A probabilistic model for interpolation Macdonald polynomials

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joint work with
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Overview

Macdonald polynomials (homogeneous polynomials)

- Introduced by Macdonald in 1989,
- Related to the geometry Hilbert scheme (Haiman '03).
- When $q = 1$, they encode the distributions of the ASEP model (Cantini–de Gier–Wheeler '15), and the t-PushTASEP model (Ayyer–Martin–Williams '25).
- Have combinatorial interpretation in terms of tableaux (Haglund–Haiman–Loehr '05), vertex-models (Borodin–Wheeler '19), multiline queues (Corteel–Mandelshtam–Williams '22).

Interpolation Macdonald polynomials (inhomogeneous polynomials)

- Introduced by Knop and Sahi in 1996,
- In the Jack limit, they have been shown to be monomial positive (Naqvi–Sahi–Sergel '23).
- Related to the knot theory of $\mathfrak{gl}_n$ (Beliakova-Gorsky '24).
- A combinatorial formula in terms of *signed multiline queues*.
- **Today:** A probabilistic model (analogue to t-PushTASEP).

Plan of the talk

- ASEP polynomials $F_\mu(x_1, \ldots, x_n; q, t)$.
- $F_\mu(x_1 = \cdots x_n = q = 1; t)$: The ASEP model (Cantini–de Gier–Wheeler).
- $F_\mu(x_1, \cdots x_n; q = 1, t)$: The PushTASEP model (Ayyer–Martin–Williams).
- Interpolation ASEP polynomials $F^*_\mu(x_1, \ldots, x_n; q, t)$.
- $F^*_\mu(x_1, \ldots, x_n; q = 1, t)$: the interpolation PushTASEP model (B.D–Williams).
- Main ingredient of the proof: a combinatorial formula for interpolation polynomials in terms of multiline queues.

Notation

Fix an integer $n \geq 1$. We say that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a partition of k (with n parts) if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = k$. The integer $k = |\lambda|$ is the size of λ.

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A an element $\mu = (\mu_1, \ldots, \mu_n)$ of $\mathbb{N}^n$ is called a composition.

We say that a composition $\mu = (\mu_1, \ldots, \mu_n)$ is a permutation of a partition λ if it can be obtained by permuting the parts of λ. We denote $S_n(\lambda)$ the set of permutations of λ.

Example: If $n = 3$, and $\lambda = (2, 1, 0)$ then

$$S_3(2,1,0) = \{(2,1,0), (2,0,1), (1,2,0), (1,0,2), (0,2,1), (0,1,2)\}.$$

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We consider the space of polynomials in n variables $x_1, \dots, x_n$ with coefficients in $\mathbb{Q}(q,t)$.
For a composition μ, write

$$x^\mu := x_1^{\mu_1} \dots x_n^{\mu_n}.$$

The family $\{x^\mu : |\mu| = d\}$ is a basis for the space of polynomials with degree d.

Hecke operators

For $1 \leq i \leq n-1$, define the linear operator on the polynomial ring

$$T_i := t - \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(1 - s_i),$$

where $s_i \cdot f(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)$

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These operators satisfy the relations of the Hecke algebra of type A_{n-1}

$$\begin{array}{ll} (T_i - t)(T_i + 1) = 0 & \text{for } 1 \le i \le n-1 \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} & \text{for } 1 \le i \le n-2 \\ T_i T_j = T_j T_i & \text{for } |i-j| > 1. \end{array}$$

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Definition (qKZ equations)

Fix a partition λ. We say that a family of homogeneous polynomials $(f_\mu(x_1, \ldots, x_n; q, t))_{\mu \in S_n(\lambda)}$ of degree $|\lambda|$ is a qKZ family if they satisfy the equations:

$$T_i f_\mu = \begin{cases} f_{s_i \mu} & \text{if } \mu_i > \mu_{i+1}, \\ t f_\mu & \text{if } \mu_i = \mu_{i+1}, \\ t f_{s_i \mu} - (1-t) f_\mu & \text{if } \mu_i < \mu_{i+1} \end{cases}$$

$$\text{and} \quad q^{\mu_n} f_\mu(x_1, \ldots, x_n) = f_{\mu_n, \mu_1, \ldots, \mu_{n-1}}(q x_n, x_1, \ldots, x_{n-1}).$$

Here $s_i \mu := (\ldots, \mu_{i+1}, \mu_i, \ldots)$.

At $q = 1$, the last equation implies that the polynomials are “invariant” under rotation.

ASEP polynomials

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and $q^{\mu_n} f_\mu(x_1,\dots,x_n) = f_{\mu_n,\mu_1,\dots,\mu_{n-1}}(qx_n,x_1,\dots,x_{n-1})$.

Proposition (Cantini–de Gier–Wheeler '15)

For a fixed partition λ, there exists a unique qKZ family $(F_\mu)_{\mu\in S_n(\lambda)}$ with the normalization $[x^\lambda]F_\lambda = 1$. Moreover,

$$\sum_{\mu\in S_n(\lambda)} F_\mu(x_1,\dots,x_n;q,t) = P_\mu(x_1,\dots,x_n;q,t),$$

where $P_\mu(x_1,\dots,x_n;q,t)$ is the symmetric Macdonald polynomial.

The polynomials $(F_\mu)_{\mu\in S_n(\lambda)}$ are called the ASEP polynomials.

ASEP polynomials

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For a fixed partition λ*, there exists a unique qKZ family* $(F_\mu)_{\mu \in S_n(\lambda)}$ *with the normalization* $[x^\lambda]F_\lambda = 1$*. Moreover,*

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Example: : When $\lambda = (2, 1, 0)$, we have

$$F_{2,1,0} = x_1^2 x_2 + \frac{qt - q}{qt^2 - 1} x_1 x_2 x_3, \qquad F_{2,0,1} = x_1^2 x_3 + \frac{qt^2 - qt}{qt^2 - 1} x_1 x_2 x_3,$$

$$F_{1,2,0} = x_1 x_2^2 + \frac{qt^2 - qt}{qt^2 - 1} x_1 x_2 x_3, \qquad F_{1,0,2} = x_1 x_3^2 + \frac{t - 1}{qt^2 - 1} x_1 x_2 x_3,$$

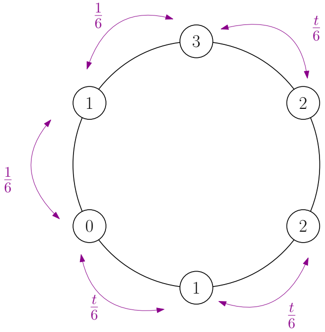
$$F_{0,2,1} = x_2^2 x_3 + \frac{t - 1}{qt^2 - 1} x_1 x_2 x_3, \qquad F_{0,1,2} = x_2 x_3^2 + \frac{t^2 - t}{qt^2 - 1} x_1 x_2 x_3.$$

The (multi-species) ASEP model

Fix $n \geq 1$ and a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ and a parameter $0 < t < 1$.

The mASEP (multi-species Asymmetric Simple Exclusion Process) with content λ is the Markov chain such that each state is indexed by a composition $\mu = (\mu_1, \ldots, \mu_n) \in S_n(\lambda)$. We interpret μ as n particles on a ring, labeled $\mu_1, \ldots, \mu_n$. At each step:

- We choose an index $1 \leq i \leq n$ uniformly at random.
- If $\mu_i > \mu_{i+1}$, we swap $(\mu_i, \mu_{i+1}) \to (\mu_{i+1}, \mu_i)$ with probability t; otherwise we swap the two particles with probability 1 (indices taken modulo n).



The transition probabilities on the state indexed by $\mu = (3, 2, 2, 1, 0, 1)$.

The mASEP model

Theorem (Cantini–de Gier–Wheeler '15)

The stationary distribution of the mASEP with content λ is proportional to the ASEP polynomials evaluated at $q = x_1 = \cdots = x_n = 1$. Equivalently,

$$\pi_{mASEP(\lambda)}(\mu) = \frac{F_\mu(x_1 = \cdots = x_n = 1; q = 1, t)}{P_\lambda(x_1 = \cdots = x_n = 1; q = 1, t)}.$$

Remark: The transition probabilities in the ASEP model are closely related to the coefficients which appear in the qKZ equations:

$$T_i F_\mu = \begin{cases} F_{s_i\mu} & \text{if } \mu_i > \mu_{i+1}, \\ tF_\mu & \text{if } \mu_i = \mu_{i+1}, \\ tF_{s_i\mu} - (1-t)F_\mu & \text{if } \mu_i < \mu_{i+1}. \end{cases}$$

The system is invariant under rotation at $q = 1$

$$F_\mu(x_1, \ldots, x_n; q = 1, t) = F_{\mu_n, \mu_1, \ldots, \mu_{n-1}}(x_n, x_1, \ldots, x_{n-1}; q = 1, t).$$

The t-PushTASEP

Fix a partition $\lambda = (\lambda_1, \dots, \lambda_n)$. The *PushTASEP* (Push Totally Asymmetric Simple Exclusion Process) with content λ is a Markov process such that each state is indexed by a composition $\mu \in S_n(\lambda)$. We interpret μ as n particles on a ring, labeled $\mu_1, \dots, \mu_n$.

Conventions and notation:

- A particle labeled 0 will be called a vacancy.
- There exists at least one part of size 0 in λ (the system has at least one vacancy).
- We assume that $0 < t < 1$ and that $x_i > 0$ for $1 \leq i \leq n$.
- We denote $[m]_t$ the t-integer

$$[m]_t = 1 + t + \cdots + t^{m-1}.$$

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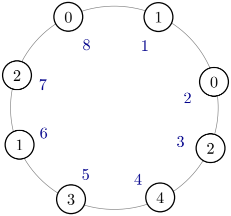
The dynamics are as follows:

- We choose the particle in the j-th position with probability proportional to $\frac{1}{x_j}$.
- This particle at position j starts traveling clockwise. Suppose there are m weaker particles in the system, then with probability $\frac{t^{k-1}}{[m]_t}$ the activated particle will move to the location of the kth of these weaker particles. If this location contains a particle, then that particle becomes active, and chooses a weaker particle to displace in the same way. The procedure continues until the active particle arrives at a vacancy.



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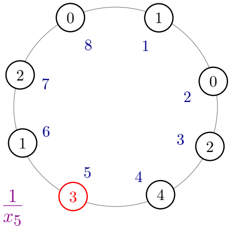

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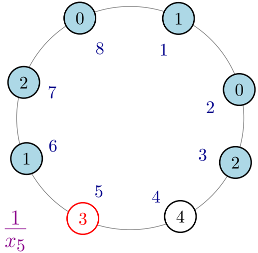

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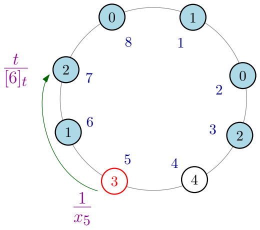

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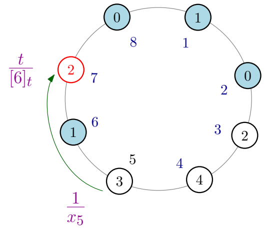

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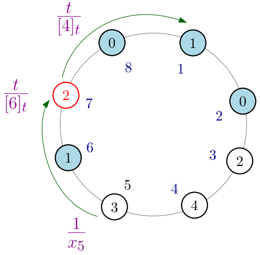

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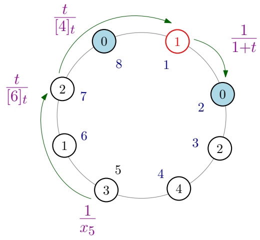

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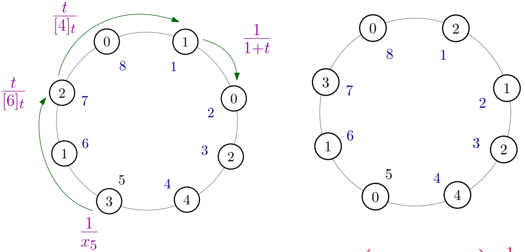

A transition of t-PushTASEP with probability $\frac{1}{x_5}\left(\sum_{1\le i\le 6}\frac{1}{x_i}\right)^{-1}\frac{t}{[6]_t}\,\frac{t}{[6]_t}\,\frac{1}{1+t}$.

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Remark: The dynamics can equivalently be described as follows: each time the active particle passes a site with a weaker particle,

- it continues to move with probability t,
- and settles at that site with probability $(1 - t)$, displacing and activating the particle that is located there.

If it passes the mth such site, then it continues cyclically around the ring.

Totally asymmetric: particles only move clockwise.

The t-PushTASEP

Theorem (Ayyer–Martin–Williams '25)

The stationary distribution of the t-PushTASEP with content λ is proportional to the ASEP polynomials evaluated at $q = 1$. Equivalently,

$$\pi_\lambda(\mu) = \frac{F_\mu(x_1, \ldots, x_n; q = 1, t)}{P_\lambda(x_1, \ldots, x_n; q = 1, t)}.$$

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The proof is based on a combinatorial formula for the ASEP polynomials in terms of multiline queues.

Interpolation polynomials

Recall: Fix an integer $n \geq 1$. We consider the space of polynomials in n variables $x_1, \dots, x_n$ with coefficients $\mathbb{Q}(q,t)$.

We say that $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ is a composition of size k (with n parts) if $\mu_1 + \mu_2 + \cdots + \mu_n = k$. The integer $k = |\mu|$ is the size of μ.

For a composition μ, write

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Given a composition $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$ and $1 \leq i \leq n$, we define

$$k_i(\mu) := \#\{j : j < i \text{ and } \mu_j > \mu_i\} + \#\{j : j > i \text{ and } \mu_j \geq \mu_i\}, \text{ and}$$
$$\widetilde{\mu} := \left(q^{\mu_1} t^{-k_1(\mu)}, \ldots, q^{\mu_n} t^{-k_n(\mu)}\right).$$

Example: When $\mu = (4,2,0,1,4)$ we have $\widetilde{\mu} = (q^4 t^{-1}, q^2 t^{-2}, t^{-4}, q t^{-3}, q^4)$. For a polynomial $f(x_1, \ldots, x_n)$ and $\mu \in \mathbb{N}^n$, define $f(\widetilde{\mu}) = f(\widetilde{\mu}_1, \ldots, \widetilde{\mu}_n)$.

Interpolation ASEP polynomials

Theorem (B.D–Williams '25)

For any composition $\mu \in \mathbb{N}^n$*, there exists a unique polynomial* $F^*_\mu(x_1, \ldots, x_n; q, t)$ *such that:*

- $\deg(F^*_\mu) \leq |\mu|$,
- *for any* $\tau \in S_n(\mu)$*, we have* $[x^\tau]F^*_\mu = \delta_{\tau,\mu}$.
- $F^*_\mu(\widetilde{\nu}) = 0$*, for any* ν *satisfying* $|\nu| \leq |\mu|$ *and* $\nu \notin S_n(\mu)$.

Moreover, the top homogeneous part of F^*_μ *is the ASEP polynomial* F_μ.

We call these polynomials the interpolation ASEP polynomials.

Interpolation ASEP polynomials

Theorem (B.D–Williams '25)

For any composition $\mu \in \mathbb{N}^n$, there exists a unique polynomial $F^*_\mu(x_1, \ldots, x_n; q, t)$ such that:

- $\deg(F^*_\mu) \leq |\mu|$,
- *for any* $\tau \in S_n(\mu)$, *we have* $[x^\tau]F^*_\mu = \delta_{\tau,\mu}$.
- $F^*_\mu(\widetilde{\nu}) = 0$, *for any* ν *satisfying* $|\nu| \leq |\mu|$ *and* $\nu \notin S_n(\mu)$.

Moreover, the top homogeneous part of F^*_μ is the ASEP polynomial F_μ.

We call these polynomials the interpolation ASEP polynomials.

This result is based on the theory of interpolation polynomials of Knop and Sahi.

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Example: $n = 1$ and $\mu = (k)$. We want

$$F^*_{(k)}(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_0,$$
$$F^*_{(k)}(q^m) = 0 \qquad \text{for } 0 \leq m < k,$$

We then have

$$F^*_{(k)}(x) = (x-1)(x-q)\ldots(x-q^{k-1}).$$

Interpolation ASEP polynomials

Example: $n = 2$ and $\mu = (0, 2)$.

$$F^*_{(0,2)}(x_1, x_2) = x_2^2 + ax_1x_2 + bx_1 + cx_2 + d,$$

$$F^*_{(0,2)}(q/t, q) = 0 \qquad \nu = (1, 1)$$

$$F^*_{(0,2)}(q, 1/t) = 0 \qquad \nu = (1, 0)$$

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We then have

$$F^*_{(0,2)}(x_1, x_2) = x_2^2 + \frac{1-t}{1-qt}x_1x_2 + q\frac{1-t}{1-qt}x_1 + \frac{1+qt-qt^2-q^2t^2}{t(1-qt)}x_2 + \frac{q(1-qt)}{t(1-qt^2)}.$$

In particular,

$$F_{(0,2)}(x_1, x_2) = x_2^2 + \frac{1-t}{1-qt}x_1x_2.$$

Interpolation symmetric Macdonald polynomials

Theorem (B.D–Williams '25)

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Moreover, the top homogeneous part of F^*_μ is the ASEP polynomial F_μ.

Proposition

For any partition λ, we have

$$P^*_\lambda(x_1, \dots, x_n; q, t) = \sum_{\mu \in S_n(\lambda)} F^*_\mu(x_1, \dots, x_n; q, t),$$

where P^*_λ is the symmetric interpolation Macdonald polynomial.

Action of the Hecke operators

The action of the Hecke operators on interpolation polynomials is similar to their action on the homogeneous polynomials.

Proposition

For any composition μ, we have

$$T_i F^*_\mu = \begin{cases} F^*_{s_i\mu} & \text{if } \mu_i > \mu_{i+1}, \\ tF^*_\mu & \text{if } \mu_i = \mu_{i+1}, \\ tF^*_{s_i\mu} - (1-t)F^*_\mu & \text{if } \mu_i < \mu_{i+1} \end{cases}.$$

However, the interpolation polynomials do not satisfy the circular symmetry. In particular, they cannot be characterized by the qKZ equations.

The Interpolation PushTASEP model

Fix a partition $\lambda = (\lambda_1, \dots, \lambda_n)$. The interpolation *PushTASEP* with content λ is a Markov process such that each state is indexed by a composition $\mu \in S_n(\lambda)$. We interpret μ as n particles on a ring, labeled $\mu_1, \dots, \mu_n$.

Conventions and notation:

- There exists at least one part of size 0 in λ.
- We denote $[m]_t$ the t-integer

$$[m]_t = 1 + t + \cdots + t^{m-1}.$$

- We define for $1 \leq k \leq n$, the following elements in $\mathbb{Q}(t, x_1, \dots, x_n)$:

$$\mathfrak{p}_k := \frac{t^{-n+1}(1-t)}{x_k - t^{-n+2}}, \qquad \text{and} \qquad \mathfrak{q}_k := \frac{(1-t)x_k}{x_k - t^{-n+2}}.$$

- We assume that $0 < t < 1$ and that $x_i > t^{-n+1}$ for $1 \leq i \leq n$. Under these hypotheses on the parameters, the quantities $\mathfrak{p}_k$ and $\mathfrak{q}_k$ are probabilities.

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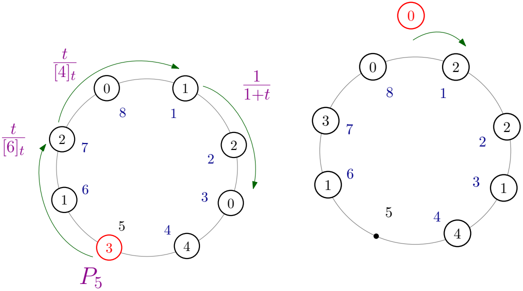
The dynamics are as follows:

Step 0 We choose the particle in the j-th position with probability P_j proportional to $\prod_{k<j}\left(x_k - \frac{1}{t^{n-2}}\right)\prod_{k>j}\left(x_k - \frac{1}{t^{n-1}}\right)$.

Step 1 The particle at position j, say with label a, is activated, and starts traveling clockwise according to the rules of the (classical) t-Push TASEP.



Step 1 of the interpolation t-PushTASEP

The Interpolation PushTASEP model

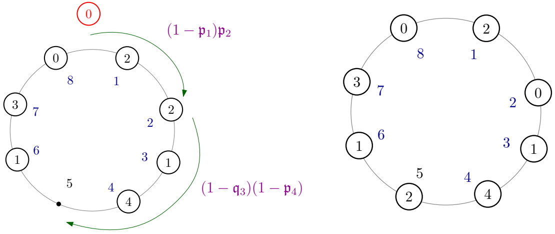
Fix a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$. The interpolation *PushTASEP* with content λ is a Markov process such that each state is indexed by a composition $\mu \in S_n(\lambda)$. We interpret μ as n particles on a ring, labeled $\mu_1, \ldots, \mu_n$.

The dynamics are as follows:

Step 2 The last activated particle in Step 1 labeled $a := 0$, now goes to position 1 and starts traveling clockwise. When the activated particle labeled a gets to site k for $1 \leq k \leq j-1$ containing a particle with label $b \geq 0$, it settles at that site (displacing and activating the site's particle) with probability

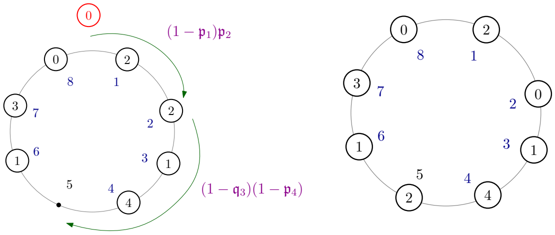
$$\begin{cases} \mathfrak{p}_k & \text{if } b > a, \\ \mathfrak{q}_k & \text{if } b < a \end{cases}$$

The activated particle always settles at position j with probability 1.



Step 2 of the interpolation t-PushTASEP

The Interpolation PushTASEP model



Step 2 of the interpolation t-PushTASEP

Remarks:

- Unlike in the classical PushTASEP, this interpolation model is not invariant under rotation.
- In Step 2, the active particle can push both weaker and stronger particles.
- When $x_i \gg 1$, we recover the classical PushTASEP:

$$\mathfrak{p}_k := \frac{t^{-n+1}(1-t)}{x_k - t^{-n+2}} \longrightarrow 0,$$

the active particle cannot push a stronger particle and Step 2 is trivial.

The interpolation PushTASEP

Theorem (B.D–Williams '25)

The stationary distribution of the interpolation PushTASEP with content λ is proportional to the interpolation ASEP polynomials evaluated at $q = 1$. Equivalently,

$$\pi^*_\lambda(\mu) = \frac{F^*_\mu(x_1, \ldots, x_n; q = 1, t)}{P_\lambda(x_1, \ldots, x_n; q = 1, t)}.$$

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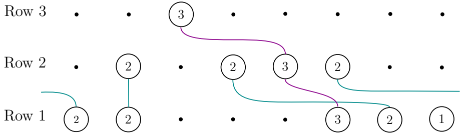
The proof is based on a combinatorial formula for the interpolation ASEP polynomials in terms of multiline queues.

Main ingredient of the proof: Multiline queues

A ball system is an $L \times n$ array (for some $L, n \geq 1$), with rows labeled from bottom to top as $1, 2, \ldots, L$, and columns labeled from left to right from 1 to n, in which each of the Ln positions is either empty or occupied by a ball labeled by $a > 0$.

A multiline queue is a ball system such that:

- each ball in row $r > 1$ is paired with a ball in the row below, using the shortest strand traveling (weakly) from left to right, allowing the strand to wrap around if necessary.
- a ball in a strand of height k is labeled by k,
- it does not contain the forbidden configuration.



A multiline queue.

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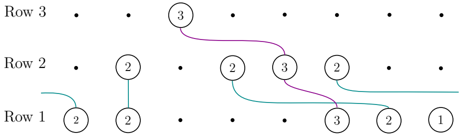

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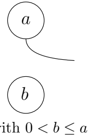

The forbidden configuration for multiline queues.

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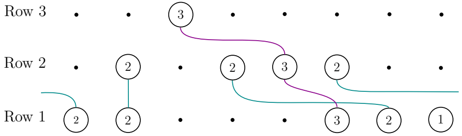


A multiline queue of type $(2, 2, 0, 0, 0, 3, 2, 1)$.

The type of a multiline queue is the composition μ obtained by reading the labels in row 1.

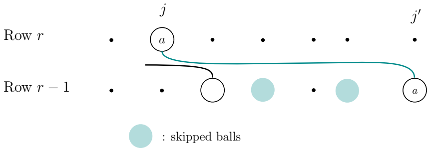
- $n = \#$columns in the multiline queue $= \#$parts of μ.
- $L = \#$rows in the multiline queue $=$ the size of the maximal part in μ.

Weights of Multiline queues (at $q = 1$)

- A ball in column i has the weight x_i.
- **We define a specific order on the pairings of the multiline queue.** Each nontrivial pairing p, connecting balls labeled a, between rows $r > 1$ and $r-1$, and columns j and j', has weight $\mathrm{wt}_{\mathrm{pair}}(p)$:

$$\mathrm{wt}_{\mathrm{pair}}(p) = \frac{(1-t)t^{\mathrm{skip}(p)}}{1-t^{\mathrm{free}(p)}}$$

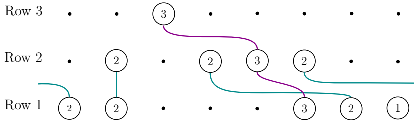
free(p) : balls not yet paired in row $r-1$,
skip(p): free balls which have been skipped by p.



- The total weight of a multiline queue Q is denoted $\mathrm{wt}(Q)$:

$$\mathrm{wt}(Q) = \prod_{\text{balls } B} \mathrm{wt}(B) \prod_{\text{pairings } p} \mathrm{wt}_{\mathrm{pair}}(p).$$

The multiline queue formula for Macdonald polynomials



A multiline queue of type $(2, 2, 0, 0, 0, 3, 2, 1)$.

$$\mathrm{wt}(Q) = x_1 x_2^2 x_3 x_4 x_5 x_6^2 x_7 x_8 \frac{(1-t)t}{1-t^4} \frac{1-t}{1-t^5} \frac{(1-t)t^2}{1-t^3} \frac{1-t}{1-t^2}.$$

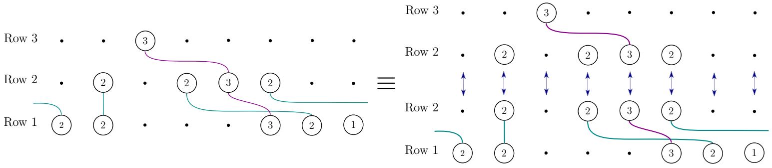
Theorem (Corteel–Mandelshtam–Williams ’23)

For any composition μ, we have

$$F_\mu(x_1, \ldots, x_n; q = 1, t) = \sum_{\substack{\text{multiline-queues} \\ Q \text{ of type } \mu}} \mathrm{wt}(Q).$$

Correspondence between MLQ and transitions in the PushTASEP

Write a combinatorial decomposition/branching rule.



Correspondence between MLQ and transitions in the PushTASEP

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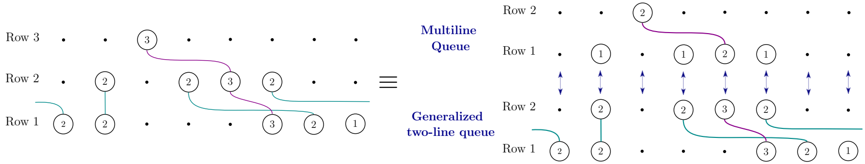
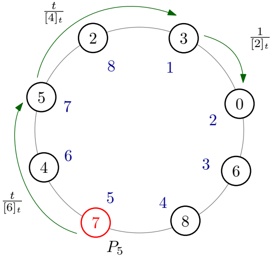
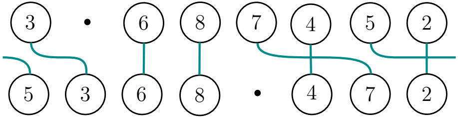


Correspondence between MLQ and transitions in the PushTASEP

We can encode the transitions in the PushTASEP using Generalized two-line queues.



A transition of the t-PushTASEP



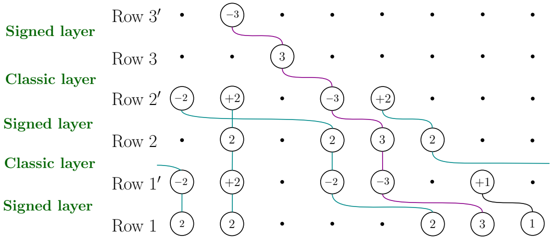
The corresponding two-line queue.

Signed Multiline queues

An enhanced ball system is a $2L \times n$ array ($L, n \geq 1$), with rows labeled from bottom to top as $1, 1', 2, 2', \ldots, L, L'$, and columns labeled from left to right from 1 to n, in which each of the $2Ln$ positions is either empty or occupied by a ball. A ball in row r is labeled by $a > 0$, and a ball in row r' is labeled $\pm a$, where $a > 0$.

A signed multiline queue is an enhanced ball system, satisfying the conditions:

- each pair connects two balls with the same absolute value,
- after forgetting the signs, classic layers correspond to layers from classic MLQ.
- each ball in row r' is paired with a ball in row r, using the shortest strand traveling (weakly) from left to right, without wrapping around.
- it does not contain any forbidden configuration.



A signed multiline queue of type $(2, 2, 0, 0, 0, 3, 2, 1)$.

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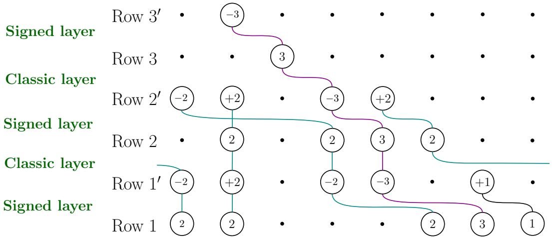

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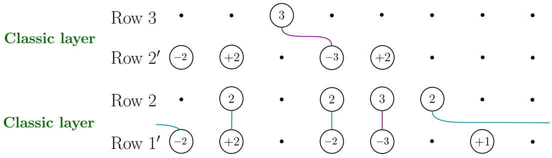

Classic layers of a signed multiline queue.

Signed Multiline queues

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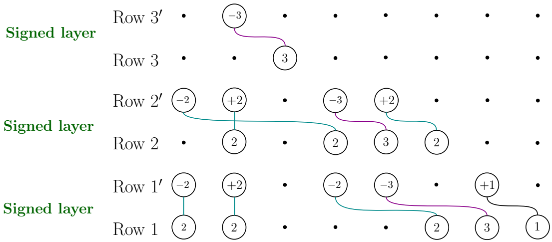

Signed layers of a signed multiline queue.

Signed Multiline queues

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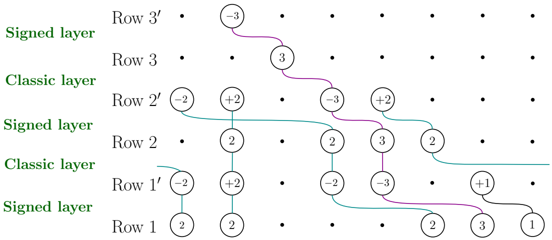

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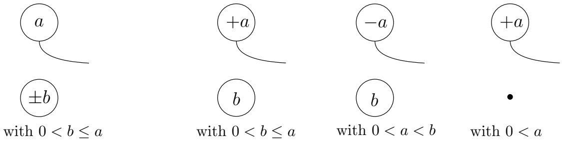


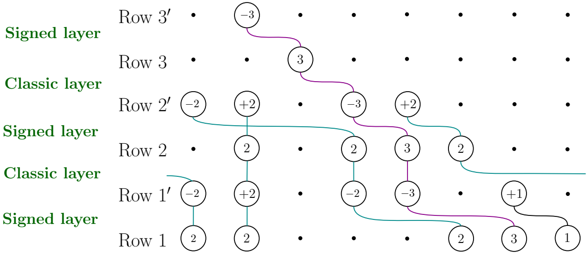
The forbidden configuration for signed multiline queues.

Weights of Signed Multiline Queues

- Only signed balls have weights. A ball in column i, row r', has the weight

$$\begin{cases} x_i & \text{if it is positive,} \\ \frac{-1}{t^{n-1}} & \text{if it is negative.} \end{cases}$$

- Each nontrivial pairing p has a weight $\mathrm{wt}_{\mathrm{pair}(p)}$ (we use the same order to place pairings)



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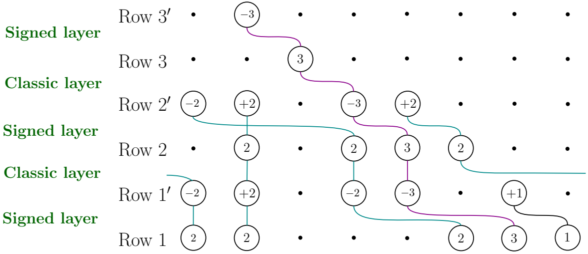

A signed multiline queue with ball weight: $x_2^2 x_5 x_7 \left(\frac{-1}{t^7}\right)^3 \left(\frac{-1}{t^7}\right)^2 \left(\frac{-1}{t^7}\right)$

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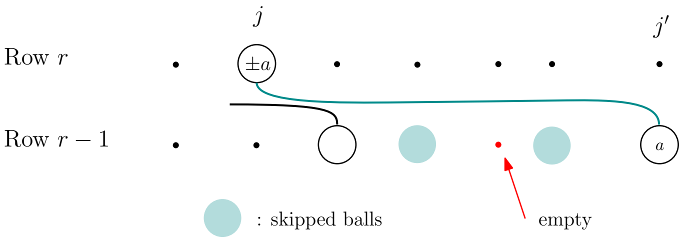
- Weights of pairings in classic layers are defined as before,
- weight of a nontrivial pairing in a signed layer is given by

$$\mathrm{wt}_{\mathrm{pair}}(p) = \begin{cases} (1-t)t^{\mathrm{skip}(p)+\mathrm{empty}(p)} & \text{if } p \text{ connects a positive ball and a regular ball} \\ -(1-t)t^{\mathrm{skip}(p)+\mathrm{empty}(p)} & \text{if } p \text{ connects a negative ball and a regular ball.} \end{cases}$$



Multiline queue formula for interpolation polynomials

Theorem (BD–Williams)

For any composition μ, we have

$$F^*_\mu(x_1, \ldots, x_n; q = 1, t) = \sum_{\substack{\text{signed multiline-queues} \\ Q \text{ of type } \mu}} \mathrm{wt}(Q).$$

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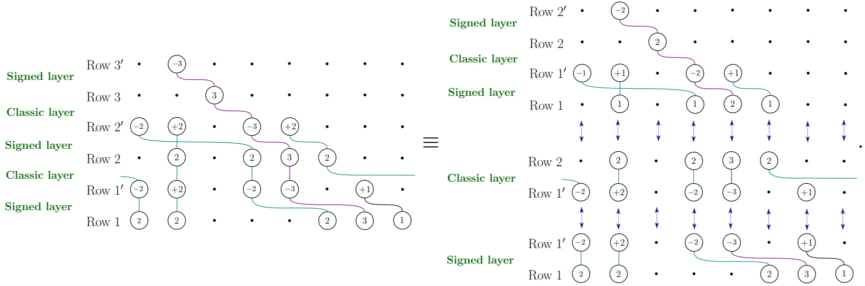
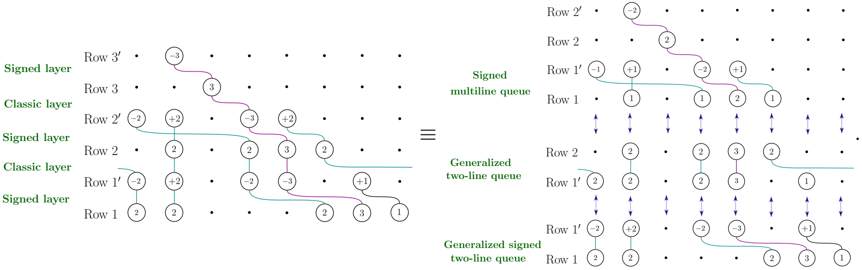
We also give a tableau formula for F^*_μ and P^*_μ.

Correspondence between signed multiline queues and transition in the interpolation PushTASEP

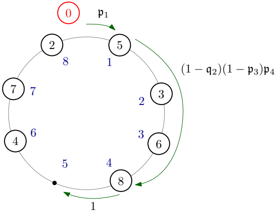


Correspondence between signed multiline queues and transition in the interpolation PushTASEP

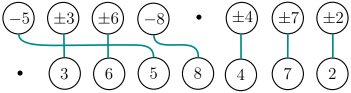


Correspondence between signed multiline queues and transition in the interpolation PushTASEP

- We can encode the transitions in Step 1 of the interpolation PushTASEP using generalized two-line queues.
- We can encode the transitions in Step 2 of the interpolation PushTASEP using signed generalized two-line queues:



A transition in Step 2 of the interpolation PushTASEP



The corresponding signed two-line queues.