu,\nu}$. As in Remark 5.4.4, we can think of M as a pre-hypermap which we denote M'. First, notice that each degree 2 face in M' (degree 1 face in M) is incident exactly to one edge of type 1 and one edge of type 2. Notice also that the only case in which an edge of type 2 is incident to two faces of degree 2 is the case of an isolated loop.

By deleting in M' all edges of type 2 incident to marked faces and forgetting the colors of these faces, we obtain a map N in $\mathcal{OPH}^{\pi}_{\mu,\nu}$, see Fig. 5.4. Indeed, in this procedure the degree of a black vertex (as defined in Eq. (5.28)) is unchanged. Moreover, each one of the faces of N inherits a color from M and its degree is unchanged.

Conversely, from $N \in \mathcal{OPH}^{\pi}_{\mu,\nu}$ we obtain a map $M \in \mathcal{OH}^{\pi}_{\mu,\nu}$ as follows; first we transform each white leaf into a loop, then we mark the formed 2-degree face and finally, we color it so that the added edge is bicolor. □

An example of the correspondence described above is given in Fig. 5.5.

Proposition 5.4.10. *For any partitions π,μ and ν, we have*

$$g^{\pi}_{\mu,\nu}(1) = |\mathcal{OPH}^{\pi}_{\mu,\nu}|.$$

Equivalently, $(|\mu| + |\nu|)!/z_\pi g^{\pi}_{\mu,\nu}(1)$ is the number of labelled orientable pre-hypermaps of profile (π, μ, ν) (see Definition 5.4.1).

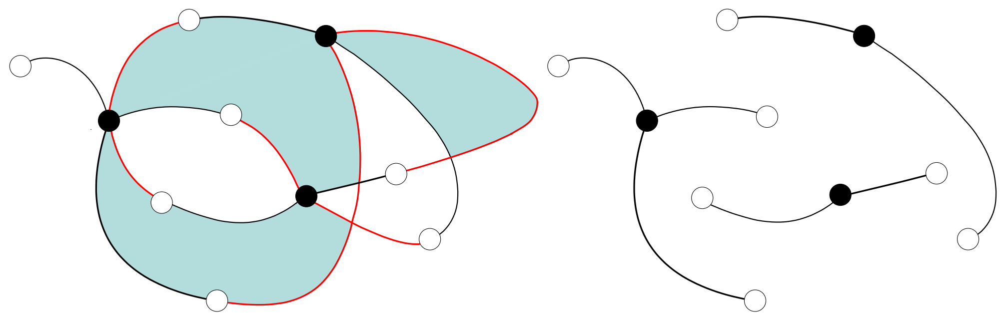


Figure 5.6: On the left an oriented pre-hypermap, faces colored in $(-)$ are represented in blue. Bicolor edges of type 2 are represented in red. On the right the $[3, 2, 2]$-star map obtained by deleting edges of type 2.

Proof. We know from Lemma 5.4.9 that $|\mathcal{OPH}^\pi_{\mu,\nu}| = |\mathcal{OH}^\pi_{\mu,\nu}|$. On the other hand $g^\pi_{\mu,\nu}(1) = |\mathcal{OH}^\pi_{\mu,\nu}|$ by Eq. (5.20), this gives the first part of the proposition.

To obtain the second part, we start by noticing that a pre-hypermap of profile (π, μ, ν) has $|\mu| + |\nu|$ edges. Moreover, to pass from a vertex-labelled hypermap of vertex-type π to a labelled hypermap, we start by labelling edges and then we forget vertex labels which corresponds to multiplying by $(|\mu| + |\nu|)!/z_\pi$. □

5.4.3 Combinatorial interpretation of $\mathcal{G}^{(1)}$

In this subsection, we give a second combinatorial interpretation for $g^\pi_{\mu,\nu}(1)$ which generalizes Proposition 5.4.10. This interpretation consists in seeing the series of hypermaps as an operator (see Corollary 5.4.13) rather than a "static" generating series.

Fix a partition π. We call *π-star* map the unique bipartite map with only white vertices of degree 1 and black vertices of type π. Notice that labelled π-star maps are in bijection with permutations of cycle type π. In particular, there are $|\pi|!/z_\pi$ such maps.

Lemma 5.4.11. *Let M be a pre-hypermap of profile (π, μ, ν). Then the map obtained by deleting all edges of type 2 is the π-star map.*

Conversely, let M be a BFC *map such that $\lambda^+(M) = \mu$ and $\lambda^-(M) = \nu$. Assume that M is obtained by adding $|\mu| + |\nu| - |\pi|$ edges to the π-star map and by coloring the faces, such that the added edges are bicolor of type 2. Then all white vertices of M have degree 1 or 2, white vertices of degree 2 are incident to two faces of different colors, and $\lambda^\bullet(M) = \pi$. In other terms, M is a pre-hypermap of profile (π, μ, ν).*

Proof. We start by proving the first assertion. Let M be a pre-hypermap of profile (π, μ, ν) and let N be the map obtained by deleting all edges of type 2 (an example is given in Fig. 5.6). By definition all white vertices of M have degree 1 or 2. Moreover, each white vertex of degree two is incident to one edge of type 1 and one edge of type 2. Hence, N is a star map. Furthermore, the degree of a black vertex in a pre-hypermap is given by the number of incident edges of type 1 (see Eq. (5.29)), and is then unchanged by deleting edges of type 2. By consequence the type of black vertices in N is the same as in M, that is π.

Let us now prove the second assertion. Let M be a BFC obtained from the π-star map as above. Since all added edges are of type 2, we cannot add two edges incident to the same white corner. Hence, all white vertices in M have degree 1 or 2. Moreover, white vertices of degree 2 are incident to two faces of different colors since the added edges are bicolor. This proves that M is a pre-hypermap. The type of black vertices of M is π by the same argument as above. This finishes the proof of the lemma. □

We deduce the following proposition.

Proposition 5.4.12. *Fix three partitions π, μ and ν, and set $m := |\pi|$ and $n := |\mu| + |\nu| - |\pi|$. Let N be a labelled orientable bipartite map of face-type π. Then $\frac{n!}{m!} g^{\pi}_{\mu,\nu}(1)$ is the number of ways of obtaining a labelled* BFC *map M from N by:*

1. *adding n edges to N to obtain a map of face-type $\mu \cup \nu$,*
2. *coloring the faces such that the added edges are bicolor of type 2, and such that the obtained* BFC *map M satisfies $\lambda^+(M) = \mu$ and $\lambda^-(M) = \nu$,*
3. *relabelling all the edges.*

Proof. We start by proving the result when N is a labelled π-star map. We know from Proposition 5.4.10 that the number of labelled pre-hypermaps of profile (π, μ, ν) is $n!/z_\pi g^{\pi}_{\mu,\nu}(1)$. We now count in a different way the number of labelled pre-hypermaps of profile (π, μ, ν).

Let $f^{\pi}_{\mu,\nu}$ be the number of ways of realizing the steps (1), (2) and (3) described above starting from a fixed labelled star map of face-type π.

In order to obtain a pre-hypermap of profile (π, μ, ν), we start by choosing a labelled π-star map (we have $m!/z_\pi$ choices), we then have $f^{\pi}_{\mu,\nu}$ ways to add edges to obtain a labelled pre-hypermap of profile (π, μ, ν) (we use here Lemma 5.4.11). Hence

$$n!/z_\pi g^{\pi}_{\mu,\nu}(1) = m!/z_\pi f^{\pi}_{\mu,\nu}.$$

We deduce that $f^{\pi}_{\mu,\nu} = \frac{n!}{m!} g^{\pi}_{\mu,\nu}(1)$. This finishes the proof for star maps.

In order to obtain the assertion for any bipartite labelled map N of face-type π, we prove that the number of ways to realize the steps (1), (2) and (3) on a map N depends only on the face-type of the map and not on its structure. Indeed, when we have two maps of the same face-type, one can always find a bijection between the corners of the two maps which "preserves the face structures": two corners are consecutive when we travel along a face (in the direct orientation) in the first map, if and only if their images in the second map satisfy the same property. Once such a bijection is fixed, each way of adding edges and coloring faces on one map can be copied on the second map in a unique way that respects the bijection of the corners. The two maps obtained have necessarily the same $(+)$ and $(-)$ types (but not necessarily the same vertex degrees). We give an example in Fig. 5.7. □

Given Proposition 5.4.12, it is possible to think of pre-hypermaps as partially constructed hypermaps. Indeed, hypermaps are obtained by adding a maximal number of edges on π-star maps. We now deduce a combinatorial interpretation of $\mathcal{G}^{(1)}$.

Corollary 5.4.13. *Fix a labelled orientable bipartite map M. Then,*

$$\mathcal{G}^{(1)}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) \cdot \frac{p_{\lambda^\circ(N)}}{|N|!} = \sum_{M} t^{|M|-|N|} \frac{q_{\lambda^+(M)} r_{\lambda^-(M)}}{|M|!}$$

where the sum is taken over all ways to add edges to N and to color faces, in order to obtain a BFC *orientable map M such that the added edges are of type 2.*

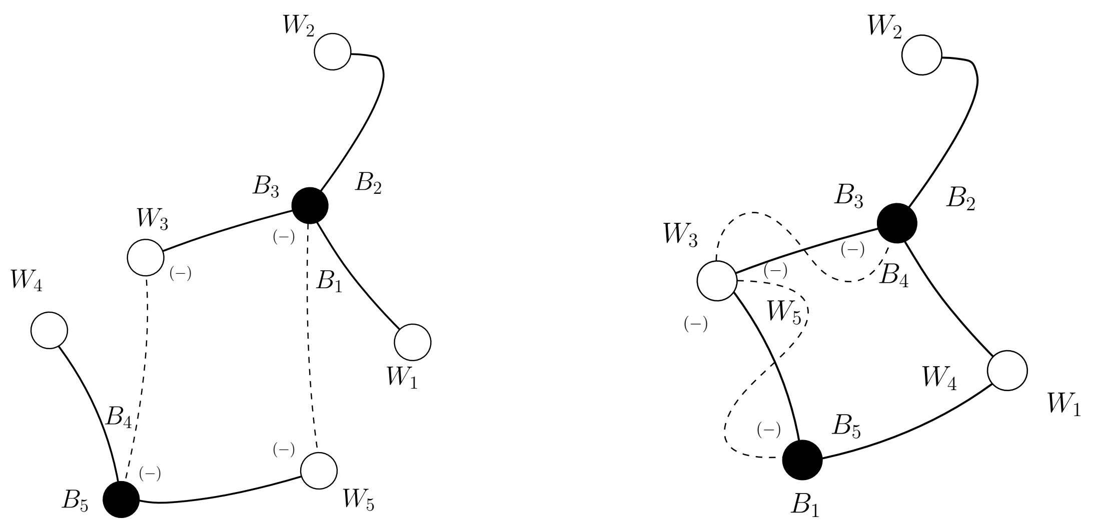


Figure 5.7: The plain edges represent two initial maps with the same face-type $\pi = [3, 2]$. The labels W_i and B_i give a bijection between the corners of the two maps which preserves the face structure.
The two dashed edges are added and faces colored with respect to this bijection; an edge between corners (B_4, W_3) and an edge between (B_1, W_5). Corners incident to a face of color $(-)$ are marked with a sign $(-)$. The maps obtained satisfy $\lambda^+ = [5]$ and $\lambda^- = [2]$.

5.4.4 End of the combinatorial proof

Through this subsection, we fix once and for all an integer $\ell \geq 0$, a partition π and a labelled bipartite map N of face-type π. Our goal is to use Proposition 5.4.2 and Corollary 5.4.13 to prove that, for $\alpha = 1$, both sides of Eq. (5.25) act in the same way on the weight of N given by $\frac{p_\pi}{|\pi|!} = \frac{p_{\lambda^\diamond(N)}}{|N|!}$. This implies the commutation relation of Eq. (5.25) since power-sum functions are a basis of $\mathcal{S}_\alpha$.

The following definition will be useful in the combinatorial proof of Eq. (5.25); all along this subsection, $\mathcal{M}$ will be the (infinite) set of labelled BFC maps which are obtained from N by

- adding one black vertex v and ℓ white vertices,
- adding some edges, such that each one of the new vertices is incident at least to one edge,
- choosing a color for each face,
- relabelling edges.

Moreover, in such map we distinguish the edges of the initial map N.

Fix a BFC map M in $\mathcal{M}$. We denote by $\mathcal{E}_v(M)$ the set of edges incident to v in M, and we denote $\mathcal{T}_2(M \backslash N)$ the set of edges of type 2 in M not contained in N.

On the one hand, we have

$$\left(\mathcal{C}_\ell^{(1)}(-t, \boldsymbol{q}) + \mathcal{C}_\ell^{(1)}(-t, \boldsymbol{r})\right) \cdot \mathcal{G}^{(1)}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) \cdot \frac{p_{\lambda^\diamond(N)}}{|N|!}$$
$$= \sum_{n \geq \ell} (-1)^n \sum_M t^{|M|-|N|} \frac{q_{\lambda^+(M)} r_{\lambda^-(M)}}{|M|!}, \quad (5.30)$$

where the second sum is taken over BFC maps in $\mathcal{M}$ obtained from N as follows:

- **Step 1:** We add edges to N and we color the faces of the obtained map, such that the added edges are of type 2.
- **Step 2:** We start by choosing either the $(+)$ or the $(-)$ part of the map, and we add a black vertex v of degree n and ℓ white vertices only connected to v, such that all added edges are incident to faces of the chosen color.

After each one of these operations we relabel all the edges.

Note that in this construction we cannot obtain a map $M \in \mathcal{M}$ in two different ways, since all edges added in **Step 1** are bicolor while those added in **Step 2** are not. More precisely, the right-hand side of Eq. (5.30) can be rewritten as follows

$$\sum_{M \in \mathcal{M}^{(1)}} (-1)^{|\mathcal{E}_v(M)|} \frac{q_{\lambda^+(M)} r_{\lambda^-(M)}}{|M|!},$$

where

$$\begin{aligned}\mathcal{M}^{(1)} := \{ & M \in \mathcal{M} \text{ such that each added white vertex is only connected to } v \\ & \text{and such that } \mathcal{E}(M \backslash N) = \mathcal{T}_2(M \backslash N) \uplus \mathcal{E}_v(M), \text{ the union being disjoint.} \}\end{aligned}$$

Actually, the following lemma allows to simplify the definition of $\mathcal{M}^{(1)}$.

Lemma 5.4.14. *Fix a* BFC *map* $M \in \mathcal{M}$. *If*

$$\mathcal{E}(M \backslash N) = \mathcal{T}_2(M \backslash N) \uplus \mathcal{E}_v(M) \tag{5.31}$$

then the new white vertices are all only connected to v.

Proof. Let us suppose that there exists an added white vertex w which is incident to a black vertex different from v. Since added edges are either incident to v or of type 2 (see Eq. (5.31)), this implies that w is incident to a bicolor edge e, and by consequence is incident to faces of different colors. As e is of type 2, w cannot have degree 1. Let e_1 and e_2 denote the edges forming a corner in w with e (e_1 and e_2 are not necessarily distinct). But since we cannot have two consecutive edges of type 2 around a vertex, then e_1 and e_2 are both incident to v, and are besides incident to faces of different colors. We deduce that v is also incident to faces of different colors. By Remark 5.4.6 we get that v is incident to an edge of type 2, this contradicts the fact that the union is disjoint in Eq. (5.31). □

We deduce that

$$\mathcal{M}^{(1)} := \{ M \in \mathcal{M} \text{ such that } \mathcal{E}(M \backslash N) = \mathcal{T}_2(M \backslash N) \uplus \mathcal{E}_v(M) \}.$$

On the other hand, we have

$$\mathcal{G}^{(1)}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) \cdot \mathcal{C}_\ell^{(1)}(-t, \boldsymbol{p}) \cdot \frac{p_{\lambda^\diamond(N)}}{|N|!} = \sum_{n \geq \ell} (-1)^n \sum_M t^{|M|-|N|} \frac{q_{\lambda^+(M)} r_{\lambda^-(M)}}{|M|!} \tag{5.32}$$

where the second sum runs over the set of labelled BFC maps obtained from N as follows;

- **Step 1'**: we add a black vertex v of degree n and ℓ white vertices only connected to v.
- **Step 2'**: we add edges and color faces such that the edges added in this step are bicolor and of type 2.

Note that all these maps are in the set

$$\mathcal{M}^{(2)} := \{M \in \mathcal{M} \text{ such that } \mathcal{E}(M\backslash N) = \mathcal{E}_v(M) \cup \mathcal{T}_2(M\backslash N), \\ \text{the union not necessarily disjoint}\}.$$

It is straightforward that $\mathcal{M}^{(1)} \subseteq \mathcal{M}^{(2)}$. Our goal is to prove that the total contribution of maps in $\mathcal{M}^{(2)}\backslash\mathcal{M}^{(1)}$ in Eq. (5.32) is 0. Indeed, any map in $M \in \mathcal{M}^{(2)}$ contributes in Eq. (5.32) with a coefficient

$$\sum_{\mathcal{E}(M\backslash N)=\mathcal{I}^{(1)}\uplus\mathcal{I}^{(2)}} (-1)^{|\mathcal{I}^{(1)}|}\frac{q_{\lambda^+(M)}r_{\lambda^-(M)}}{|M|!}, \tag{5.33}$$

where the sum runs over all possible ways to decompose the set edges of $M\backslash N$ into $\mathcal{I}^{(1)}$ and $\mathcal{I}^{(2)}$ such that $\mathcal{I}^{(1)} \subseteq \mathcal{E}_v(M)$ and $\mathcal{I}^{(2)} \subseteq \mathcal{T}_2(M)$. The only edges for which we have a choice (they can be either in $\mathcal{I}^{(1)}$ or in $\mathcal{I}^{(2)}$) are exactly the edges in $\mathcal{T}_2(M\backslash N) \cap \mathcal{E}_v(M)$. Let $r(M) := |\mathcal{T}_2(M\backslash N) \cap \mathcal{E}_v(M)|$. Then, Eq. (5.33) can be rewritten as follows

$$\sum_{i=0}^{r(M)} (-1)^{i+(|M|-|N|-\mathcal{T}_2(M\backslash N))}\binom{r(M)}{i}\frac{q_{\lambda^+(M)}r_{\lambda^-(M)}}{|M|!} = \begin{cases} 0 & \text{if } r(M) > 0 \\ (-1)^{|\mathcal{E}_v(M)|}\frac{q_{\lambda^+(M)}r_{\lambda^-(M)}}{|M|!} & \text{if } r(M) = 0. \end{cases}$$

This finishes the combinatorial proof of Theorem 5.1.9 for $\alpha = 1$.

5.5 Solution of the differential equation

The main purpose of this section is to solve the differential equation of the main theorem to give an explicit expression of the structure coefficients $g^{\pi}_{\mu,\nu}(\alpha)$, see Theorem 5.5.1. As a byproduct of this result we construct an algebra isomorphism between the space of symmetric functions and space of shifted symmetric functions (Corollary 5.5.3). Finally, we prove Theorem 5.1.11 in Section 5.5.3.

5.5.1 Explicit expression of coefficients $g^{\pi}_{\mu,\nu}$

We define the coefficients a^{λ}_{ξ} for any partitions λ and ξ by

$$a^{\lambda}_{\xi} := [t^{|\xi|}p_{\xi}]\left(\prod_{i=1}^{\ell(\lambda)} \alpha\lambda_i\mathcal{C}_{\lambda_i}(t,\boldsymbol{p})\right)\cdot 1. \tag{5.34}$$

Note that by Theorem 4.5.1, the product in the last equation can be taken in any order. It follows from the definition of the operators $\mathcal{C}_\ell$ that the coefficients a^{λ}_{ξ} are polynomials in b with non-negative coefficients. Using the combinatorial interpretation of the operators $\mathcal{C}_\ell$

(which can be derived from Proposition 4.3.4), it is also possible to give a combinatorial interpretation for these coefficients in terms of layered maps as in Section 4.3.

We also consider the coefficients $d^{\lambda}_{\mu,\nu}$ defined by

$$d^{\lambda}_{\mu,\nu} := \sum_{\xi\cup\pi=\lambda} a^{\xi}_{\mu} a^{\pi}_{\nu} = [t^{|\mu|+|\nu|} q_{\mu} r_{\nu}] \prod_{1\le i\le \ell(\mu)} \alpha\lambda_i \left(\mathcal{C}_{\lambda_i}(t,\boldsymbol{q}) + \mathcal{C}_{\lambda_i}(t,\boldsymbol{r})\right)\cdot 1,$$

where the sum is taken over all possible ways of grouping the parts of λ into two partitions ξ and π.

It follows from the definition of operators $\mathcal{C}_\ell$ (Eq. (1.48) and Eq. (4.29)) that

$$[t^k]\mathcal{C}_\ell(t,\boldsymbol{p}) = \begin{cases} 0 & \text{if } k<\ell \\ p_\ell/(\alpha\ell) & \text{if } k=\ell, \end{cases} \tag{5.35}$$

see also Eq. (5.44a) for details. We deduce that,

$$a^{\lambda}_{\xi} = \begin{cases} 0 & \text{if } |\xi|<|\lambda| \\ \delta_{\lambda,\xi} & \text{if } |\xi|=|\lambda|. \end{cases} \tag{5.36}$$

As a consequence, $d^{\lambda}_{\mu,\nu}=0$ if $|\mu|+|\nu|-|\lambda|<0$.
We now state the main result of this section.

Theorem 5.5.1. *For any partitions λ,μ and ν we have*

$$g^{\lambda}_{\mu,\nu} = (-1)^{|\mu|+|\nu|-|\lambda|} \sum_{m\ge 0} (-1)^m \sum_{\substack{\pi_1,\dots,\pi_m \\ |\lambda|<|\pi_1|<\dots<|\pi_m|}} a^{\lambda}_{\pi_1} a^{\pi_1}_{\pi_2} \dots a^{\pi_{m-1}}_{\pi_m} d^{\pi_m}_{\mu,\nu}. \tag{5.37}$$

The term $m=0$ in the second sum is interpreted as $d^{\lambda}_{\mu,\nu}$.

Proof. We fix μ and ν, and we proceed by induction on $|\mu|+|\nu|-|\lambda|$. If $|\mu|+|\nu|-|\lambda|<0$ then the equality holds since $g^{\lambda}_{\mu,\nu}=0$ (see Lemma 5.1.7). Using Corollary 5.3.6, we write

$$\prod_{1\le i\le s} \left(\mathcal{C}_{\lambda_i}(-t,\boldsymbol{q}) + \mathcal{C}_{\lambda_i}(-t,\boldsymbol{r})\right)\cdot \mathcal{G}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\cdot 1 = \mathcal{G}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) \prod_{1\le i\le s} \mathcal{C}_{\lambda_i}(-t,\boldsymbol{p})\cdot 1.$$

But we know from Example 5.1.10 and Lemma 5.1.7 that $g^{\emptyset}_{\mu,\nu}=\delta_{\mu,\emptyset}\delta_{\nu,\emptyset}$. Equivalently,

$$\mathcal{G}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\cdot 1 = 1. \tag{5.38}$$

Hence,

$$\prod_{1\le i\le s} \left(\mathcal{C}_{\lambda_i}(-t,\boldsymbol{q}) + \mathcal{C}_{\lambda_i}(-t,\boldsymbol{r})\right)\cdot 1 = \mathcal{G}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) \prod_{1\le i\le s} \mathcal{C}_{\lambda_i}(-t,\boldsymbol{p})\cdot 1.$$

We multiply by $\prod_{1\le i\le 1}\alpha\lambda_i$ and we extract the coefficient of $[t^{|\mu|+|\nu|}q_\mu r_\nu]$. Using Eq. (5.36) and Lemma 5.1.7 we obtain

$$\begin{aligned}(-1)^{|\mu|+|\nu|} d^{\lambda}_{\mu,\nu} &= \sum_{\substack{\kappa \\ |\lambda|\le|\kappa|\le|\mu|+|\nu|}} (-1)^{|\kappa|} a^{\lambda}_{\kappa} g^{\kappa}_{\mu,\nu} \\ &= (-1)^{|\lambda|} g^{\lambda}_{\mu,\nu} + \sum_{\substack{\kappa \\ |\lambda|<|\kappa|\le|\mu|+|\nu|}} (-1)^{|\kappa|} a^{\lambda}_{\kappa} g^{\kappa}_{\mu,\nu}.\end{aligned} \tag{5.39}$$

Hence

$$g^{\lambda}_{\mu,\nu} = (-1)^{|\mu|+|\nu|-|\lambda|} d^{\lambda}_{\mu,\nu} - \sum_{\substack{\kappa \\ |\lambda|<|\kappa|\leq|\mu|+|\nu|}} (-1)^{|\kappa|-|\lambda|} a^{\lambda}_{\kappa} g^{\kappa}_{\mu,\nu}. \tag{5.40}$$

Using the induction hypothesis, we obtain

$$g^{\lambda}_{\mu,\nu} = (-1)^{|\mu|+|\nu|-|\lambda|} d^{\lambda}_{\mu,\nu} - \sum_{\substack{\kappa \\ |\lambda|<|\kappa|\leq|\mu|+|\nu|}} (-1)^{|\mu|+|\nu|-|\lambda|} a^{\lambda}_{\kappa} \sum_{m\geq 0} (-1)^m \sum_{\substack{\pi_1,\dots,\pi_m \\ |\kappa|<|\pi_1|<\dots<|\pi_m|}} a^{\kappa}_{\pi_1} \dots a^{\pi_{m-1}}_{\pi_m} d^{\pi_m}_{\mu,\nu}.$$

Finally, this can be rewritten as

$$g^{\lambda}_{\mu,\nu} = (-1)^{|\mu|+|\nu|-|\lambda|} \sum_{m\geq 0} (-1)^m \sum_{\substack{\pi_1,\dots,\pi_m \\ |\lambda|<|\pi_1|<\dots<|\pi_m|}} a^{\lambda}_{\pi_1} a^{\pi_1}_{\pi_2} \dots a^{\pi_{m-1}}_{\pi_m} d^{\pi_m}_{\mu,\nu}.$$ □

We deduce from the last proof the following proposition.

Proposition 5.5.2. *Fix $n \geq 0$. Then, Eq.* (5.38) *and equations*

$$[t^k] \left(\mathcal{C}_\ell(-t, \boldsymbol{q}) + \mathcal{C}_\ell(-t, \boldsymbol{r}) \right) \cdot \mathcal{G}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) = [t^k] \mathcal{G}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) \cdot \mathcal{C}_\ell(-t, \boldsymbol{p}). \tag{5.41}$$

for $\ell \geq 1$ and $\ell \leq k \leq n+\ell$, characterize the operators $\mathcal{G}_i$ for $i \leq n$.

Note that by definition, the lowest term in $\mathcal{C}_\ell$ as a formal power-series in t has degree ℓ. Hence, the previous equations involve only operators $\mathcal{G}_i$ for $i \leq n$.

Proof. Fix $n \geq 0$. First notice that Eqs. (5.41) imply by induction that for any partition $\lambda = [\lambda_1, \dots, \lambda_s]$ and for any k such that $|\lambda| \leq k \leq |\lambda| + n$, we have

$$[t^k] \prod_{1\leq i\leq s} \left(\mathcal{C}_{\lambda_i}(-t, \boldsymbol{q}) + \mathcal{C}_{\lambda_i}(-t, \boldsymbol{r}) \right) \cdot \mathcal{G}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) = [t^k] \mathcal{G}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) \cdot \prod_{1\leq i\leq s} \mathcal{C}_{\lambda_i}(-t, \boldsymbol{p}).$$

In the proof of Theorem 5.5.1 these equations are used to obtain the explicit formula (5.37) of $g^{\lambda}_{\mu,\nu}$ for $0 \leq |\mu| + |\nu| - |\lambda| \leq n$. In particular, they characterize operators $\mathcal{G}_0, \dots, \mathcal{G}_n$. □

Actually, the operators $\mathcal{C}_\ell$ for $\ell \geq 1$, can be generated using only operators $\mathcal{C}_0$ and $\mathcal{C}_1$ (see Theorem 4.5.1). One can use this result to show that Equations (5.41) for $\ell \in \{0, 1\}$ and $1 \leq k \leq n+1$ also characterize the operators $\mathcal{G}_i$ for $i \leq n$. In particular, each operator $\mathcal{G}_i$ is characterized by finitely many equations.

5.5.2 $g^{\pi}_{\mu,\nu}$ as structure coefficients of symmetric functions

We denote for every μ

$$A^{(\alpha)}_{\mu} := \sum_{\lambda} (-1)^{|\mu|} a^{\lambda}_{\mu} p_{\lambda},$$

where a^{λ}_{μ} are the coefficients defined in Eq. (5.34). Multiplying Eq. (5.39) by p_λ and summing over all λ gives

$$A^{(\alpha)}_{\mu} \cdot A^{(\alpha)}_{\nu} = \sum_{\kappa} g^{\kappa}_{\mu,\nu} A^{(\alpha)}_{\kappa}.$$

When p_λ is thought of as a power-sum symmetric function in an underlying alphabet $\boldsymbol{x}$ (see Eq. (1.8)), $A^{(\alpha)}_{\mu}$ becomes a symmetric function. We then have the following corollary.

Corollary 5.5.3. *The map*

$$\begin{array}{ccc} \mathcal{S}_\alpha & \longrightarrow & \mathcal{S}_\alpha^* \\ A_\mu^{(\alpha)} & \longmapsto & \theta_\mu^{(\alpha)} \end{array}$$

is an algebra isomorphism between $\mathcal{S}_\alpha$ and $\mathcal{S}_\alpha^*$.

For $\alpha = 1$, such an isomorphism has been obtained in [CGS04] using a different approach.

5.5.3 A differential expression for the lower terms of $\mathcal{G}$

In this section we prove Theorem 5.1.11. This proof represents no difficulty but involves some lengthy computation. For any $\ell, k \geq 0$, we consider the operator

$$\mathcal{C}_{\ell,k}(\boldsymbol{p}) = (\ell + k)[t^{k+\ell}]\mathcal{C}_\ell(t, \boldsymbol{p}).$$

If $\mathcal{X}$ and $\mathcal{X}'$ are two vector spaces, we denote by $\mathcal{O}(\mathcal{X},\mathcal{X}')$ the space of linear operators from $\mathcal{X}$ to $\mathcal{X}'$. We also set $\mathcal{O}(\mathcal{X}) := \mathcal{O}(\mathcal{X}, \mathcal{X})$. Let O be an operator in one alphabet such that $O(\boldsymbol{p}) \in \mathcal{O}(\mathbb{Q}(\alpha)[\boldsymbol{p}])$, and let $P(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) \in \mathcal{O}(\mathbb{Q}(\alpha)[\boldsymbol{p}], \mathbb{Q}(\alpha)[\boldsymbol{q}, \boldsymbol{r}])$. We introduce their *three alphabet commutator* $[O, P]_{\boldsymbol{p}}^{\boldsymbol{q},\boldsymbol{r}} \in \mathcal{O}(\mathbb{Q}(\alpha)[\boldsymbol{p}], \mathbb{Q}(\alpha)[\boldsymbol{q}, \boldsymbol{r}])$ defined by

$$[P, O]_{\boldsymbol{p}}^{\boldsymbol{q},\boldsymbol{r}} := P(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) \cdot O(\boldsymbol{p}) - (O(\boldsymbol{q}) + O(\boldsymbol{r})) \cdot P(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}).$$

If $Q_1 \in \mathcal{O}(\mathbb{Q}(\alpha)[\boldsymbol{p}])$ and $Q_2 \in \mathcal{O}(\mathbb{Q}(\alpha)[\boldsymbol{q}, \boldsymbol{r}])$, it is easy to check that this commutator satisfies the relation

$$\begin{aligned}[Q_2 \cdot P \cdot Q_1, O]_{\boldsymbol{p}}^{\boldsymbol{q},\boldsymbol{r}} =& [Q_2(\boldsymbol{q}, \boldsymbol{r}), O(\boldsymbol{q})] \cdot P \cdot Q_1 + [Q_2(\boldsymbol{q}, \boldsymbol{r}), O(\boldsymbol{r})] \cdot P \cdot Q_1 \\ &+ Q_2(\boldsymbol{q}, \boldsymbol{r}) \cdot [P, O]_{\boldsymbol{p}}^{\boldsymbol{q},\boldsymbol{r}} \cdot Q_1(\boldsymbol{p}) + Q_2(\boldsymbol{q}, \boldsymbol{r}) \cdot P \cdot [Q_1(\boldsymbol{p}), O(\boldsymbol{p})] .\end{aligned} \tag{5.42}$$

We denote by $\widetilde{\mathcal{G}}_0, \widetilde{\mathcal{G}}_1, \widetilde{\mathcal{G}}_2$ the differential operators given respectively by the right-hand sides of Eqs. (5.12) to (5.14). Our goal is to prove that $\mathcal{G}_i = \widetilde{\mathcal{G}}_i$ for $0 \leq i \leq 2$. Applying Proposition 5.5.2, it is enough to show that for any $\ell \geq 1$ and $0 \leq i \leq 2$

$$\sum_{0 \leq j \leq i} \frac{(-1)^{\ell+j}}{\ell + j} \left[\widetilde{\mathcal{G}}_{i-j}, \mathcal{C}_{\ell,j}\right]_{\boldsymbol{p}}^{\boldsymbol{q},\boldsymbol{r}} = 0 \tag{5.43}$$

The following lemma gives a differential expression for the operators $\mathcal{C}_{\ell,j}$ for $j \leq 2$, which does not involve the alphabet Y. We refer to [Ben24, Appendix A] for a proof.

Lemma 5.5.4. *For any $\ell \geq 1$, we have*

$$\mathcal{C}_{\ell,0} = p_\ell/\alpha, \tag{5.44a}$$

$$\mathcal{C}_{\ell,1} = \binom{\ell+1}{2} \frac{b}{\alpha} \cdot p_{\ell+1} + (\ell + 1) \sum_{m \geq 1} p_{m+\ell+1} \frac{m\partial}{\partial p_m} + \frac{\ell + 1}{2\alpha} \sum_{1 \leq i \leq \ell} p_i p_{\ell+1-i}, \tag{5.44b}$$

$$\mathcal{C}_{\ell,2} = \frac{1}{3\alpha}\binom{\ell+2}{2}\sum_{\substack{i_1+i_2+i_3=\ell+2\\ i_1,i_2,i_3\geq 1}} p_{i_1}p_{i_2}p_{i_3} + \binom{\ell+2}{3}\frac{(3\ell+5)b^2}{4\alpha}p_{\ell+2} \tag{5.44c}$$
$$+ \alpha\binom{\ell+2}{2}\sum_{k,m\geq 1} p_{\ell+k+m+2}\frac{m\partial}{\partial p_m}\frac{k\partial}{\partial p_k} + \binom{\ell+2}{2}\sum_{m\geq 1} b(\ell+m+1)p_{m+\ell+2}\frac{m\partial}{\partial p_m}$$
$$+ \sum_{\substack{i_1+i_2=\ell+2\\ i_1,i_2\geq 1}} \frac{b\cdot(\ell+2)\left((\ell+1)^2 - i_1 i_2\right)}{2\alpha}p_{i_1}p_{i_2} + \binom{\ell+3}{4}p_{\ell+2}$$
$$+ \binom{\ell+2}{2}\sum_{m\geq 1}\sum_{\substack{i_1+i_2=\ell+m+2\\ i_1,i_2\geq 1}} p_{i_1}p_{i_2}\frac{m\partial}{\partial p_m}.$$

In the following lemma we establish some useful commutation relations for the operator Ψ.

Lemma 5.5.5. *We have the following relations between operators in* $\mathcal{O}(\mathbb{Q}(\alpha)[\boldsymbol{p}], \mathbb{Q}(\alpha)[\boldsymbol{q},\boldsymbol{r}])$

$$\Psi \cdot p_\ell = (q_\ell + r_\ell)\cdot\Psi, \quad \textit{i.e.} \quad [\Psi, p_\ell]_{\boldsymbol{p}}^{\boldsymbol{q},\boldsymbol{r}} = 0$$

$$\Psi\cdot\frac{\partial}{\partial p_\ell} = \frac{\partial}{\partial q_\ell}\cdot\Psi = \frac{\partial}{\partial r_\ell}\cdot\Psi.$$

Moreover,

$$\left[\Psi, p_i\frac{\partial}{\partial p_j}\right]_{\boldsymbol{p}}^{\boldsymbol{q},\boldsymbol{r}} = \left[\Psi, p_i\frac{\partial}{\partial p_{j_1}}\frac{\partial}{\partial p_{j_2}}\right]_{\boldsymbol{p}}^{\boldsymbol{q},\boldsymbol{r}} = 0,$$
$$[\Psi, p_{i_1}p_{i_2}]_{\boldsymbol{p}}^{\boldsymbol{q},\boldsymbol{r}} = (q_{i_1}r_{i_2} + r_{i_1}q_{i_2})\Psi,$$
$$\left[\Psi, p_{i_1}p_{i_2}\frac{\partial}{\partial p_j}\right]_{\boldsymbol{p}}^{\boldsymbol{q},\boldsymbol{r}} = q_{i_1}r_{i_2}\Psi\frac{\partial}{\partial p_j} + r_{i_1}q_{i_2}\Psi\frac{\partial}{\partial p_j}.$$

Proof. The first equation is immediate from the definition of the operator Ψ. Let us show the second equation. Fix a partition λ. If $m_\ell(\lambda) = 0$ then

$$\Psi\cdot\frac{\partial}{\partial p_\ell}p_\lambda = \frac{\partial}{\partial q_\ell}\cdot\Psi p_\lambda = 0.$$

Otherwise, we denote by μ the partition obtained from λ by erasing a part of size ℓ. Then

$$\Psi\cdot\frac{\partial}{\partial p_\ell}p_\lambda = m_\ell(\lambda)\Psi p_\mu = m_\ell(\lambda)\prod_{i\in\mu}(q_i + r_i).$$

On the other hand,

$$\frac{\partial}{\partial q_\ell}\cdot\Psi p_\lambda = \frac{\partial}{\partial q_\ell}\prod_{i\in\lambda}(q_i + r_i) = m_\ell(\lambda)\prod_{i\in\mu}(q_i + r_i).$$

The last three equations follow from the first ones. □

We now prove Theorem 5.1.11.

Proof of Theorem 5.1.11. From Lemma 5.5.5 and Eqs. (5.44a) and (5.44b), we have

$$\frac{1}{\ell}\left[\widetilde{\mathcal{G}}_0, \mathcal{C}_{\ell,0}\right]_{\boldsymbol{p}}^{\boldsymbol{q},\boldsymbol{r}} = 0,$$

$$\frac{1}{\ell+1}\left[\widetilde{\mathcal{G}}_0, \mathcal{C}_{\ell,1}\right]_{\boldsymbol{p}}^{\boldsymbol{q},\boldsymbol{r}} = \frac{1}{\alpha}\sum_{\substack{m_1+m_2=\ell+1\\ m_1,m_2\geq 1}} q_{m_1} r_{m_2}\Psi = \frac{1}{\ell}\left[\widetilde{\mathcal{G}}_1, \mathcal{C}_{\ell,0}\right]_{\boldsymbol{p}}^{\boldsymbol{q},\boldsymbol{r}}.$$

These two equations together with Proposition 5.5.2 give Eqs. (5.12) and (5.13). Let us now prove Eq. (5.14). Using Lemma 5.5.5 and Eq. (5.44c), we get

$$\begin{aligned}\frac{1}{\ell+2}\left[\widetilde{\mathcal{G}}_0, \mathcal{C}_{\ell,2}\right]_{\boldsymbol{p}}^{\boldsymbol{q},\boldsymbol{r}} =& \frac{\ell+1}{2\alpha}\sum_{\substack{i_1+i_2+i_3=\ell+2\\ i_1,i_2,i_3\geq 1}} (q_{i_1}q_{i_2}r_{i_3} + q_{i_1}r_{i_2}r_{i_3})\cdot\Psi \\ &+ \sum_{\substack{i_1+i_2=\ell+2\\ i_1,i_3\geq 1}} \frac{b\cdot((\ell+1)^2 - i_1 i_2)}{2\alpha} q_{i_1} r_{i_2}\cdot\Psi \\ &+ (\ell+1)\sum_{m\geq 1}\sum_{\substack{i_1+i_2=\ell+m+2\\ i_1,i_2\geq 1}} q_{i_1} r_{i_2}\cdot\Psi\cdot\frac{m\partial}{\partial p_m}.\end{aligned}$$

Moreover,

$$\begin{aligned}\frac{1}{\ell}\left[\widetilde{\mathcal{G}}_2, \mathcal{C}_{\ell,0}\right]_{\boldsymbol{p}}^{\boldsymbol{q},\boldsymbol{r}} =& \frac{1}{2\alpha}\sum_{\substack{m_1+m_2=\ell+2\\ m_1,m_2\geq 1}} b(m_1-1)(m_2-1) q_{m_1} r_{m_2}\Psi \\ &+ \frac{1}{2\alpha}\sum_{\substack{m_1+m_2+m_3=\ell+2\\ m_1,m_2,m_3\geq 1}} (m_1-1)(q_{m_1}r_{m_2}r_{m_3} + r_{m_1}q_{m_2}q_{m_3})\Psi \\ &+ \sum_{m\geq 1}\sum_{\substack{i_1+i_2=m+\ell+2\\ i_1,i_2\geq 1}} \min(\ell, m, i_1-1, i_2-1) q_{i_1} r_{i_2}\Psi\frac{m\partial}{\partial p_m} \\ &+ \frac{1}{\alpha}\sum_{m\geq 1}\sum_{\substack{k_1+k_2=\ell+1\\ k_1,k_2\geq 1}}\sum_{\substack{m_1+m_2=m+1\\ m_1,m_2\geq 1}} q_{m_1}q_{k_1}r_{m_2}r_{k_2}\Psi\frac{m\partial}{\partial p_m}.\end{aligned}$$

Applying Eq. (5.42), we get

$$\begin{aligned}\left[\widetilde{\mathcal{G}}_1, \mathcal{C}_{\ell,1}\right]_{\boldsymbol{p}}^{\boldsymbol{q},\boldsymbol{r}} =& \sum_{m\geq 1}\sum_{\substack{m_1+m_2=m+1\\ m_1,m_2\geq 1}} \Bigg([q_{m_1}r_{m_2}, \mathcal{C}_{\ell,1}(\boldsymbol{q})]\cdot\Psi\cdot\frac{m\partial}{\partial p_m} + [q_{m_1}r_{m_2}, \mathcal{C}_{\ell,1}(\boldsymbol{r})]\cdot\Psi\cdot\frac{m\partial}{\partial p_m} \\ &+ q_{m_1}r_{m_2}\cdot[\Psi, \mathcal{C}_{\ell,1}]_{\boldsymbol{p}}^{\boldsymbol{q},\boldsymbol{r}}\cdot\frac{m\partial}{\partial p_m} + q_{m_1}r_{m_2}\cdot\Psi\cdot\left[\frac{m\partial}{\partial p_m}, \mathcal{C}_{\ell,1}(\boldsymbol{p})\right]\Bigg).\end{aligned}$$

Hence,

$$\frac{1}{\ell+1}\left[\widetilde{\mathcal{G}}_1,\mathcal{C}_{\ell,1}\right]_{\boldsymbol{p}}^{\boldsymbol{q},\boldsymbol{r}} = \sum_{m\geq 1}\sum_{\substack{m_1+m_2=m+1\\ m_1,m_2\geq 1}}\Bigg(-\left(m_1q_{m_1+\ell+1}r_{m_2}+m_2q_{m_1}r_{m_2+\ell+1}\right)\cdot\Psi\cdot\frac{m\partial}{\partial p_m}$$
$$+\frac{1}{\alpha}q_{m_1}r_{m_2}\sum_{\substack{k_1+k_2=\ell+1\\ k_1,k_2\geq 1}}q_{k_1}r_{k_2}\Psi\frac{m\partial}{\partial p_m}\Bigg)+\frac{b(\ell+1)\ell}{2\alpha}\sum_{\substack{m_1+m_2=\ell+2\\ m_1,m_2\geq 1}}q_{m_1}r_{m_2}\cdot\Psi$$
$$+\sum_{m\geq 1}\sum_{\substack{m_1+m_2=m+\ell+2\\ m_1,m_2\geq 1}}(m+\ell+1)q_{m_1}r_{m_2}\cdot\Psi\cdot\frac{m\partial}{\partial p_m}$$
$$+\frac{1}{\alpha}\sum_{\substack{m_1+m_2+m_3=\ell+2\\ m_1,m_2,m_3\geq 1}}(m_1+m_2-1)(q_{m_1}r_{m_2}q_{m_3}+q_{m_1}r_{m_2}r_{m_3})\cdot\Psi.$$

The last sum can be symmetrized as follows

$$\frac{1}{2\alpha}\sum_{\substack{m_1+m_2+m_3=\ell+2\\ m_1,m_2,m_3\geq 1}}(2m_1+m_2+m_3-2)(r_{m_1}q_{m_2}q_{m_3}+q_{m_1}r_{m_2}r_{m_3})\cdot\Psi.$$

One may also notice that for any $m,\ell,i_1,i_2\geq 1$, such that $i_1+i_2=m+\ell+2$ we have

$$m-\mathbb{1}_{i_1\geq\ell+2}(i_1-\ell-1)-\mathbb{1}_{i_2\geq\ell+2}(i_2-\ell-1)=\min(m,\ell,i_1-1,i_2-1).$$

Using these two remarks and combining the three equations above, we get

$$\frac{1}{\ell+2}\left[\widetilde{\mathcal{G}}_0,\mathcal{C}_{\ell,2}\right]_{\boldsymbol{p}}^{\boldsymbol{q},\boldsymbol{r}}-\frac{1}{\ell+1}\left[\widetilde{\mathcal{G}}_1,\mathcal{C}_{\ell,1}\right]_{\boldsymbol{p}}^{\boldsymbol{q},\boldsymbol{r}}+\frac{1}{\ell}\left[\widetilde{\mathcal{G}}_2,\mathcal{C}_{\ell,0}\right]_{\boldsymbol{p}}^{\boldsymbol{q},\boldsymbol{r}}=0.$$

which gives Eq. (5.43) for $n=2$ and finishes the proof of the theorem. □

5.6 Equations for connected series

In this section we consider a connected version $\widehat{G}^{(\alpha)}$ of the series $G^{(\alpha)}$. We establish some general properties about the series $\widehat{G}^{(\alpha)}$ and we derive from the main theorem a family of differential equation for this series.

5.6.1 Connected series

We recall that

$$\widehat{\tau}\equiv\widehat{\tau}^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}):=\alpha\cdot\log(\tau^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})).$$

We introduce in a similar way the series

$$\widehat{G}\equiv\widehat{G}^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}):=\alpha\cdot\log(G^{(\alpha)}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})).$$

The series $\widehat{G}$ and $\widehat{\tau}$ are well defined in $\mathbb{Q}(\alpha)[t,\boldsymbol{p}][\![\boldsymbol{q},\boldsymbol{r}]\!] \cap \mathbb{Q}(\alpha)[t,\boldsymbol{q},\boldsymbol{r}][\![\boldsymbol{p}]\!]$, since

$$[p_\emptyset]G^{(\alpha)} = [p_\emptyset]\,\widehat{\tau}^{(\alpha)} = [q_\emptyset r_\emptyset]G^{(\alpha)} = [q_\emptyset r_\emptyset]\tau^{(\alpha)} = 1,$$

where $\emptyset$ denotes the empty integer partition, see Lemma 5.1.7.

By Proposition 5.1.5 and Theorem 5.1.3, the series $\tau^{(\alpha)}$ (resp. $\widehat{G}^{(\alpha)}$) is a generating series of connected hypermaps (resp. connected hypermaps with marked faces) when $\alpha \in \{1,2\}$. These two series are related by a variant of Eq. (5.19).

Lemma 5.6.1. *We have:*

$$\widehat{G}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) = -\frac{p_1}{t} + \exp\left(\frac{\partial}{t\partial q_1}\right)\exp\left(\frac{\partial}{t\partial r_1}\right)\widehat{\tau}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}).$$

Proof. First, notice that the operator $\exp\left(\frac{\partial}{t\partial q_1}\right)$ is well defined on $\mathbb{Q}(\alpha)[\boldsymbol{q},\boldsymbol{r},t,1/t][\![\boldsymbol{p}]\!]$, and for any $A, B \in \mathbb{Q}(\alpha)[\boldsymbol{q},\boldsymbol{r},t,1/t][\![\boldsymbol{p}]\!]$,

$$\exp\left(\frac{\partial}{t\partial q_1}\right)\cdot(AB) = \left(\exp\left(\frac{\partial}{t\partial q_1}\right)\cdot A\right)\left(\exp\left(\frac{\partial}{t\partial q_1}\right)\cdot B\right).$$

Since $\widehat{\tau}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) \in \mathbb{Q}(\alpha)[\boldsymbol{q},\boldsymbol{r},t][\![\boldsymbol{p}]\!] \subset \mathbb{Q}(\alpha)[\boldsymbol{q},\boldsymbol{r},t,1/t][\![\boldsymbol{p}]\!]$, we get

$$\begin{aligned}\exp\left(\frac{\partial}{t\partial q_1}\right)\exp\left(\frac{\partial}{t\partial r_1}\right)\cdot\tau(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) &= \exp\left(\frac{\partial}{t\partial q_1}\right)\exp\left(\frac{\partial}{t\partial r_1}\right)\cdot\sum_{k\geq 0}\frac{\widehat{\tau}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})^k}{\alpha^k k!}\\ &= \sum_{k\geq 0}\frac{1}{k!}\left(\exp\left(\frac{\partial}{t\partial q_1}\right)\exp\left(\frac{\partial}{t\partial r_1}\right)\cdot\frac{\widehat{\tau}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})}{\alpha}\right)^k\\ &= \exp\left(\exp\left(\frac{\partial}{t\partial q_1}\right)\exp\left(\frac{\partial}{t\partial r_1}\right)\cdot\frac{\widehat{\tau}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})}{\alpha}\right).\end{aligned}$$

We conclude by substituting this formula in Eq. (5.19). □

We recall that the coefficients of the b-conjecture $h^\pi_{\mu,\nu}$ are defined by

$$\widehat{\tau}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) = \sum_{n\geq 1} t^n \sum_{\pi,\mu,\nu\vdash n}\frac{h^\pi_{\mu,\nu}(\alpha)}{n}p_\pi q_\mu r_\nu.$$

Similarly, we consider the coefficients $\hat{g}^\pi_{\mu,\nu}(\alpha)$

$$\widehat{G}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) = \sum_{\pi,\mu,\nu}\frac{\hat{g}^\pi_{\mu,\nu}(\alpha)}{|\pi|}t^{|\mu|+|\nu|-|\pi|}p_\pi q_\mu r_\nu.$$

As explained in Section 3.6.1, the polynomiality of the coefficients $h^\pi_{\mu,\nu}$ has been established by Féray and Dołęga in [DF16]. The following is a more precise version of Theorem 1.3.9.

Theorem 5.6.2 ([DF17, Theorem 1.2])**.** *For any partitions $\pi,\mu,\nu \vdash n \geq 1$, the coefficient $h^\pi_{\mu,\nu}$ is polynomial in b and*

$$\deg(h^\pi_{\mu,\nu}) \leq n+2-(\ell(\pi)+\ell(\mu)+\ell(\nu)).$$

We deduce the following corollary.

Corollary 5.6.3. *For any partitions π, μ and ν, the coefficients $\hat{g}^{\pi}_{\mu,\nu}$ is polynomial in b, and*

$$\deg(\hat{g}^{\pi}_{\mu,\nu}) \leq 2 + |\mu| - \ell(\mu) + |\nu| - \ell(\nu) - (|\pi| + \ell(\pi)).$$

Proof. From Lemma 5.6.1, we get that for any $|\pi| \geq \max(|\mu|, |\nu|)$, the coefficient $\hat{g}^{\pi}_{\mu,\nu}$ is given by

$$\hat{g}^{\pi}_{\mu,\nu} = \begin{cases} 0 & \text{if } (\pi, \mu, \nu) = ([1], [0], [0]) \\ \binom{m_1(\mu)+|\pi|-|\mu|}{m_1(\mu)} \binom{m_1(\nu)+|\pi|-|\nu|}{m_1(\nu)} h^{\pi}_{\mu \cup 1^{|\pi|-|\mu|}, \nu \cup 1^{|\pi|-|\nu|}}, & \text{otherwise.} \end{cases}$$

From Theorem 5.6.2, we get that $\hat{g}^{\pi}_{\mu,\nu}$ is polynomial and

$$\begin{aligned} \deg\left(\hat{g}^{\pi}_{\mu,\nu}\right) &= \deg(h^{\pi}_{\mu \cup 1^{|\pi|-|\mu|}, \nu \cup 1^{|\pi|-|\pi|}}) \\ &\leq |\pi| + 2 - \left(\ell(\pi) + \ell\left(\mu \cup 1^{|\pi|-|\mu|}\right) + \ell\left(\nu \cup 1^{|\pi|-|\nu|}\right)\right) \\ &= 2 + |\mu| - \ell(\mu) + |\nu| - \ell(\nu) - (|\pi| + \ell(\pi)). \end{aligned}$$

□

5.6.2 Dual operators

The purpose of this subsection is to give a catalytic differential expression for the dual operators $\mathcal{B}_n$. For this, we introduce the scalar product $\langle , \rangle_Y$ on $\widetilde{\mathcal{P}}_Y$ (see Section 1.4.2), defined by

$$\begin{cases} \langle p_\lambda, p_\mu \rangle_Y = \delta_{\lambda,\mu} \alpha^{\ell(\lambda)} z_\lambda = \langle p_\lambda, p_\mu \rangle; \\ \langle p_\lambda, y_i p_\mu \rangle_Y = 0; \\ \langle y_i p_\lambda, y_j p_\mu \rangle_Y = \delta_{i,j} \delta_{\lambda,\mu} \alpha^{\ell(\lambda)+1} z_\lambda, \end{cases}$$

for any $\lambda, \mu \in \mathbb{Y}$ and $i, j \geq 0$.

If A_Y is an operator on $\mathcal{P}_Y$, then we denote by $A_Y^\perp$ its dual operator with respect to $\langle , \rangle_Y$. We deduce from the definitions the following differential expressions for the catalytic operators:

$$(y_i)^\perp = \frac{\alpha \partial}{\partial y_i}, \text{ for any } i \geq 0.$$

$$Y_+^\perp = \sum_{i \geq 2} y_{i-1} \frac{\partial}{\partial y_i},$$

$$\Theta_Y^\perp(\boldsymbol{p}) = \sum_{i \geq 1} y_i \frac{i \partial}{\partial p_i},$$

$$\Gamma_Y^\perp(\boldsymbol{p}) = \sum_{i,j \geq 1} y_{i-1} p_j \frac{\partial}{\partial y_{i+j}} + (1+b) \cdot \sum_{i,j \geq 1} y_{i+j-1} \frac{j \partial^2}{\partial y_i \partial p_j} + b \cdot \sum_{i \geq 2} y_{i-1} \frac{(i-1)\partial}{\partial y_i},$$

$$\mathcal{B}_n^\perp(\boldsymbol{p}, u) = \frac{\partial}{\partial y_0} \left(\Gamma_Y^\perp + u Y_+^\perp\right)^n \Theta_Y^\perp. \tag{5.45}$$

5.6.3 Differential equation for the series of connected maps

We denote for each $m \geq 1$

$$\widehat{G}_{\boldsymbol{p}}^{[m]} = \frac{m\partial}{\partial p_m}\,\widehat{G}, \qquad \widehat{G}_{\boldsymbol{q}}^{[m]} = \frac{m\partial}{\partial q_m}\,\widehat{G}, \qquad \widehat{G}_{\boldsymbol{r}}^{[m]} = \frac{m\partial}{\partial r_m}\,\widehat{G}\,.$$

We recall that $\mathcal{A} = \mathbb{Q}(\alpha)[t, \boldsymbol{p}][\![\boldsymbol{q}, \boldsymbol{r}]\!]$.

Proposition 5.6.4. *Fix $n \geq 1$. Then, we have the equality between operators in $\mathcal{O}(\mathcal{A})$*

$$\mathcal{B}_n(\boldsymbol{q}, u) \cdot G^{(\alpha)} = G^{(\alpha)}.\Theta_Y(\boldsymbol{q}) \left(\Gamma_Y(\boldsymbol{q}) + uY_+ + \sum_{i,j\geq 1} y_{i+j}\frac{\partial}{\partial y_{i-1}}\,\widehat{G}_{\boldsymbol{q}}^{[j]}\right)^n \frac{y_0}{1+b}.$$

Here, $G^{(\alpha)}$ acts on $\mathcal{A}$ by multiplication. Similarly,

$$\mathcal{B}_n^{\perp}(\boldsymbol{p}, u) \cdot G^{(\alpha)}$$
$$= G^{(\alpha)} \cdot \frac{\partial}{\partial y_0}\left(\Gamma_Y^{\perp}(\boldsymbol{p}) + uY_+^{\perp} + \sum_{i,j\geq 1} y_{i+j-1}\frac{\partial}{\partial y_i}\,\widehat{G}_{\boldsymbol{p}}^{[j]}\right)^n \left(\Theta_Y^{\perp}(\boldsymbol{p}) + \sum_{i\geq 1} y_i\,\widehat{G}_{\boldsymbol{p}}^{[i]}\right).$$

Proof. This is a consequence of the catalytic differential expressions of operators $\mathcal{B}_n$ and ${\mathcal{B}_n}^{\perp}$ given resp. in Eq. (1.48) and Eq. (5.45). We also use the fact that

$$\left[(1+b)\frac{j\partial}{\partial p_j}, G\right] = (1+b)\frac{j\partial G}{\partial p_j} = G.\,\widehat{G}_{\boldsymbol{p}}^{[j]} \quad \text{and} \quad \left[\frac{\partial}{\partial y_i}, G\right] = 0. \qquad \square$$

We deduce from Theorem 5.1.6 and Proposition 5.6.4 the following theorem.

Theorem 5.6.5. *The series $\widehat{G}$ satisfies the following differential equation:*

$$\sum_{n\geq 1}\frac{(-1)^n}{n}\frac{\partial}{\partial y_0}\left(\Gamma_Y^{\perp}(\boldsymbol{p}) + uY_+^{\perp} + \sum_{i,j\geq 1} y_{i+j-1}\frac{\partial}{\partial y_i}\,\widehat{G}_{\boldsymbol{p}}^{[j]}\right)^n \left(\sum_{i\geq 1} y_i\,\widehat{G}_{\boldsymbol{p}}^{[i]}\right)\cdot 1$$
$$= \sum_{n\geq 1}\frac{(-1)^n}{n}\Theta_Y(\boldsymbol{q})\left(\Gamma_Y(\boldsymbol{q}) + uY_+ + \sum_{i,j\geq 1} y_{i+j}\frac{\partial}{\partial y_{i-1}}\,\widehat{G}_{\boldsymbol{q}}^{[j]}\right)^n \frac{y_0}{1+b}\cdot 1$$
$$+ \sum_{n\geq 1}\frac{(-1)^n}{n}\Theta_Y(\boldsymbol{r})\left(\Gamma_Y(\boldsymbol{r}) + uY_+ + \sum_{i,j\geq 1} y_{i+j}\frac{\partial}{\partial y_{i-1}}\,\widehat{G}_{\boldsymbol{r}}^{[j]}\right)^n \frac{y_0}{1+b}\cdot 1.$$

Chapter 6

Open problems

In this chapter, we state some open problems which extend the ones addressed in this thesis. In Section 6.1, we formulate some intermediate questions in the direction of the b-conjecture. Section 6.2 is dedicated to problems related to Jack characters. Finally, we present in Section 6.3 analogs of Goulden–Jackson and Lassalle's conjectures for Macdonald polynomials. The last section is based on [BD24].

6.1 Problems related to the b-conjecture

6.1.1 Find a statistic for the b-conjecture

We recall that the b-conjecture (Conjecture 2) remains open. An important step towards this conjecture would be to find (even conjecturally) a statistic of non-orientability which answers the problem, *i.e.* a statistic ϑ on rooted connected maps such that for any profile (π, μ, ν), one has

$$h^{\pi}_{\mu,\nu} = \sum_{M} b^{\vartheta(M)},$$

where the sum is taken over rooted connected maps of profile (π, μ, ν).

We recall that Chapuy and Dołęga introduced in [CD22] a family of statistics which answer the marginal sum case in the b-conjecture (see Theorem 1.3.10). Unfortunately these statistics do not work when we fix the three partitions of the profile. The same problem holds for the Matching-Jack conjecture with the statistics considered in Chapter 2.

In this section, we explain with a *concrete* (minimal) example that the statistics of [CD22] do not work for the "full" b-conjecture. The example consists of a map of size 9 and genus 1, for which the type of SONs that we consider here does not agree with the corresponding coefficient $h^{\pi}_{\mu,\nu}$.

Some properties of the SONs of [CD22]

The statistics introduced by Chapuy and Dołęga have the same flavor as the ones considered in Section 2.1 and Section 4.3 except for the fact that they are defined on rooted connected maps and not vertex-labelled maps. We recall that such a statistic is obtained by:

- fixing a decomposition algorithm[1] $\prec$ on rooted connected maps,
- fixing a measure of non-orientability ρ (see Definition 4.3.1).

We denote by $\vartheta_{\rho,\prec}$ the associated statistic of non-orientability. We refer to [CD22, Section 3] for more details about their construction.

For any map M, this statistic satisfies the following properties

$$\#\text{twist}_{\prec}(M) \leq \vartheta_{\rho,\prec}(M) \leq \#\text{twist}_{\prec}(M) + \#\text{handle}_{\prec}(M), \tag{6.1}$$

where $\text{twist}_{\prec}(M)$ and $\text{handle}_{\prec}(M)$ denote respectively the set of twists and handles in the decomposition of M with respect to $\prec_M$ (see the classification of edges Definition 2.1.1). Moreover, we require that if M is not orientable, then

$$\vartheta_{\rho,\prec}(M) > 0. \tag{6.2}$$

On the other hand, one can check from the definitions that the numbers of handles and twists are always related to the genus of the map by the relation

$$2g(M) = 2\#\text{twist}_{\prec}(M) + \#\text{handle}_{\prec}(M). \tag{6.3}$$

In particular, a map of genus 1 is decomposed with 2 twists or one handle. Combining Eqs. (6.1) to (6.3) we obtain the following property: let M be a rooted connected non-orientable map of genus 1, then

$$\vartheta_{\rho,\prec}(M) = \begin{cases} 1 & \text{if } \#\text{twist}_{\prec}(M) = 0 \text{ and } \#\text{handle}_{\prec}(M) = 1 \\ 2 & \text{if } \#\text{twist}_{\prec}(M) = 2 \text{ and } \#\text{handle}_{\prec}(M) = 0. \end{cases} \tag{6.4}$$

The counter example

We consider the profile $([4,3,2],[3,3,3],[3,3,3])$ of genus 1. The associated coefficient is given by

$$h^{\pi}_{\mu,\nu} = 27b^2.$$

In particular, all maps with this profile are non-orientable.

We consider the map of Fig. 6.1. This map is non-orientable, with profile $([4,3,2],[3,3,3],[3,3,3])$ and has 9 different rootings, which we denote $(M_i)_{1\leq i\leq 9}$.

If we assume that there exists a SON $\vartheta_{\rho,\prec}$ obtained as above, and which answers the b-conjecture, then it necessarily satisfies $\vartheta_{\rho,\prec}(M_i) = 2$ for any $1 \leq i \leq 9$. But using Eq. (6.4) this implies that $\#\text{twist}_{\prec}(M_i) = 2$ and $\#\text{handle}_{\prec}(M_i) = 0$.

Unfortunately, we have tested several "natural" decomposition algorithms $\prec$ (including the one used in [CD22]) and we could not find one which gives the right statistic for the map of the example. In other terms, for each one of these algorithms, there exists always a rooting i of the map for which there is a handle in the decomposition of the map M_i.

[1] Actually, the decomposition algorithm defined in [CD22] involves some duality operations which we dot not present here.

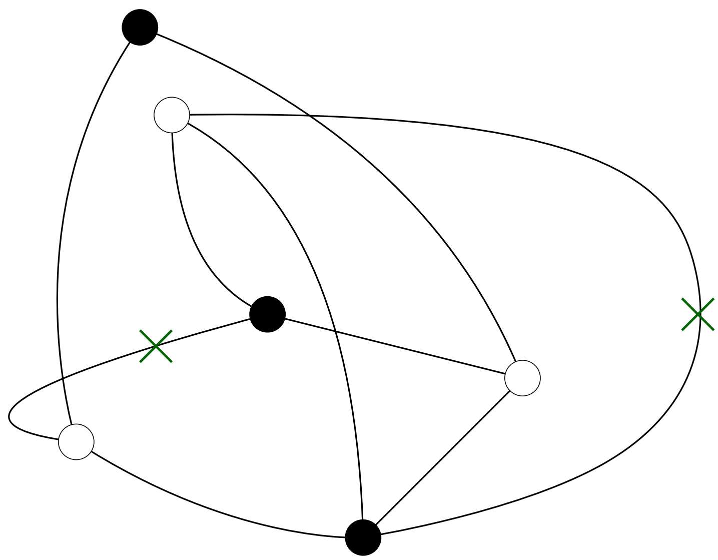


Figure 6.1: A map of profile $([4,3,2],[3,3,3],[3,3,3])$.

6.1.2 The genus 1 in the b-conjecture

In this section, we give a purely combinatorial reformulation of the b-conjecture in the case of genus 1. We start by recalling a property of the coefficients of the b-conjecture due to La Croix.

A property of La Croix and a refinement of the b-conjecture

The following is a consequence of [La 09, Theorem 5.18 and Lemma 5.7] and the polynomiality result of Dołęga–Féray (Theorem 1.3.9).

Theorem 6.1.1 ([La 09]). *Let $n \geq 1$ and let $\pi, \mu, \nu \vdash n$. Let g be the half integer defined by*

$$g := \frac{1}{2}\left(n + 2 - \ell(\pi) - \ell(\mu) - \ell(\nu)\right).$$

Then

$$h^{\pi}_{\mu,\nu} = \sum_{0 \leq i \leq g} a_i b^{2g-2i}(1+b)^i, \tag{6.5}$$

for some coefficients a_i in $\mathbb{Q}$.

Notice that if M is a connected map of profile (π, μ, ν), then the associated half integer g corresponds to the genus of M.

This property has led La Croix to formulate the following refinement of the b-conjecture [La 09, Conjecture 5.2].

Conjecture 10 ([La 09]). *For any partitions π, μ, ν, the coefficients a_i defined in Eq. (6.5) are non-negative integers.*

Remark 6.1.2. Actually, the combinatorial formula established by Chapuy–Dołęga for the marginal coefficients in the b-conjecture (Theorem 1.3.10) gives also the positivity in Conjecture 10 in the marginal case: fix two partitions μ and ν of size n and $1 \leq k \leq n$, then

$$\overline{h}^{\pi}_{\mu,k} = \sum_{0 \leq i \leq g} \bar{a}_i b^{2g-2i}(1+b)^i,$$

for some non-negative integers $\bar{a}_i$, and where

$$g := \frac{1}{2}\left(n + 2 - \ell(\pi) - \ell(\mu) - k\right).$$

Remark 6.1.3. La Croix also gives in [La 09] analogs of Theorem 6.1.1 and Conjecture 10 for the Matching-Jack conjecture.

When $g = 0$, the coefficient $h^{\pi}_{\mu,\nu}$ is independent of b; all the maps are oriented. When $g = 1/2$, we have $h^{\pi}_{\mu,\nu} = a_0 b$, and a_0 counts rooted connected maps of profile (π, μ, ν) (see Eq. (1.26)). These maps are embedded on the projective plan and hence non-oriented. We now focus on the case $g = 1$, which is in this sense the first non trivial case of the b-conjecture.

A reformulation of the b-conjecture for genus 1

We fix three partition π, μ, ν such that

$$\frac{1}{2}\left(n + 2 - \ell(\pi) - \ell(\mu) - \ell(\nu)\right) = 1.$$

We get from Eq. (6.5) that:

$$h^{\pi}_{\mu,\nu} = a_0 b^2 + a_1 b + a_1.$$

The b-conjecture for genus 1 is then equivalent to proving that a_0 and a_1 are non-negative integers. But we know from Eq. (1.25) that

$$a_1 = \left|\{\text{Oriented rooted connected bipartite maps of profile } (\pi, \mu, \nu)\}\right|, \tag{6.6}$$

and from Eq. (1.26) that

$$a_0 + 2a_1 = \left|\{\text{Rooted connected bipartite maps of profile } (\pi, \mu, \nu),\ \text{oriented or not}\}\right|.$$

This proves that a_0 and a_1 are both integers and that $a_1 \geq 0$. It remains to prove that $a_0 \geq 0$, or also that

$$2\left|\{\text{Oriented rooted connected bipartite maps of profile } (\pi, \mu, \nu)\}\right| \leq \\ \left|\{\text{Rooted connected bipartite maps of profile } (\pi, \mu, \nu),\ \text{oriented or not}\}\right|.$$

Hence the b-conjecture for genus 1 is equivalent to the following conjecture.

Conjecture 11. *Fix three partitions π, μ and ν such that*

$$\frac{1}{2}\left(n + 2 - \ell(\pi) - \ell(\mu) - \ell(\nu)\right) = 1.$$

There exists an injection from the set of rooted connected oriented maps of profile (π, μ, ν) to the set of rooted connected non-oriented maps of profile (π, μ, ν).

Note that finding such an injection gives a natural SON ϑ on rooted connected maps of genus 1;

$$\vartheta(M) = \begin{cases} 0 & \text{if } M \text{ is orientable.} \\ 1 & \text{if } M \text{ is in the image of the injection of Conjecture 11,} \\ 2 & \text{otherwise.} \end{cases}$$

Hence, understanding this case of the b-conjecture could give an idea on how to construct SONs in general. Unfortunately, despite some effort, we could not find an injection as predicted in Conjecture 11.

6.2 Other problems related to Jack characters

6.2.1 Lassalle's conjecture on Kerov's polynomials

As explained in Section 4.2, the space of shifted symmetric functions $\mathcal{S}^*_\alpha$ has several interesting bases obtained from Kerov's transition measure. We are interested here in the free cumulants of this measure $(R^{(\alpha)}_k)_{k\geq 2}$; see Eq. (4.7). The Kerov polynomial $K^{(\alpha)}_\mu$ is the polynomial expressing $\theta^{(\alpha)}_\mu$ in terms of the free cumulants. We consider here a "connected version" of these polynomials. First, let $\widehat{\theta}^{(\alpha)}_\mu$ denote the **connected** Jack characters, defined by

$$\alpha \log(F^{(\alpha)}(t, \boldsymbol{x}, \lambda)) = \sum_\mu \widehat{\theta}^{(\alpha)}_\mu(\lambda) p_\mu(\boldsymbol{x}),$$

where $F^{(\alpha)}(t, \boldsymbol{x}, \lambda)$ is the generating series of Jack characters,

$$F^{(\alpha)}(t, \boldsymbol{x}, \lambda) := \sum_{\mu \in \mathbb{Y}} t^{|\mu|} \theta^{(\alpha)}_\mu(\lambda) p_\mu(\boldsymbol{x}).$$

It is straightforward from Theorem 1.5.3 that $\widehat{\theta}^{(\alpha)}_\mu$ counts weighted **connected** layered maps of face-type μ. The *connected Kerov polynomial* $\widehat{K}_\mu$ is the polynomial defined by

$$(-1)^{|\mu|} z_\mu \widehat{\theta}^{(\alpha)}_\mu = \widehat{K}_\mu(-R^{(\alpha)}_2/\alpha, R^{(\alpha)}_3/\alpha, -R^{(\alpha)}_4/\alpha, R^{(\alpha)}_5/\alpha \dots).$$

The following conjecture is due to Lassalle.

Conjecture 12 ([Las09]). *The polynomial $\widehat{K}_\mu$ is a polynomial in b with non-negative integer coefficients.*

Note that the **integrality part** for general b is a consequence of Theorem 4.2.12. The positivity part remains however open.

Based on Theorem 1.5.3 and some tools of the theory of shifted symmetric functions from [DFŚ10], it is possible to obtain a **signed** combinatorial formula for $\widehat{K}_\mu$ for general α in terms of weighted *layered maps*. Unfortunately, understanding combinatorially the positivity in these formulas seems to be out of reach because of the non-orientability weight $b^{\vartheta(M)}$.

6.2.2 Alexandersson-Féray conjecture

Before stating Alexandersson–Féray's conjecture, we need to introduce some notation. For $k \geq 0$, we denote $(x)_k := x(x-1)\dots(x-k+1)$ the falling factorial. We consider the ring of polynomials in the variables $\alpha, s_1, \dots, s_k, r_1, \dots, r_k$. Then

$$\left(\alpha^\ell (s_1)_{i_1} \dots (s_k)_{i_k} (r_1)_{j_1} \dots (r_k)_{j_k}\right)_{\ell, i_1, \dots, i_k, j_1, \dots, j_k \geq 0},$$

is the associated *α-falling factorial basis*. We recall that if $\lambda \in \mathbb{Y}$, then we write $\lambda = \boldsymbol{s}^{\boldsymbol{r}}$, if $\boldsymbol{s}$ and $\boldsymbol{r}$ are multirectangular coordinates of λ; see Definition 1.3.20. Alexandersson and Féray have considered in [AF17] a monomial version of Jack characters, defined by the expansion

$$F^{(\alpha)}(t, \boldsymbol{x}, \lambda) = \sum_{\xi \in \mathbb{Y}} t^{|\xi|} \mathfrak{K}^{(\alpha)}_\xi(\lambda) m_\xi(\boldsymbol{x}),$$

where m_ξ denotes the monomial symmetric functions.

Conjecture 13 ([AF17]). *The expansion of $\mathfrak{K}^{(\alpha)}_\nu(\boldsymbol{s}^{\boldsymbol{r}})$ in the α-falling factorial basis has non-negative rational coefficients.*

Using a change of basis from power-sum to monomial functions, we get from Theorem 1.5.3 a combinatorial interpretation of the coefficients $\mathfrak{K}_\nu$ in terms of layered maps, whose faces are colored with positive integers. A natural question is then to try to use this formula to understand Conjecture 13.

6.3 Macdonald version of some Jack problems

A natural setting to generalize the Goulden–Jackson and Lassalle's conjectures is provided by Macdonald polynomials. In [BD24], we develop the tools which allow us to formulate a Macdonald version of Goulden–Jackson and Lassalle's conjectures. We briefly present these extensions here.

Macdonald has introduced in the 90's a family of symmetric functions which depend on two formal parameters q and t, now known as *Macdonald polynomials*. These polynomials have been thoroughly studied in [Mac95]. Jack polynomials can be obtained as a limit of a particular normalization of Macdonald polynomials, namely the *integral form* denoted $J^{(q,t)}_\lambda$.

During the last two decades, several positivity conjectures have been formulated and proved connecting Macdonald polynomials to some variants of decorated Dyck paths (Schuffle conjecture, Delta conjecture,...) [HMZ12, CM18, HRW18, DM22]. However, these problems remained disconnected from the Jack polynomials problems presented in the previous sections.

6.3.1 Macdonald characters

Inspired by the creation formula of Theorem 1.5.5, we have been able in a joint work with Michele D'Adderio [BD24] to obtain a similar **creation formula** for Macdonald polynomials.

Theorem 6.3.1. *There exists a family of (explicit) operators* $(\mathbf{\Gamma}_n^{(+)})_{n\geq 1}$*, such that for any partition* $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_\ell]$*, one has*

$$J_\lambda^{(q,t)} = \mathbf{\Gamma}_{\lambda_1}^{(+)} \cdot \mathbf{\Gamma}_{\lambda_2}^{(+)} \dots \mathbf{\Gamma}_{\lambda_\ell}^{(+)} \cdot 1. \tag{6.7}$$

While the proof of the creation formula in the case of Jack polynomials (Theorem 1.5.4) is based on the combinatorics of maps, the proof we give in the case of Macdonald polynomials is algebraic. Indeed, the creation operators, $\mathbf{\Gamma}_n^{(+)}$ have the same flavor as the operators developed in the Macdonald polynomials theory (Delta operator [BGHT99], Theta operator [DIVW21],...).

Using some variants of these Macdonald creation operators, we construct a Macdonald generalization $\theta_\mu^{(q,t)}$ of Jack characters. As in the Jack case, these characters can be thought of as functions on Young diagrams but also have a structure of q,t-shifted symmetric functions. They are also related to the theory of **shifted Macdonald polynomials** developed in the 90's [Sah96, Oko97, Oko98, Las98].

6.3.2 A reparametrization and positivity conjectures

We consider the change of variables

$$q = 1 + \gamma\alpha \quad \text{and} \quad t = 1 + \gamma. \tag{6.8}$$

We have observed that this reparametrization allows one to give the natural Macdonald extension to many known results and conjectures about Jack polynomials (positivity in the monomial basis, Stanley conjecture, Goulden–Jackson conjectures).

In particular, Macdonald characters seem to satisfy a variant of Lassalle's conjecture (Conjecture 8) tested for $n \leq 7$ and $k \leq 3$;

Conjecture 14. *Let* μ *be a partition. Then* $(-1)^{|\mu|}(1+\gamma)^{(k+1)|\mu|}\theta_\mu^{(\alpha,\gamma)}(\lambda_1, \dots, \lambda_k)$ *is positive in the variables* $\alpha, \gamma, -\alpha\lambda_1, \dots, -\alpha\lambda_k$ *with non-negative integer coefficients.*

We also obtain a Macdonald version of the Matching-Jack and the b-conjecture. First, let $n(.)$ be the statistic on Young diagram given by

$$n(\lambda) := \sum_{1\leq i\leq \ell(\lambda)} \lambda_i(i-1).$$

We consider the coefficients $\boldsymbol{c}_{\mu,\nu}^\pi(\alpha,\gamma)$ and $\boldsymbol{h}_{\mu,\nu}^\pi(\alpha,\gamma)$ defined by

$$\sum_{\lambda\in\mathbb{Y}} u^{|\lambda|} t^{-2n(\lambda)} q^{n(\lambda')} \frac{J_\lambda^{(q,t)}(\boldsymbol{p}) J_\lambda^{(q,t)}(\boldsymbol{q}) J_\lambda^{(q,t)}(\boldsymbol{r})}{(1-t)^{|\lambda|} j_\lambda^{(q,t)}} = \sum_{m\geq 0} u^m \sum_{\pi,\mu,\nu\vdash m} \frac{\boldsymbol{c}_{\mu,\nu}^\pi(\alpha,\gamma)}{z_\pi(\alpha,\gamma)} p_\pi q_\mu r_\nu, \tag{6.9}$$

and

$$\log\left(\sum_{\lambda\in\mathbb{Y}} u^{|\lambda|} t^{-2n(\lambda)} q^{n(\lambda')} \frac{J_\lambda^{(q,t)}(\boldsymbol{p}) J_\lambda^{(q,t)}(\boldsymbol{q}) J_\lambda^{(q,t)}(\boldsymbol{r})}{(1-t)^{|\lambda|} j_\lambda^{(q,t)}}\right) = \sum_{m\geq 1} u^m \sum_{\pi,\mu,\nu\vdash m} \frac{\boldsymbol{h}_{\mu,\nu}^\pi(\alpha,\gamma)}{\alpha[m]_q} p_\pi q_\mu r_\nu.$$

Here $z_\pi(\alpha,\gamma)$ is a deformation of z_π and $j_\lambda^{(q,t)}$ is the squared norm of Macdonald polynomials with respect to a q,t-deformed scalar product.

The corresponding Jack coefficients are then obtained by simple specializations;

$$\boldsymbol{c}^\pi_{\mu,\nu}(\alpha,\gamma=0)=c^\pi_{\mu,\nu}(\alpha) \quad \text{and} \quad \boldsymbol{h}^\pi_{\mu,\nu}(\alpha,\gamma=0)=h^\pi_{\mu,\nu}(\alpha).$$

We formulate the following conjectures.

Conjecture 15 (A Macdonald version of the Matching-Jack conjecture)**.** *For any positive integer m and partitions π,μ,ν of m, the quantity $(1+\gamma)^{m(m-1)}z_\mu z_\nu \boldsymbol{c}^\pi_{\mu,\nu}(\alpha,\gamma)$ is a polynomial in $b:=\alpha-1$ and γ with non-negative integer coefficients.*

Conjecture 16 (A Macdonald version of the b-conjecture)**.** *For any positive integer m and partitions π,μ,ν of m, the quantity*

$$(1+\gamma)^{m(m-1)}z_\pi z_\mu z_\nu \boldsymbol{h}^\pi_{\mu,\nu}(\alpha,\gamma)$$

is a polynomial in b and γ with non-negative integer coefficients.

Conjecture 15 has been tested for $m\leq 8$ and Conjecture 16 for $m\leq 9$.

Remark 6.3.2. As in the Jack case, the coefficients $\boldsymbol{c}^\pi_{\mu,\nu}$ correspond to a particular case of the *structure coefficients* of Macdonald characters. These coefficients seem to satisfy a generalized version of Conjecture 15.

6.3.3 Towards a 2-parameter generalization of map enumeration

The generalized conjectures formulated above might be the starting point to connect the Macdonald polynomials theory to the combinatorics of Jack polynomials presented in this thesis. In fact, such connections can be useful in two directions.

In one direction, generalizing the combinatorial formulas of Theorems 1.5.1 and 1.5.3 would lead to a natural 2-parameter generalization of the theory of weighted maps. For instance, the similarity between the creation formulas Eq. (1.52) and Eq. (6.7) makes it natural to ask if the combinatorics of the Jack creation operators $\mathcal{B}_n^{(+)}$ can be generalized in some sense to the Macdonald ones $\boldsymbol{\Gamma}_n^{(+)}$.

In the other direction, having a Macdonald version of open Jack problems allows one to use the tools provided by the theory of Macdonald polynomials and which do not all exist in the Jack case. Indeed, the Macdonald generalized conjectures presented above are connected to the **theory of Macdonald operators** initiated by Bergeron, Garsia, Haiman and Tesler. This connection might give a new perspective to understand open problems about Jack polynomials.

For example, the Macdonald generalization of the Matchings-Jack conjecture (Conjecture 15) turned out to be closely related to positivity problems considered recently in some works on Macdonald operators. Indeed, Bergeron, Haglund, Iraci and Romero have introduced in [BHIR23] the super Nabla operator $\nabla_{\boldsymbol{y}}$, which is a diagonal operator on *modified Macdonald polynomials*.

This operator has a natural analog $\boldsymbol{\nabla}_{\boldsymbol{y}}$ for the Macdonald integral form $J_\lambda^{(q,t)}$, defined by

$$\boldsymbol{\nabla}_{\boldsymbol{y}}\cdot J_\lambda^{(q,t)}(\boldsymbol{x})=t^{-n(\lambda)}J_\lambda^{(q,t)}(\boldsymbol{x})J_\lambda^{(q,t)}(\boldsymbol{y}).$$

This is actually a generalized version of the nabla operator;

$$\boldsymbol{\nabla} \cdot J_\lambda^{(q,t)}(\boldsymbol{x}) = q^{n(\lambda')} t^{-n(\lambda)} J_\lambda^{(q,t)}(\boldsymbol{x}).$$

With these definitions in hand, one can prove that the coefficients $\boldsymbol{c}^\pi_{\mu,\nu}(\alpha,\gamma)$ are equivalently given by

$$\gamma^n \boldsymbol{\nabla}_{\boldsymbol{y}} \boldsymbol{\nabla} \cdot p_\pi(\boldsymbol{x}) = \sum_{\mu,\nu \vdash |\pi|} \boldsymbol{c}^\pi_{\mu,\nu}(\alpha,\gamma) p_\mu(\boldsymbol{x}) p_\nu(\boldsymbol{y}).$$

Another interesting tool in this direction is given by **integrable hierarchies**. Indeed, Bourgine and Garbali have proved in a recent work [BG23] that some specializations of the Macdonald series of Eq. (6.9) satisfy a **$\mathbf{q},\mathbf{t}$-deformation of the 2-Toda hierachy**. This hierarchy has been an efficient tool in the case of Schur functions to obtain equations satisfied by the generating series of some families of maps which are out of reach combinatorially [MJD00, GJ08, CC15]. Since finding a Jack analog of this hierarchy is still an open problem, it seems natural to start by considering the Macdonald case and then take the Jack limit.

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