Path operators and (q, t)-tau functions

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joint work with
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Weighted double Hurwitz numbers

Fix a power series $G(z) = 1 + \sum_{k\geq 1} u_k z^k$.

$$\log\left(\sum_{\lambda} s_\lambda(X) s_\lambda(Y) \prod_{\square\in\lambda} G(c(\square))\right) = \sum_{n\geq 1}\sum_{\mu,\nu\vdash n} H_G(\mu,\nu) p_\mu(X) p_\nu(Y),$$

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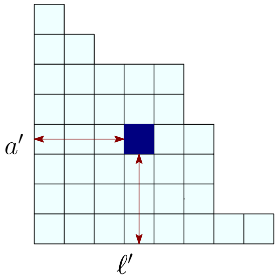

$a'(\square)$: the co-arm of $\square$,

$\ell'(\square)$: the co-leg of $\square$,

$c(\square) := a'(\square) - \ell'(\square)$ is the content of $\square$.

$G(c(\square))$ is a formal power-series in the variables u_k.

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$H_G(\mu,\nu)$ are called the double G-weighted Hurwitz numbers.

Well studied cases:

- classical Hurwitz numbers $G(z) = \exp(uz)$,
- monotone Hurwitz numbers $G(z) = 1/(1-uz)$,
- dessins d'enfants $G(z) = 1 + uz$.

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$H_G(\mu, \nu)$ count equivalently:

- some families of maps on orientable surfaces,
- branched coverings of the sphere by an orientable surface,

with weights which depend on the variables u_k.

Jack deformation

$$\log\left(\sum_{\lambda}\frac{J_\lambda^{(\alpha)}(X)J_\lambda^{(\alpha)}(Y)}{j_\lambda^{(\alpha)}}\prod_{\square\in\lambda}G(c_\alpha(\square))\right)=\sum_{n\geq 1}\sum_{\mu,\nu\vdash n}H_G^{(\alpha)}(\mu,\nu)p_\mu(X)p_\nu(Y),$$

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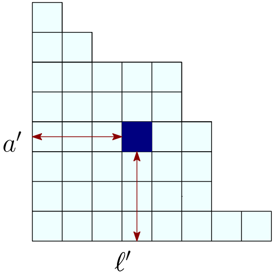

$a'(\square)$: the co-arm of $\square$,
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$H_{G}^{(\alpha)}(\mu,\nu)$ are α-deformed double Hurwitz numbers.

In the one alphabet case, they count some families of maps on non-orientable surfaces, with an α-weight correlated to their non-orientability (Bonzom–Chapuy–Dołęga 2023, Ruza 2023, Chidambaram–Dołęga–Osuga 2024).

The (q,t)-tau function

$$\tau_G(X,Y) := \sum_{\lambda} \frac{\widetilde{H}_\lambda^{(q,t)}[X]\widetilde{H}_\lambda^{(q,t)}[Y]}{w_\lambda} \prod_{\square\in\lambda} G(c_{q,t}(\square)),$$

$\widetilde{H}_\lambda^{(q,t)}$: Modified Macdonald polynomials,
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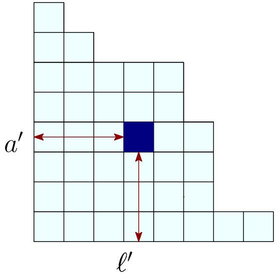

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This series is related to:

- series in enumerative geometry (Hausel –Letellier–Villegas '11, Carlsson–Villegas '16, Mellit '18),
- a (q,t)-extension of the 2D Toda hierarchy (Bourgine–Garbali '24).

Main result

We associate to a formal power-series $G(z) = 1 + \sum_{k\geq 1} u_k z^k$ and an integer $n \geq 1$, a combinatorial differential operator $\mathcal{A}_G^{(n)}$ on the space of symmetric functions.

Theorem (BD–Bonzom–Dołęga)

Write $G = G_1/G_2$ where G_1 and G_2 are series with constant term 1. For any $n \geq 1$ we have

$$\mathcal{A}_{G_1}^{(n)}(X) \cdot \tau_G(X,Y) = \left(\mathcal{A}_{G_2}^{(n)}(Y)\right)^* \cdot \tau_G(X,Y),$$

where $\left(\mathcal{A}_{G_2}^{(n)}\right)^*$ is the adjoint of $\mathcal{A}_{G_2}^{(n)}$ with respect to $\langle .,. \rangle_*$. Moreover, these equations fully characterize the function $\tau_G(X,Y)$.

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Remark

One can always take $G_2 = 1$.

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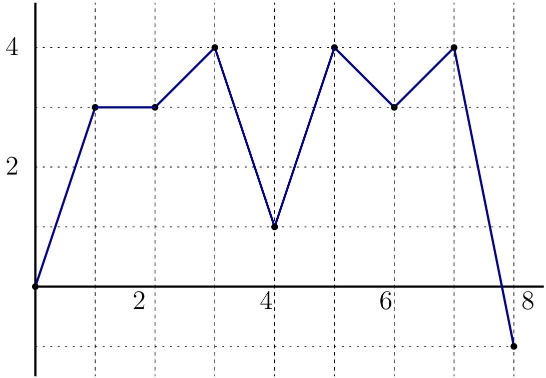
Example: If $G = \frac{1}{1-u}$, we can take

- $G_1 = 1 + u + u^2 + \ldots$ and $G_2 = 1$.
- $G_1 = 1$ and $G_2 = 1 - u$.

Alternating paths

An alternating path of **length** $2\ell > 0$ and **degree** $n \in \mathbb{Z}$, is a path in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}$, starting at $(0,0)$ ending at $(2\ell, n)$, and such that an odd (resp. even) step is a weakly up step (resp. weakly down step).



An alternating path of length 8 and degree -1.

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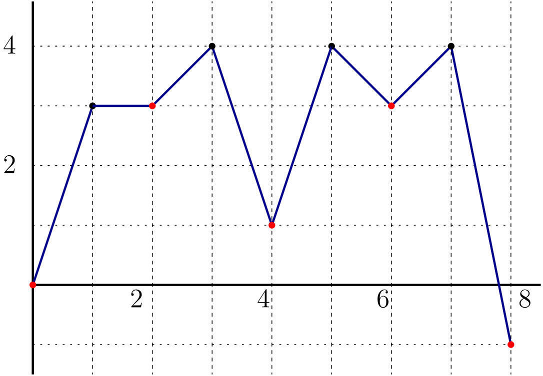
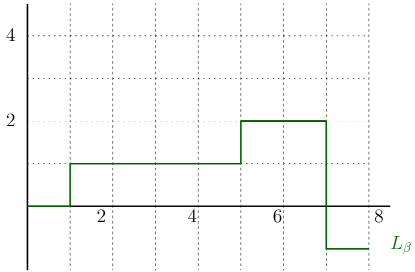
A valley is a point of the path with an even x-coordinate.
The other points are peaks.



The valleys of an alternating path of length 8 and degree -1.

The paths $\mathbf{R}_\beta$

Fix a sequence $\beta := (\beta_1, \ldots, \beta_\ell) \in \mathbb{Z}^\ell$. Define L_β as the path starting at $(0, 0)$ ending at $(2\ell, |\beta|)$ and with vertical increment β_j at x-coordinate $2j - 1$.



The path L_β for $\beta = (1, 0, 1, -3)$.

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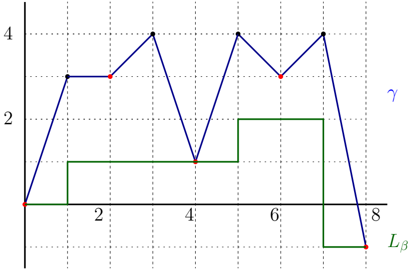
Define $\mathbf{R}_\beta$ as the set of alternating paths of length 2ℓ, degree $|\beta|$, staying weakly above L_β.



A path $\gamma \in \mathbf{R}_\beta$ for $\beta = (1, 0, 1, -3)$.

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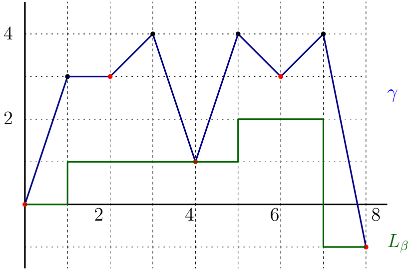

A path $\gamma \in \mathbf{R}_\beta$ for $\beta = (1, 0, 1, -3)$.

If $\gamma \in \mathbf{R}_\beta$ and V is a valley of γ, we define its β-height $\mathrm{ht}_\beta(V) \geq 0$ as the height w.r.t to L_β.

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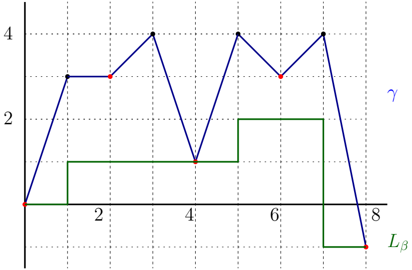

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The five valleys of the path in the example have respective β-heights $0, 2, 0, 1, 0$.

Path operators

Plethystic identities

$$\sum_{n\geq 0} z^n h_n[X] = \exp\left(\sum_{k\geq 1} z^k \frac{p_k[X]}{k}\right)$$

h_n : complete homogeneous symmetric functions.

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$$\sum_{n\geq 0} z^n h_n[-X] = \exp\left(\sum_{k\geq 1} \frac{-z^k p_k[X]}{k}\right),$$

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$$\sum_{n\geq 0} z^n h_n[MX] = \exp\left(\sum_{k\geq 1} \frac{(1-q^k)(1-t^k)p_k[X]}{k}\right)$$

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$$\sum_{n\geq 0} z^n h_n^{\perp}[MX] = \exp\left(\sum_{k\geq 1} \frac{(1-q^k)(1-t^k) z^k p_k^{\perp}[X]}{k}\right)$$

with $p_k^{\perp} = \frac{k\partial}{\partial p_k}$.

We associate to each one-step path of degree k an operator:

$$\mathcal{O}(k) := \begin{cases} h_k[-X] & \text{if } k > 0 \\ h_{\overline{k}}^{\perp}[MX] & \text{if } k < 0 \\ 1 & \text{if } k = 0, \end{cases}$$

Remark

Each one of these operators is differential in the variables p_k.

Path operators

To a path $\gamma \in \mathbf{R}_\beta$ with steps $(\gamma_1, \ldots, \gamma_{2\ell})$, we associate the operator

$$\mathcal{O}_\beta(\gamma) := \left(\prod_{V \, \text{valley}} (qt)^{\text{ht}_\beta(V)} \right) \mathcal{O}(\gamma_1) \ldots \mathcal{O}(\gamma_{2\ell}).$$

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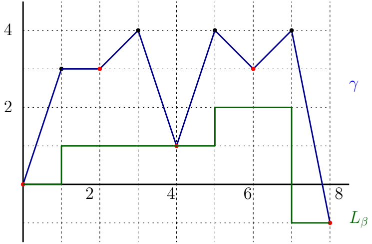
If γ has degree n, then $\mathcal{O}_\beta(\gamma)$ is homogeneous of degree n.



A path $\gamma \in \mathbf{R}_\beta(1, 0, 1, -3)$.

Example: $\mathcal{O}_\beta(\gamma) = (qt)^2(qt)^1 h_3[-X]h_1[-X]h_3^\perp[MX]h_3[-X]h_1^\perp[MX]h_1[-X]h_5^\perp[MX].$

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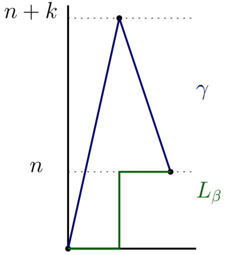
Define the operator $\mathcal{R}_\beta$ by

$$\mathcal{R}_\beta := \sum_{\gamma \in \mathbf{R}_\beta} \mathcal{O}_\beta(\gamma).$$

Examples

- Case $\ell = 1$: when $\beta = (n)$

$$\mathcal{R}_{(n)} = \sum_{k \geq 0} h_{k+n}[-X] h_k^{\perp}[MX] =: D_n.$$



Paths corresponding to the operator $\mathcal{R}_{(n)} = D_n$.

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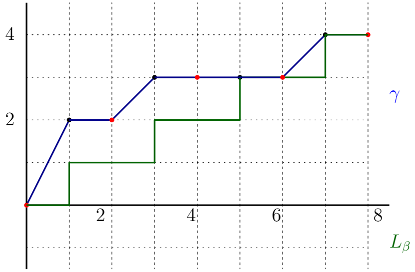
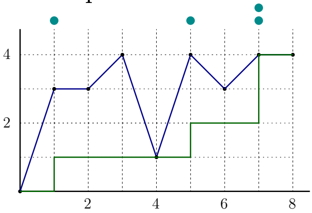
- Case $t = 1$ and $\beta = (1, 1, \ldots, 1)$: we then have $M = (1-q)(1-t) = 0$ all even steps should be horizontal, and alternating paths become Dyck paths counted with q^{area} and a weight $(-1)^k e_k$ for any up step of degree k.



A path appearing in the expansion of $\mathcal{R}_{(1,1,1,1)}$ at $t = 1$.

A reparametrization: the operators $\mathcal{Q}_\alpha$

To study the tau function τ_G, we consider sequences $\beta = (\beta_1, \ldots, \beta_\ell) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}^{\ell-1}$. We decorate peaks of alternating paths by particles: one particle corresponds to a unit increment of L_β. In other terms, there are β_i particles on the i-th peak.



An alternating path γ in $\mathbf{R}_{1,0,1,2}$.

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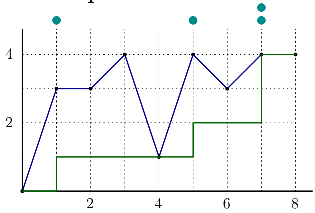

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Fix $\beta \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}^{\ell-1}$ of size n, we define a sequence $\alpha \in \mathbb{Z}_{\geq 0}^{n}$ such that:

- $2\alpha_i$ is the distance between the i-th and the $i+1$-th particles, for $1 \leq i \leq n-1$,
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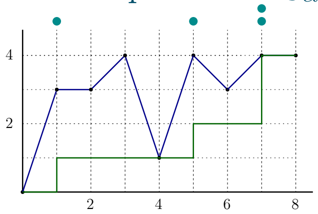

An alternating path γ in $\mathbf{Q}_{(2,1,0,0)} = \mathbf{R}_{1,0,1,2}$.

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We then define the set $\mathbf{Q}_\alpha$ by

$$\mathbf{Q}_\alpha := \mathbf{R}_\beta$$

and the operator $\mathcal{Q}_\alpha := \mathcal{R}_\beta = \sum_{\gamma \in \mathbf{Q}_\alpha} \mathcal{O}_\beta(\gamma)$.

Operators $\mathcal{A}_G^{(n)}$

If $G(z) = u_0 + u_1 z + u_2 z^2 + \dots$ (with $u_0 = 1$) we define

$$\mathcal{A}_G^{(n)} := \sum_{\alpha \in (\mathbb{Z}_{\geq 0})^n} u_{\alpha_1} \dots u_{\alpha_n} \mathcal{Q}_\alpha,$$

defined as formal power-series in the variables u_i.
$\rightarrow$ the series of all path operators of degree n with an extra weight u_i for two particles separated by distance i.

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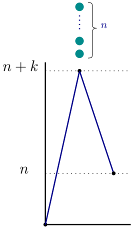
Example: When $G = 1$, $\mathcal{A}_1^{(n)} = D_n$.



Paths corresponding to the operator $\mathcal{A}_1^{(n)}$.

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For any partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$, define

$$\mathfrak{a}_{G,\lambda} := \mathcal{A}_G^{(\lambda_1)} \dots \mathcal{A}_G^{(\lambda_{\ell(\lambda)})} \cdot 1.$$

Fact: The family $(\mathfrak{a}_{G,\lambda})_\lambda$ is a basis for the space of symmetric functions.

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$$\tau_G(X,Y) = \sum_{\lambda} \mathfrak{a}_{G_1,\lambda}(X)\mathfrak{b}_{G_2,\lambda}(Y).$$

When $G = 1$, this corresponds to the Macdonald Cauchy identity.

Idea of the proof:

Construct the path operators from simple operators which act nicely on power-sums and on Macdonald polynomials in the same time.

Theorem (Pieri rule, Macdonald '96)

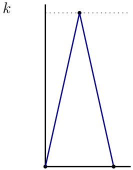
For any λ,

$$h_1 \cdot \widetilde{H}_{\lambda}^{(q,t)} = \sum_{\lambda \nearrow \mu} d^{\lambda,\mu} \widetilde{H}_{\mu}^{(q,t)},$$

where the sum runs over partitions λ *obtained from* μ *by adding one cell.*

The Macdonald operator:

$$D_0 := \sum_{k\geq 0} h_k[-X]h_k^{\perp}[MX]$$



Paths corresponding to the operator D_0.

Theorem (Macdonald '96, Garsia–Haiman–Tesler '99)

$$D_0 \cdot \widetilde{H}_{\lambda}^{(q,t)} = \left(1 - M \sum_{\square\in\lambda} c_{q,t}(\square)\right) \widetilde{H}_{\lambda}^{(q,t)},$$

Macdonald polynomials are the unique functions satisfying this property up to normalization.

Commutators

$$h_1 \cdot \widetilde{H}_\lambda^{(q,t)} = \sum_{\lambda \nearrow \mu} d^{\lambda,\mu} \widetilde{H}_\mu^{(q,t)},$$

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Example:

$$\begin{aligned}[h_1, D_0] \cdot \widetilde{H}_\lambda^{(q,t)} &= M \sum_{\lambda \nearrow \mu} \left(\sum_{\square \in \mu} c_{q,t}(\square) - \sum_{\square \in \lambda} c_{q,t}(\square) \right) d^{\lambda,\mu} \widetilde{H}_\mu^{(q,t)} \\ &= M \sum_{\lambda \nearrow \mu} c_{q,t}\left(\mu/\lambda\right) d^{\lambda,\mu} \widetilde{H}_\mu^{(q,t)},\end{aligned}$$

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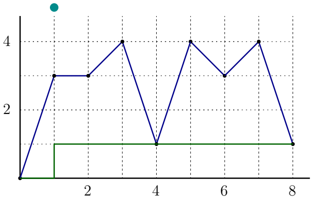
Example:

$$\underbrace{[[[h_1, D_0], D_0], \ldots D_0]}_{k \text{ times}} \cdot \widetilde{H}_\lambda^{(q,t)} = M^k \sum_{\lambda \nearrow \mu} \left(c_{q,t}\left(\mu/\lambda\right)\right)^k d^{\lambda,\mu} \widetilde{H}_\mu^{(q,t)},$$

The first commutation relation (one particle paths)

The case $\alpha = (n)$.



One particle alternating path in $\mathbf{Q}_3$.

Theorem (BD–Bonzom–Dołęga)

For any $n \geq 0$,

$$\mathcal{Q}_{(n+1)} = \frac{1}{M}\left[D_0, \mathcal{Q}_{(n)}\right].$$

As a consequence,

$$\mathcal{Q}_{(n)} = \frac{-1}{M^n} \underbrace{[D_0, [D_0, \ldots [D_0, h_1]]]}_{n+1 \text{ times}}$$

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$$\mathcal{A}_G^{(1)} := \mathcal{Q}_0 + \sum_{n\geq 1} u_n \mathcal{Q}_{(n)}.$$

$$\Longrightarrow \mathcal{A}_G^{(1)} \cdot \widetilde{H}_\lambda^{(q,t)} = \sum_{\lambda \nearrow \mu} G(c_{q,t}(\mu/\lambda))\, d^{\lambda,\mu}\, \widetilde{H}_\mu^{(q,t)}$$

We obtain the first differential equation

$$\mathcal{A}_{G_1}^{(1)}(X) \cdot \tau_G(X,Y) = \left(\mathcal{A}_{G_2}^{(1)}(Y)\right)^* \cdot \tau_G(X,Y).$$

The second commutation relation

Theorem (BD–Bonzom–Dołęga)

For any $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$

$$\sum_{\sigma \in \mathfrak{S}_n} \mathcal{Q}_{\sigma(\alpha)} = \frac{1}{M} \sum_{\sigma \in \mathfrak{S}_n} \left[\mathcal{Q}_{\alpha_{\sigma(n)}-1}, \mathcal{Q}_{\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(n-1)}} \right],$$

where $\sigma(\alpha) := (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)})$, *and* $\mathcal{Q}_{-1} = -h_1$.

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As a consequence, for any power-series G, we have

$$\mathcal{A}_G^{(n+1)} = \frac{1}{M} \left[\mathcal{A}_{G/z}^{(1)}, \mathcal{A}_G^{(n)} \right].$$

By induction on n, we obtain the higher order differential equations:

$$\mathcal{A}_{G_1}^{(n)}(X) \cdot \tau_G(X, Y) = \left(\mathcal{A}_{G_2}^{(n)}(Y) \right)^* \cdot \tau_G(X, Y).$$

Connection to the Shuffle algebra and Negut elements

A vertex operator formula:

Proposition (BD–Bonzom–Dołęga)

For any $\beta \in (\mathbb{Z}_{\geq 0})^\ell$,

$$\mathcal{R}_\beta = [z_1^{\beta_1} \cdots z_\ell^{\beta_\ell}] \frac{D(z_1) \cdots D(z_\ell)}{\prod_{i=1}^{\ell-1} \left(1 - qt z_{i+1}/z_i\right)},$$

where

$$D(z) = \sum_{m,n \geq 0} z^{m-n} h_m[-X] h_n^\perp[MX].$$

Proof: Induction on ℓ.

Connection to the Shuffle algebra and Negut elements

The shuffle algebra S is the graded algebra $S = \oplus_{n\geq 0} S_n$ where $S_0 = \mathbb{Q}(q,t)$, and for $n \geq 1$ an element F of S_n is of the form

$$F(z_1,\dots,z_n) = f(z_1,\dots,z_n) \prod_{1\leq i\neq j\leq n} \frac{z_i - z_j}{(z_i - qz_j)(z_i - tz_j)},$$

where $f(z_1,\dots,z_n)$ is a symmetric Laurent polynomial which satisfies the *wheel condition.*

$$f(z_1,\dots,z_n) = 0 \qquad \text{whenever} \qquad \left\{\frac{z_1}{z_2}, \frac{z_2}{z_3}, \frac{z_3}{z_1}\right\} = \left\{q, t, \frac{1}{qt}\right\}.$$

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The shuffle algebra S is equipped with the following product: for $F \in S_n, G \in S_m$, then $F * G \in S_{n+m}$ with

$$(F*G)(z_1,\ldots,z_{n+m}) = \frac{1}{n!m!} \operatorname{Sym}\left(F(z_1,\ldots,z_n) G(z_{n+1},\ldots,z_{n+m}) \prod_{i=1}^{n} \prod_{j=n+1}^{n+m} \omega(z_j/z_i) \right),$$

where Sym denotes the symmetrization with respect to the variables $z_1,\ldots,z_{n+m}$, and

$$\omega(x) := \frac{(1-x)(1-qtx)}{(1-qx)(1-tx)}.$$

Connection to the Shuffle algebra and Negut elements

Negut elements: Negut '14 introduced a family of elements indexed by $\beta \in \mathbb{Z}^\ell$:

$$X_\beta(z_1, \ldots, z_\ell) := \mathrm{Sym}\left(\frac{z_1^{-\beta_1} \cdots z_\ell^{-\beta_\ell}}{\prod_{i=1}^{\ell-1}(1 - qt z_{i+1}/z_i)} \prod_{1 \le i < j \le \ell} \omega(z_j/z_i) \right) \quad \in S_\ell.$$

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The shuffle algebra acts on the space of symmetric functions through the representation $F \mapsto \hat{F}$ where

$$\hat{F} := [z_1^0 \cdots z_n^0] F(z_1, \ldots, z_n) \frac{D(z_1) \cdots D(z_n)}{\prod_{1 \leq i < j \leq n} \omega(z_j/z_i)}.$$

Theorem (BD–Bonzom–Dołęga)

For any $\beta \in \mathbb{Z}^\ell$, we have

$$\widehat{X}_\beta = \mathcal{R}_\beta.$$

Connection to the extended delta conjecture

Let Π_G be the operator on symmetric functions defined by

$$\Pi_G \cdot \widetilde{H}_\lambda^{(q,t)} = \widetilde{H}_\lambda^{(q,t)} \prod_{\square \in \lambda} G(c_{q,t}(\square)).$$

Reformulation of the main result: for $G = G_1/G_2$, we have

$$\Pi_G \cdot \mathcal{A}_{G_2}^{(n)} \cdot \Pi_G^{-1} = \mathcal{A}_{G_1}^{(n)}, \quad \text{for } n \geq 1.$$

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Define the operators Δ_{h_n} and Δ_{e_n} by their generating series:

$$\Pi_{1+uz} = \sum_{n\geq 0} u^n \Delta_{e_n}, \quad \Pi_{(1-vz)^{-1}} = \sum_{n\geq 0} v^n \Delta_{h_n}.$$

Define also

$$\Delta'_{e_n} = \sum_{i=0}^{n} (-1)^i \Delta_{e_{n-i}}.$$

The extended Delta conjecture (Haglund–Remmel–Wilson '18), proved by Blasiak–Haiman–Morse–Pun–Seelinger '23, states that

$$\Delta_{h_l} \Delta'_{e_k} \cdot e_n$$

is positive in the monomial basis and counts Dyck paths with some weights.

Connection to the extended delta conjecture

Theorem (BD–Bonzom–Dołęga)

Fix three integers $l \geq 0$ and $0 < k \leq n$.
We have

$$(-1)^n \Delta_{h_l} \Delta'_{e_{k-1}} \cdot e_n = \sum_{\substack{\beta \in \mathbb{Z}_{\geq 0}^{k+l-1}, \\ |\beta| = n-k}} \sum_{\substack{\beta' \in \{0,1\}^{k+l-1}, \\ |\beta'| = l}} \mathcal{R}_{\beta + 1^{k+l} - (0,\beta')} \cdot 1 = \widehat{F} \cdot 1,$$

where

$$F := \mathrm{Sym}\left(\frac{z_1 \cdots z_{k+l}}{\prod_{i=1}^{l+k-1} (1 - qt z_{i+1}/z_i)} h_{n-k}(z_1, \ldots, z_{k+l}) e_l(z_2^{-1}, \ldots, z_{l+k}^{-1}) \prod_{1 \leq i < j \leq \ell} \omega(z_j/z_i) \right).$$

Idea of the proof: Write $(-1)^n e_n = D_n \cdot 1 = \mathcal{A}_1^{(n)}$.