A combinatorial formula for Interpolation Macdonald polynomials

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joint work with
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Overview

Macdonald polynomials (homogeneous polynomials)

- Introduced by Macdonald in 1989,
- Related to the space of diagonal harmonics (Bergeron–Garsia '99), to geometry of Hilbert scheme (Haiman '03).
- Have combinatorial interpretation in terms of tableaux (Haglund–Haiman–Loehr '05), vertex-models (Borodin–Wheeler '19), multiline queues (Corteel–Mandelshtam–Williams '22).
- When $q = 1$, they encode the distributions of the ASEP model (Cantini–de Gier–Wheeler '15), and the t-PushTASEP model (Ayyer–Martin–Williams '25).

Interpolation Macdonald polynomials (inhomogeneous polynomials)

- Introduced by Knop and Sahi in 1996, further studied by Lassalle and Okounkov.
- A Cauchy type identity (Olshansky '17).
- In the Jack limit, they have been shown to be monomial positive (Naqvi–Sahi–Sergel '23).
- Related to the knot theory of $\mathfrak{gl}_n$ (Beliakova-Gorsky '24).
- **Today:** A combinatorial formula in terms of *signed multiline queues*.
- **Application:** A probabilistic model (analogue to t-PushTASEP).

Plan of the talk

- Interpolation ASEP polynomials f^*_μ and Interpolation symmetric polynomials P^*_λ,
- Multiline queues and the combinatorial formula for homogeneous polynomials f_μ and P_λ (Corteel–Mandelshtam–Williams),
- Main result: Signed multiline queues and the combinatorial formula for the interpolation polynomials f^*_μ and P^*_λ,
- Sketch of the proof,
- Application: factorization property.

Notation:
g^*: will denote an interpolation polynomial (inhomogeneous)
g : the homogeneous polynomial obtained from g by taking the top degree part.

Compositions

Fix an integer $n \geq 1$. We consider the space of polynomials in n variables $x_1, \dots, x_n$ with coefficients $\mathbb{Q}(q,t)$.

We say that $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ is a **composition** of size k (with n parts) if $\mu_1 + \mu_2 + \cdots + \mu_n = k$. The integer $k = |\mu|$ is the size of μ.

Notation: For a composition μ, write

$$x^\mu := x_1^{\mu_1} \dots x_n^{\mu_n}.$$

The family $\{x^\mu : |\mu| \leq d\}$ is a basis for the space of polynomials with degree at most d.

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Given a composition $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n$, we define

$$\begin{aligned} k_i(\mu) &:= \#\{j : j < i \text{ and } \mu_j > \mu_i\} + \#\{j : j > i \text{ and } \mu_j \geq \mu_i\}, \text{ and} \\ \widetilde{\mu} &:= \left(q^{\mu_1} t^{-k_1(\mu)}, \dots, q^{\mu_n} t^{-k_n(\mu)}\right). \end{aligned}$$

Example: When $\mu = (4,2,0,1,4)$ we have $\widetilde{\mu} = (q^4 t^{-1}, q^2 t^{-2}, t^{-4}, q t^{-3}, q^4)$. For a polynomial $f(x_1, \dots, x_n)$ and $\mu \in \mathbb{N}^n$, define $f(\widetilde{\mu}) = f(\widetilde{\mu}_1, \dots, \widetilde{\mu}_n)$.

Nonsymmetric interpolation polynomials

Theorem (Knop, Sahi 1996)

For any composition μ, there exists a unique polynomial $E^*_\mu(x_1,\dots,x_n;q,t)$ such that:

- $\deg(E^*_\mu) \leq |\mu|$,
- $[x^\mu]E^*_\mu = 1$
- $E^*_\mu(\widetilde{\nu}) = 0$, *for any ν satisfying $|\nu| \leq |\mu|$ and $\nu \neq \mu$.*

Moreover, the top homogeneous part of E^*_μ is the Macdonald polynomial E_μ.

The first two conditions are equivalent to:

$$E^*_\mu = x^\mu + \sum_{\substack{\nu:|\nu|\leq|\mu| \\ \nu\neq\mu}} a_\nu x^\nu, \qquad \text{for some } a^\nu_\mu \in \mathbb{Q}(q,t).$$

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The polynomials E^*_μ are the *interpolation nonsymmetric Macdonald polynomials*.

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Theorem (BD-Williams)

For any composition $\mu \in \mathbb{N}^n$, there exists a unique polynomial $f^*_\mu(x_1,\dots,x_n;q,t)$ such that:

- $\deg(f^*_\mu) \leq |\mu|$,
- *for any $\tau \in S_n(\mu)$, we have $[x^\tau]f^*_\mu = \delta_{\tau,\mu}$.*
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Moreover, the top homogeneous part of f^*_μ is the ASEP polynomial f_μ.

$S_n(\mu)$: Compositions obtained by permuting μ.

Nonsymmetric interpolation polynomials

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For any composition $\mu \in \mathbb{N}^n$*, there exists a unique polynomial* $f^*_\mu(x_1, \ldots, x_n; q, t)$ *such that:*

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We call these polynomials the interpolation ASEP polynomials.

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Example: $n = 1$ and $\mu = (k)$. We want

$$\begin{aligned} f^*_{(k)}(x) &= x^k + a_{k-1}x^{k-1} + \cdots + a_0, \\ f^*_{(k)}(q^m) &= 0 \qquad \text{for } 0 \leq m < k, \end{aligned}$$

We then have

$$f^*_{(k)}(x) = (x-1)(x-q)\ldots(x-q^{k-1}).$$

Nonsymmetric interpolation polynomials

Example: $n = 2$ and $\mu = (0, 2)$.

$$\begin{aligned}
&f^*_{(0,2)}(x_1, x_2) = x_2^2 + ax_1x_2 + bx_1 + cx_2 + d, && \\
&f^*_{(0,2)}(q/t, q) = 0 && \nu = (1, 1) \\
&f^*_{(0,2)}(q, 1/t) = 0 && \nu = (1, 0) \\
&f^*_{(0,2)}(1/t, q) = 0 && \nu = (0, 1) \\
&f^*_{(0,2)}(1/t, 1) = 0 && \nu = (0, 0)
\end{aligned}$$

We then have

$$f^*_{(0,2)}(x_1, x_2) = x_2^2 + \frac{1-t}{1-qt}x_1x_2 + q\frac{1-t}{1-qt}x_1 + \frac{1+qt-qt^2-q^2t^2}{t(1-qt)}x_2 + \frac{q(1-qt)}{t(1-qt^2)}.$$

In particular,

$$f_{(0,2)}(x_1, x_2) = x_2^2 + \frac{1-t}{1-qt}x_1x_2.$$

Symmetric polynomials

We say that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a **partition** of k (with n parts) if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = k$. The integer $k = |\lambda|$ is the size of λ.

Example: $n = 4$, and $\lambda = (3, 3, 1, 0)$ is a partition of size 7.

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The monomial symmetric function m_λ is defined by

$$m_\lambda(x_1, \dots, x_n) = \sum_{\mu \in S_n(\lambda)} x^\mu.$$

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Example: For $n = 3$ and $\lambda = (2, 1, 0)$, we have

$$m_{(2,1,0)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2.$$

We then have that $\{m_\lambda : |\lambda| \leq d\}$ is a basis for the space of symmetric polynomials with degree at most d.

Interpolation symmetric Macdonald polynomials

Theorem (Knop, Sahi 1996)

For any partition λ, there exists a unique ***symmetric*** *polynomial $P^*_\lambda(x_1,\dots,x_n;q,t)$ such that:*

- $\deg(P^*_\lambda) \leq |\lambda|$,
- $[m_\lambda]P^*_\lambda = 1$
- $P^*_\lambda(\widetilde{\kappa}) = 0$, *for any* ***partition*** κ *satisfying* $|\kappa| \leq |\lambda|$ *and* $\kappa \neq \lambda$.

Moreover, the top homogeneous part of P^*_λ is the Macdonald polynomial P_λ.

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Proposition

For any partition λ, we have

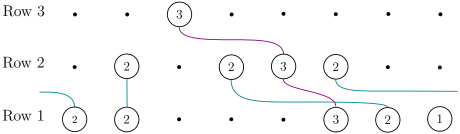
$$P^*_\lambda(x_1, \ldots, x_n; q, t) = \sum_{\mu \in S_n(\lambda)} f^*_\mu(x_1, \ldots, x_n; q, t).$$

Multiline queues

A ball system is an $L \times n$ array (for some $L, n \geq 1$), with rows labeled from bottom to top as $1, 2, \ldots, L$, and columns labeled from left to right from 1 to n, in which each of the Ln positions is either empty or occupied by a ball labeled by $a > 0$.

A multiline queue is a ball system such that:

- each ball in row $r > 1$ is paired with a ball in the row below, using the shortest strand traveling (weakly) from left to right, allowing the strand to wrap around if necessary.
- a ball in a strand of height k is labeled by k,
- it does not contain the forbidden configuration.



A multiline queue.

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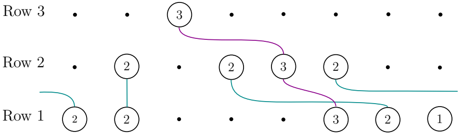

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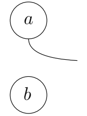

with $0 < b \leq a$

The forbidden configuration for multiline queues.

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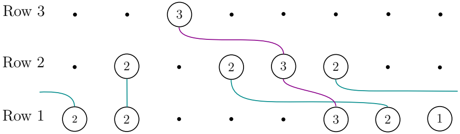


A multiline queue of type $(2, 2, 0, 0, 0, 3, 2, 1)$.

The type of a multiline queue is the composition μ obtained by reading the labels in row 1.

- n = #columns in the multiline queue = #parts of μ.
- L = #rows in the multiline queue = the size of the maximal part in μ.

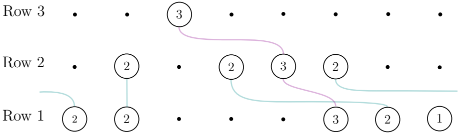
Weights of Multiline queues

Pairing order in each layer:

We read the balls in row r in decreasing order of their label; within a fixed label a:

- we start by placing the trivial a-pairings,
- we pair the remaining balls labeled a from right to left.

As we read the balls in this order, we imagine placing the strands pairing the balls one by one.



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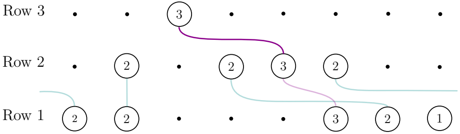

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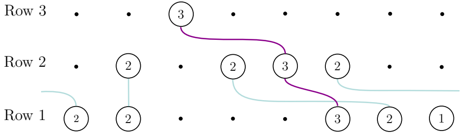

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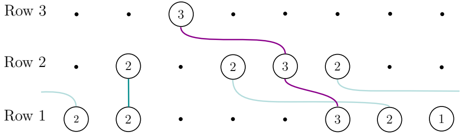

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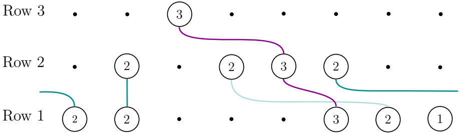

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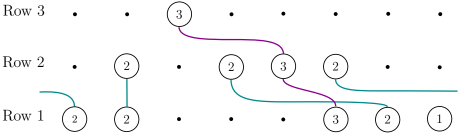


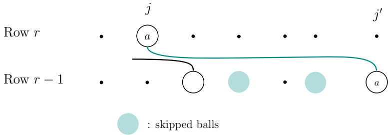
A multiline queue of type $(2, 2, 0, 0, 0, 3, 2, 1)$.

Weights of Multiline queues

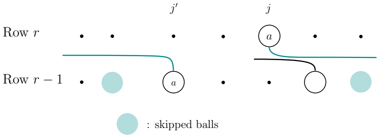
- A ball in column i has the weight x_i.
- Each nontrivial pairing p, connecting balls labeled a, between rows $r > 1$ and $r-1$, and columns j and j', has weight $\mathrm{wt}_{\mathrm{pair}}(p)$:

$$\mathrm{wt}_{\mathrm{pair}}(p) = \begin{cases} \dfrac{(1-t)t^{\mathrm{skip}(p)}}{1-q^{a-r+1}t^{\mathrm{free}(p)}} & \text{if } j' > j \\ \dfrac{(1-t)t^{\mathrm{skip}(p)}}{1-q^{a-r+1}t^{\mathrm{free}(p)}} \cdot q^{a-r+1} & \text{if } j' < j. \end{cases}$$

free(p) : balls not yet paired in row $r-1$,
skip(p): free balls which have been skipped by p.



Case $j < j'$



Case $j > j'$

- The total weight of a multiline queue Q is denoted $\mathrm{wt}(Q)$:

$$\mathrm{wt}(Q) = \prod_{\text{balls } B} \mathrm{wt}(B) \prod_{\text{pairings } p} \mathrm{wt}_{\mathrm{pair}}(p).$$

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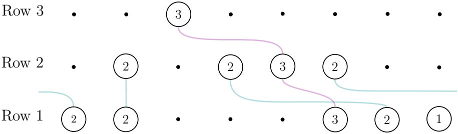

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$$\mathrm{wt}(Q) = x_1 x_2^2 x_3 x_4 x_5 x_6^2 x_7 x_8 \frac{(1-t)t}{1-qt^4} \frac{1-t}{1-q^2t^5} \frac{(1-t)qt^2}{1-qt^3} \frac{1-t}{1-qt^2}.$$

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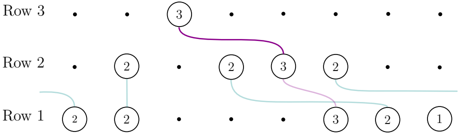

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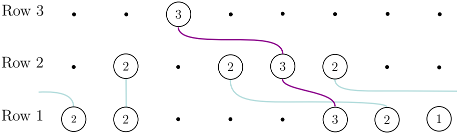

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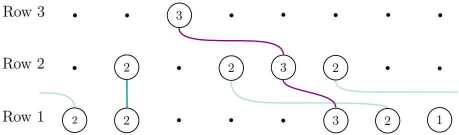

A multiline queue of type $(2, 2, 0, 0, 0, 3, 2, 1)$.

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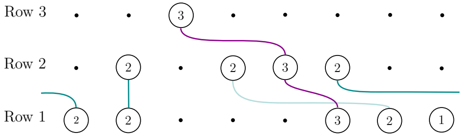

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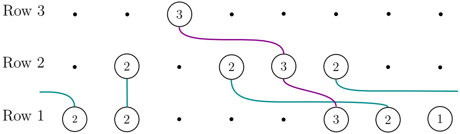

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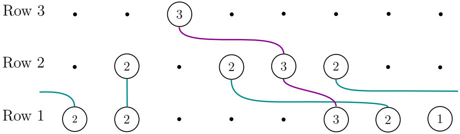

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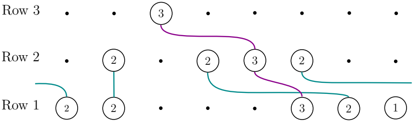
The multiline queue formula for Macdonald polynomials



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Theorem (Corteel–Mandelshtam–Williams '23)

For any composition μ, we have

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As a consequence, for any partition λ,

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Main ideas of the proof:

1. Assume that the polynomials f_μ have a decomposition of the form

$$f_\mu(x_1,\ldots,x_n)=\prod_{i:\mu_i>0}x_i\sum_\nu a_\mu^\nu f_\nu(x_1,\ldots,x_n),$$

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   Remark: This characterization has two parts:

   - an action of the Hecke operators (also known as the Demazure–Lustig operators).
   - a circular symmetry property.

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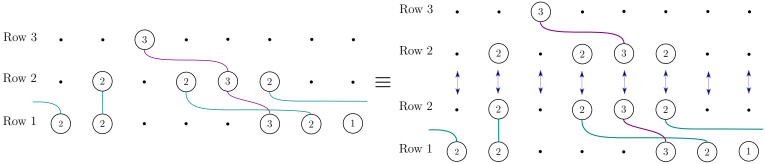

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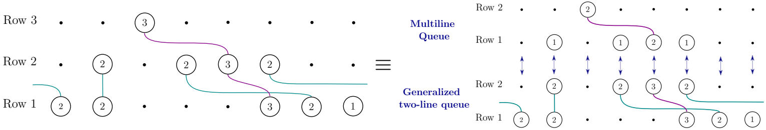

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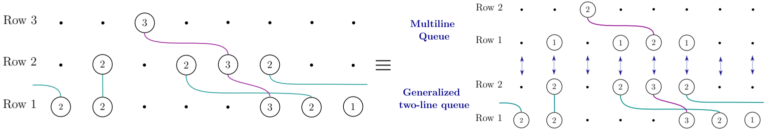

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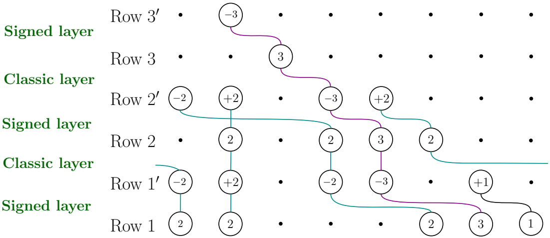
4. Show that A_μ^ν satisfy the relations established in Step 2, to get that $a_\mu^{\nu^-} = A_\mu^\nu$

Signed Multiline queues

An enhanced ball system is a $2L \times n$ array ($L, n \geq 1$), with rows labeled from bottom to top as $1, 1', 2, 2', \ldots, L, L'$, and columns labeled from left to right from 1 to n, in which each of the $2Ln$ positions is either empty or occupied by a ball. A ball in row r is labeled by $a > 0$, and a ball in row r' is labeled $\pm a$, where $a > 0$.

A signed multiline queue is an enhanced ball system, satisfying the conditions:

- each pair connects two balls with the same absolute value,
- after forgetting the signs, classic layers correspond to layers from classic MLQ.
- each ball in row r' is paired with a ball in row r, using the shortest strand traveling (weakly) from left to right, without wrapping around.
- it does not contain any forbidden configuration.



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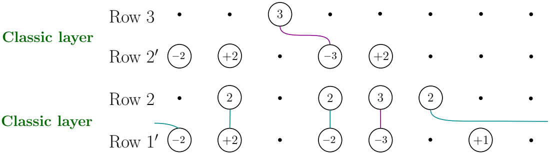

Classic layers of a signed multiline queue.

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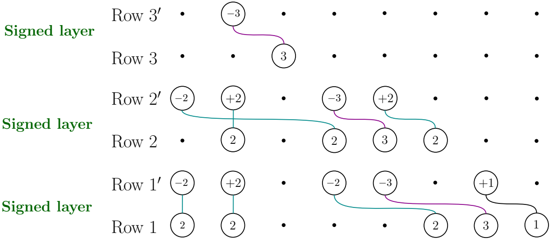

Signed layers of a signed multiline queue.

Signed Multiline queues

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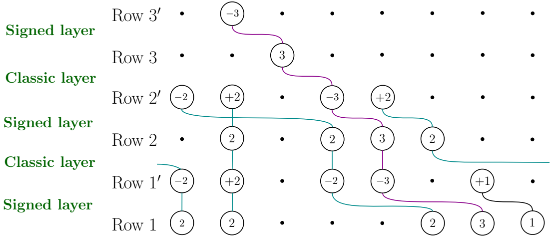

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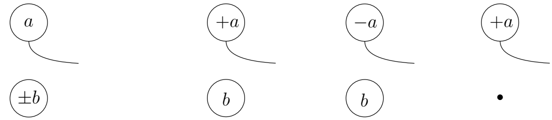
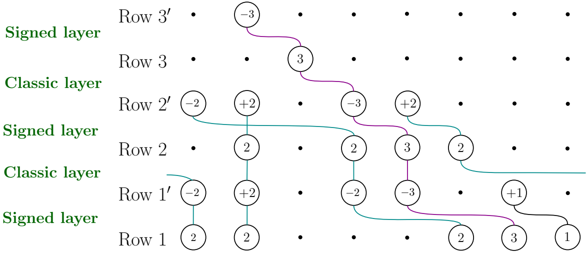


with $0 < b \leq a$ with $0 < b \leq a$ with $0 < a < b$ with $0 < a$

The forbidden configuration for signed multiline queues.

Weights of Signed Multiline Queues

- Only signed balls have weights. A ball in column i, row r', has the weight
$$\begin{cases} x_i & \text{if it is positive,} \\ \frac{-q^{r-1}}{t^{n-1}} & \text{if it is negative.} \end{cases}$$
- Each nontrivial pairing p has a weight $\mathrm{wt}_{\mathrm{pair}(p)}$ (we use the same order to place pairings)



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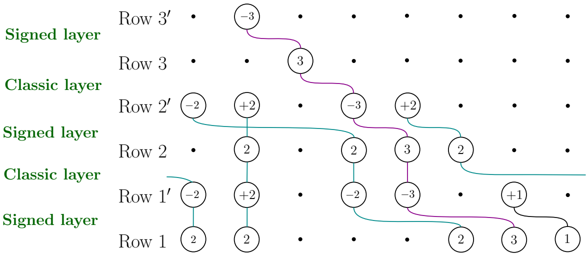

A signed multiline queue with ball weight: $x_2^2x_5x_7\left(\frac{-1}{t^7}\right)^3\left(\frac{-q}{t^7}\right)^2\left(\frac{-q^2}{t^7}\right)$

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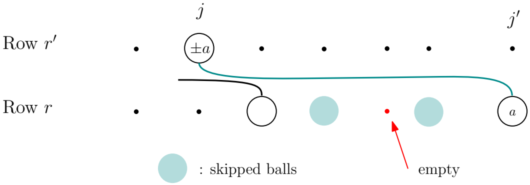
- Weights of pairings in classic layers are defined as before,
- weight of a nontrivial pairing in a signed layer is given by

$$\mathrm{wt}_{\mathrm{pair}}(p) = \begin{cases} (1-t)t^{\mathrm{skip}(p)+\mathrm{empty}(p)} & \text{if } p \text{ connects a positive ball and a regular ball} \\ -(1-t)t^{\mathrm{skip}(p)+\mathrm{empty}(p)} & \text{if } p \text{ connects a negative ball and a regular ball.} \end{cases}$$



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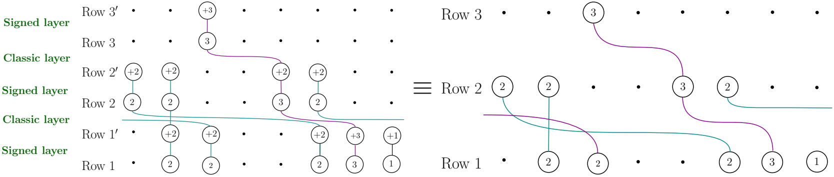
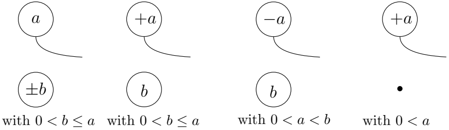
Remark: In a SMLQ contributing to the top homogeneous part, all signed balls should be positive.

Signed Multiline queues

Remark: In a SMLQ contributing to the top homogeneous part, all signed balls should be positive. A signed multiline queue without negative balls is a multiline queue.





The forbidden configuration for signed multiline queues.

Main result

Theorem (BD–Williams)

For any composition μ, we have

$$f^*_\mu(x_1,\dots,x_n;q,t) = \sum_{\substack{\textit{signed multiline-queues}\\ Q \textit{ of type } \mu}} \mathrm{wt}(Q).$$

As a consequence, for any partition λ,

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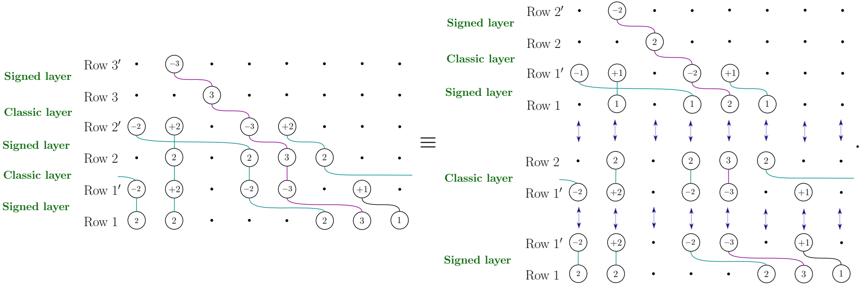
We also give a tableau formula for f^*_μ and P^*_μ.

Combinatorial decomposition



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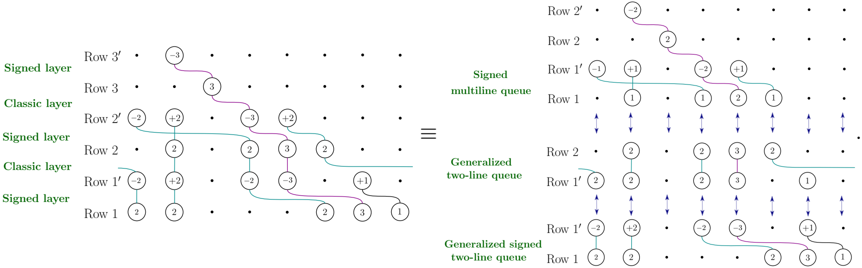

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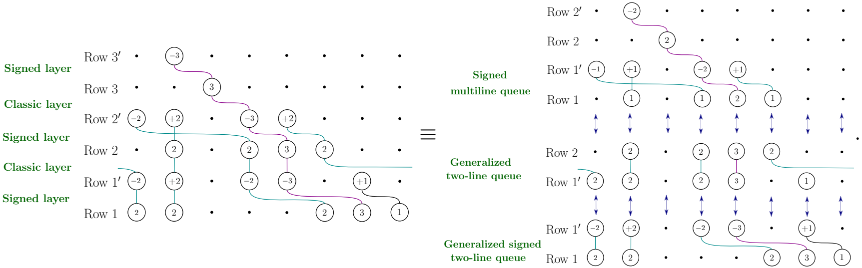
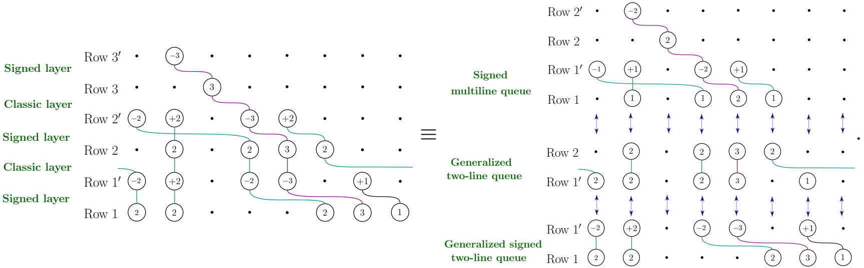


Define the generating function:

$$F^*_\mu(x_1,\dots,x_n;q,t) := \sum_{\substack{\text{signed multiline-queues}\\ Q \text{ of type } \mu}} \mathrm{wt}(Q),$$

Recall: We want to show that $f^*_\mu = F^*_\mu$.

Combinatorial decomposition



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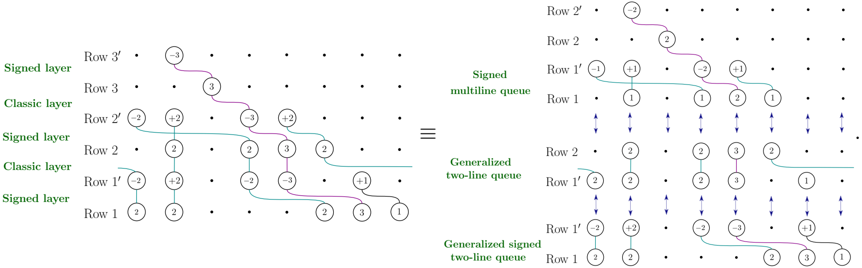
Recall: For $\lambda,\nu \in \mathbb{N}^n$, define $\mathrm{A}^\lambda_\nu(q,t) :=$ the weighted generating functions of generalized two-line queues of bottom row ν, top row λ.
For $\mu \in \mathbb{N}^n$ and $\alpha \in \mathbb{Z}^n$, define $\mathrm{B}^\alpha_\mu(t) :=$ the weighted generating functions of generalized signed two-line queues of bottom row μ, top row α,

Combinatorial decomposition



We have

$$F^*_\mu(x_1,\ldots,x_n)=\sum_{\alpha\in\mathbb{Z}^n}\sum_{\lambda\in\mathbb{N}^n}\mathrm{B}^\alpha_\mu\,\mathrm{wt}_\alpha\,\mathrm{A}^\lambda_{|\alpha|}q^{|\lambda^-|}F^*_{\lambda^-}\left(\frac{x_1}{q},\ldots,\frac{x_n}{q}\right)$$

where λ^- is the composition obtained from λ by substracting 1 from each non zero entry, and $\mathrm{wt}_\alpha=\prod_{i:\alpha_i>0}x_i\prod_{i:\alpha_i<0}\frac{-1}{t^{n-1}}$: ball weight in row $1'$.

Algebraic decomposition

Recall:
$$f_\mu(x_1,\ldots,x_n)=\prod_{i:\mu_i>0} x_i \sum_\nu a_\mu^\nu f_\nu(x_1,\ldots,x_n),$$

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Define $\mathrm{Pack}(k,n)$ as the set of compositions $\mu\in\mathbb{N}^n$ such that $\mu_i\neq 0$ if and only if $1\leq i\leq k$.

Example: $(2,3,1,0,0)\in\mathrm{Pack}(3,5)$.

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Theorem

If $\mu \in \mathrm{Pack}(k,n)$, then the polynomial f_μ^* is divisible by $\prod_{1\leq i\leq k}\left(x_i - \frac{1}{t^{n-1}}\right)$. Moreover,

$$f_\mu^* = \prod_{1\leq i\leq k}\left(x_i - \frac{1}{t^{n-1}}\right) \sum_{\substack{\nu\in\mathbb{N}^n \\ |\nu|\leq|\mu|-k}} K_\mu^\nu(q,t) f_\nu^*,$$

Hint: A polynomial $R(x_1,\dots,x_n)$ is identically 0 if and only if $R(\widetilde{\nu}) = 0$ for any $\nu \in \mathbb{N}^n$.

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$$f^*_\mu(x_1,\ldots,x_n) = \prod_{1\leq i\leq k}\left(x_i - \frac{1}{t^{n-1}}\right) \sum_{\substack{\nu\in\mathbb{N}^n \\ |\nu|=|\mu|-k}} K^\nu_\mu(q,t)q^{|\nu|} f^*_\nu\left(\frac{x_1}{q},\ldots,\frac{x_n}{q}\right).$$

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If $\mu \in \mathrm{Pack}(k,n)$, then the polynomial f^*_μ is divisible by $\prod_{1\leq i\leq k}\left(x_i - \frac{1}{t^{n-1}}\right)$. Moreover,

$$f^*_\mu(x_1,\dots,x_n) = \prod_{1\leq i\leq k}\left(x_i - \frac{1}{t^{n-1}}\right)\sum_{\substack{\nu\in\mathbb{N}^n\\ |\nu|=|\mu|-k}} K^\nu_\mu(q,t)q^{|\nu|}f^*_\nu\left(\frac{x_1}{q},\dots,\frac{x_n}{q}\right).$$

We also have $K^{\nu-}_\mu = \mathrm{A}^\nu_\mu$.

Hint: Extract the top homogeneous coefficient and use the formula of Corteel–Mandelshtam–Williams.

Algebraic decomposition

Define $\mathrm{Pack}(k,n)$ as the set of compositions $\mu \in \mathbb{N}^n$ such that $\mu_i \neq 0$ if and only if $1 \leq i \leq k$.

Example: $(2,3,1,0,0) \in \mathrm{Pack}(3,5)$.

Theorem

If $\mu \in \mathrm{Pack}(k,n)$*, then the polynomial* f^*_μ *is divisible by* $\prod_{1\leq i\leq k}\left(x_i - \frac{1}{t^{n-1}}\right)$*. Moreover,*

$$f^*_\mu(x_1,\dots,x_n) = \prod_{1\leq i\leq k}\left(x_i - \frac{1}{t^{n-1}}\right) \sum_{\substack{\nu\in\mathbb{N}^n\\ |\nu|=|\mu|-k}} K^\nu_\mu(q,t) q^{|\nu|} f^*_\nu\left(\frac{x_1}{q},\dots,\frac{x_n}{q}\right).$$

We also have $K^{\nu-}_\mu = \mathrm{A}^\nu_\mu$.

We check that this decomposition agrees with the combinatorial decomposition.

Action of Hecke operators

Do the interpolation polynomials satisfy the qKZ equations?

- We have the same action of the Hecke operators as in the homogeneous case,
- there is no analogue for the circular symmetry property.

For $1 \leq i \leq n-1$, define the operator

$$T_i := t - \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(1 - s_i).$$

Action of Hecke operators

For $1 \leq i \leq n-1$, define the operator

$$T_i := t - \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(1 - s_i).$$

These operators satisfy the relations of the Hecke algebra of type A_{n-1}

$$\begin{array}{ll} (T_i - t)(T_i + 1) = 0 & \text{for } 1 \leq i \leq n-1 \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} & \text{for } 1 \leq i \leq n-2 \\ T_i T_j = T_j T_i & \text{for } |i-j| > 1. \end{array}$$

Proposition

For any composition μ, we have

$$T_i f^*_\mu = \begin{cases} f^*_{s_i\mu} & \text{if } \mu_i > \mu_{i+1}, \\ t f^*_\mu & \text{if } \mu_i = \mu_{i+1}, \\ t f^*_{s_i\mu} - (1-t) f^*_\mu & \text{if } \mu_i < \mu_{i+1} \end{cases}.$$

Any f^*_μ can be obtained from the packed case using Hecke operators.

Unpacking procedure

Recall: We want to prove that

$$f^*_\mu(x_1,\dots,x_n) = \sum_{\alpha\in\mathbb{Z}^n}\sum_{\lambda\in\mathbb{N}^n} \mathrm{A}^\alpha_\mu\,\mathrm{wt}_\alpha\,\mathrm{B}^\lambda_{|\alpha|} q^{|\lambda^-|} f^*_{\lambda^-}\left(\frac{x_1}{q},\dots,\frac{x_n}{q}\right),$$

checked for packed μ.

Unpacking procedure

Fix μ, such that $\mu_i > 0$ and $\mu_{i+1} = 0$. Assume that

$$f^*_\mu(x_1, \dots, x_n) = \sum_{\alpha \in \mathbb{Z}^n} \sum_{\lambda \in \mathbb{N}^n} b^\alpha_\mu \, \mathrm{wt}_\alpha \, \mathrm{A}^\lambda_{|\alpha|} q^{|\lambda^-|} f^*_{\lambda-} \left(\frac{x_1}{q}, \dots, \frac{x_n}{q} \right),$$

for some coefficients $(b^\alpha_\mu)_{\alpha \in \mathbb{Z}^n}$. We then apply T_i to obtain an equation for $f^*_{s_i \mu}$:

$$f^*_{s_i \mu}(x_1, \dots, x_n) = \sum_{\alpha \in \mathbb{Z}^n} \sum_{\lambda \in \mathbb{N}^n} b^\alpha_{s_i \mu} \, \mathrm{wt}_\alpha \, \mathrm{A}^\lambda_{|\alpha|} q^{|\lambda^-|} f^*_{\lambda-} \left(\frac{x_1}{q}, \dots, \frac{x_n}{q} \right),$$

where $b^\alpha_{s_i \mu} =$

Unpacking procedure

Fix μ, such that $\mu_i > 0$ and $\mu_{i+1} = 0$. Assume that

$$f^*_\mu(x_1, \ldots, x_n) = \sum_{\alpha \in \mathbb{Z}^n} \sum_{\lambda \in \mathbb{N}^n} b^\alpha_\mu \, \mathrm{wt}_\alpha \, \mathrm{A}^\lambda_{|\alpha|} q^{|\lambda^-|} f^*_{\lambda-}\left(\frac{x_1}{q}, \ldots, \frac{x_n}{q}\right),$$

for some coefficients $(b^\alpha_\mu)_{\alpha \in \mathbb{Z}^n}$. We then apply T_i to obtain an equation for $f^*_{s_i\mu}$:

$$f^*_{s_i\mu}(x_1, \ldots, x_n) = \sum_{\alpha \in \mathbb{Z}^n} \sum_{\lambda \in \mathbb{N}^n} b^\alpha_{s_i\mu} \, \mathrm{wt}_\alpha \, \mathrm{A}^\lambda_{|\alpha|} q^{|\lambda^-|} f^*_{\lambda-}\left(\frac{x_1}{q}, \ldots, \frac{x_n}{q}\right),$$

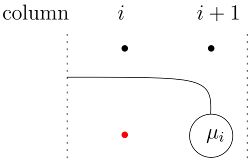
where $b^\alpha_{s_i\mu} =$

$$\begin{array}{ll}
tb^\alpha_\mu & \text{if } \alpha_i = \alpha_{i+1}, \\
tb^{s_i\alpha}_\mu & \text{if } \alpha_i > \alpha_{i+1} \geq 0,\ -\alpha_i > -\alpha_{i+1} > 0,\ \alpha_i > -\alpha_{i+1} > 0 \\
 & -\alpha_{i+1} > \alpha_i = 0 \text{ or } \alpha_i = -\alpha_{i+1} > 0 \\
b^{s_i\alpha}_\mu - (1-t)b^\alpha_\mu & \text{if } \alpha_{i+1} > \alpha_i \geq 0,\ -\alpha_{i+1} > -\alpha_i > 0, \\
 & -\alpha_i = \alpha_{i+1} > 0 \text{ or } \alpha_{i+1} > -\alpha_i > 0, \\
tb^{s_i\alpha}_\mu - (1-t)b^\alpha_\mu & \text{if } -\alpha_i > \alpha_{i+1} = 0, \\
tb^{s_i\alpha}_\mu - (1-t)\left(b^\alpha_\mu - b^{\ldots,-\alpha_i,-\alpha_{i+1},\ldots}_\mu\right) & \text{if } -\alpha_i > \alpha_{i+1} > 0, \\
b^{s_i\alpha}_\mu - (1-t)G^{\ldots,-\alpha_i,-\alpha_{i+1},\ldots}_\mu & \text{if } -\alpha_{i+1} > \alpha_i > 0.
\end{array}$$

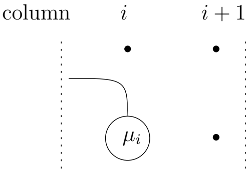
Unpacking procedure

We check that the combinatorial coefficients $\mathrm{B}^{\alpha}_{\mu}$ satisfy the recursion obtained from the Hecke algebra.

Example: Let's check that if $\alpha_i = \alpha_{i+1} = 0$, we have $\mathrm{B}^{\alpha}_{s_i\mu} = t\mathrm{B}^{\alpha}_{\mu}$.



A generalized signed two-line queue with bottom row $s_i\mu$ and top row α:



A generalized signed two-line queue of bottom row μ and top row α:

The empty statistic of the pairing in the diagram increases by 1 when we go from the right figure to the left one.

The t-PushTASEP

Fix a partition $\lambda = (\lambda_1, \dots, \lambda_n)$. The *PushTASEP* (Push Totally Asymmetric Simple Exclusion Process) with content λ is a Markov chain such that each state is indexed by a composition $\mu \in S_n(\lambda)$. We interpret μ as n particles on a ring, labeled $\mu_1, \dots, \mu_n$.

Conventions and notation:

- A particle labeled 0 will be called a vacancy.
- There exists at least one part of size 0 in λ (the system has at least one vacancy).
- We assume that $0 < t < 1$ and that $x_i > 0$ for $1 \le i \le n$.
- We denote $[m]_t$ the t-integer

$$[m]_t = 1 + t + \cdots + t^{m-1}.$$

The t-PushTASEP

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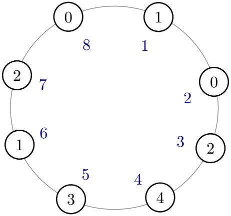
The dynamics are as follows:

- We choose the particle in the j-th position with probability proportional to $\frac{1}{x_j}$.
- This particle at position j starts traveling clockwise. Suppose there are m weaker particles in the system, then with probability $\frac{t^{k-1}}{[m]_t}$ the activated particle will move to the location of the kth of these weaker particles. If this location contains a particle, then that particle becomes active, and chooses a weaker particle to displace in the same way. The procedure continues until the active particle arrives at a vacancy.



The dynamics of t-PushTASEP.

The t-PushTASEP

Fix a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$. The *PushTASEP* (Push Totally Asymmetric Simple Exclusion Process) with content λ is a Markov chain such that each state is indexed by a composition $\mu \in S_n(\lambda)$. We interpret μ as n particles on a ring, labeled $\mu_1, \ldots, \mu_n$.

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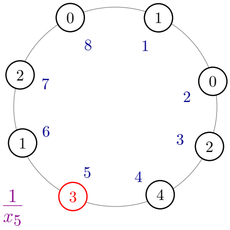

The dynamics of t-PushTASEP.

The t-PushTASEP

Fix a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$. The *PushTASEP* (Push Totally Asymmetric Simple Exclusion Process) with content λ is a Markov chain such that each state is indexed by a composition $\mu \in S_n(\lambda)$. We interpret μ as n particles on a ring, labeled $\mu_1, \ldots, \mu_n$.

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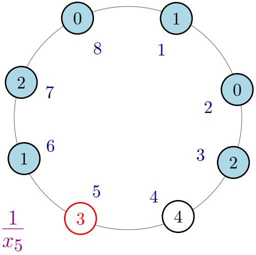

The dynamics of t-PushTASEP.

The t-PushTASEP

Fix a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$. The *PushTASEP* (Push Totally Asymmetric Simple Exclusion Process) with content λ is a Markov chain such that each state is indexed by a composition $\mu \in S_n(\lambda)$. We interpret μ as n particles on a ring, labeled $\mu_1, \ldots, \mu_n$.

The dynamics are as follows:

- We choose the particle in the j-th position with probability proportional to $\frac{1}{x_j}$.
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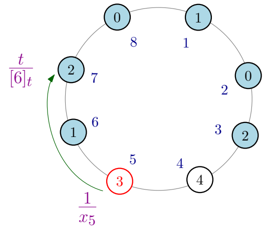

The dynamics of t-PushTASEP.

The t-PushTASEP

Fix a partition $\lambda = (\lambda_1, \dots, \lambda_n)$. The *PushTASEP* (Push Totally Asymmetric Simple Exclusion Process) with content λ is a Markov chain such that each state is indexed by a composition $\mu \in S_n(\lambda)$. We interpret μ as n particles on a ring, labeled $\mu_1, \dots, \mu_n$.

The dynamics are as follows:

- We choose the particle in the j-th position with probability proportional to $\frac{1}{x_j}$.
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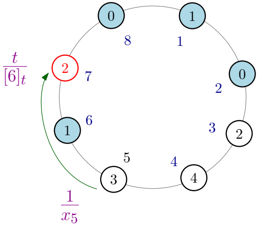

The dynamics of t-PushTASEP.

The t-PushTASEP

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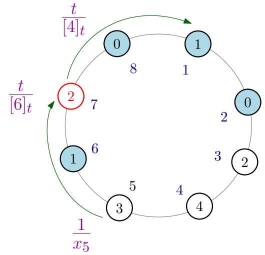


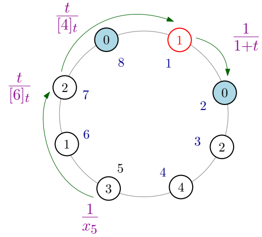
The dynamics of t-PushTASEP.

The t-PushTASEP

Fix a partition $\lambda = (\lambda_1, \dots, \lambda_n)$. The *PushTASEP* (Push Totally Asymmetric Simple Exclusion Process) with content λ is a Markov chain such that each state is indexed by a composition $\mu \in S_n(\lambda)$. We interpret μ as n particles on a ring, labeled $\mu_1, \dots, \mu_n$.

The dynamics are as follows:

- We choose the particle in the j-th position with probability proportional to $\frac{1}{x_j}$.
- This particle at position j starts traveling clockwise. Suppose there are m weaker particles in the system, then with probability $\frac{t^{k-1}}{[m]_t}$ the activated particle will move to the location of the kth of these weaker particles. If this location contains a particle, then that particle becomes active, and chooses a weaker particle to displace in the same way. The procedure continues until the active particle arrives at a vacancy.



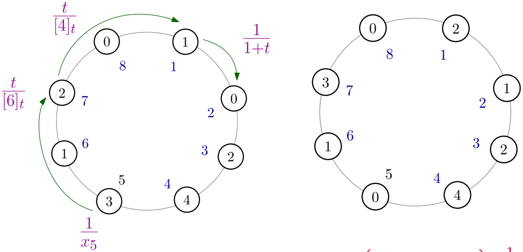
The dynamics of t-PushTASEP.

The t-PushTASEP

Fix a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$. The *PushTASEP* (Push Totally Asymmetric Simple Exclusion Process) with content λ is a Markov chain such that each state is indexed by a composition $\mu \in S_n(\lambda)$. We interpret μ as n particles on a ring, labeled $\mu_1, \ldots, \mu_n$.

The dynamics are as follows:

- We choose the particle in the j-th position with probability proportional to $\frac{1}{x_j}$.
- This particle at position j starts traveling clockwise. Suppose there are m weaker particles in the system, then with probability $\frac{t^{k-1}}{[m]_t}$ the activated particle will move to the location of the kth of these weaker particles. If this location contains a particle, then that particle becomes active, and chooses a weaker particle to displace in the same way. The procedure continues until the active particle arrives at a vacancy.



A transition of t-PushTASEP with probability $\frac{1}{x_5}\left(\sum_{1\le i\le 6}\frac{1}{x_i}\right)^{-1}\frac{t}{[6]_t}\,\frac{t}{[6]_t}\,\frac{1}{1+t}$.

The t-PushTASEP

Fix a partition $\lambda = (\lambda_1, \dots, \lambda_n)$. The *PushTASEP* (Push Totally Asymmetric Simple Exclusion Process) with content λ is a Markov chain such that each state is indexed by a composition $\mu \in S_n(\lambda)$. We interpret μ as n particles on a ring, labeled $\mu_1, \dots, \mu_n$.

The dynamics are as follows:

- We choose the particle in the j-th position with probability proportional to $\frac{1}{x_j}$.
- This particle at position j starts traveling clockwise. Suppose there are m weaker particles in the system, then with probability $\frac{t^{k-1}}{[m]_t}$ the activated particle will move to the location of the kth of these weaker particles. If this location contains a particle, then that particle becomes active, and chooses a weaker particle to displace in the same way. The procedure continues until the active particle arrives at a vacancy.

Remark: The dynamics can equivalently be described as follows: each time the active particle passes a site with a weaker particle,

- it continues to move with probability t,
- and settles at that site with probability $(1-t)$, displacing and activating the particle that is located there.

If it passes the mth such site, then it continues cyclically around the ring.

Totally asymmetric: particles only move clockwise.

The t-PushTASEP

Theorem (Ayyer–Martin–Williams '25)

The stationary distribution of the t-PushTASEP with content λ is proportional to the ASEP polynomials evaluated at $q = 1$. Equivalently,

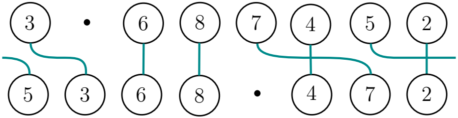
$$\pi_\lambda(\mu) = \frac{f_\mu(x_1, \ldots, x_n; q = 1, t)}{P_\lambda(x_1, \ldots, x_n; q = 1, t)}.$$

Correspondence between MLQ and transitions in the PushTASEP

We can encode the transitions in the PushTASEP using Generalized two-line queues.

`Step_1_restricted.pdf`

A transition of the t-PushTASEP



The corresponding two-line queue.

The Interpolation PushTASEP model

Fix a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$. The interpolation *PushTASEP* with content λ is a Markov chain such that each state is indexed by a composition $\mu \in S_n(\lambda)$. We interpret μ as n particles on a ring, labeled $\mu_1, \ldots, \mu_n$.

Conventions and notation:

- There exists at least one part of size 0 in λ.
- We denote $[m]_t$ the t-integer
$$[m]_t = 1 + t + \cdots + t^{m-1}.$$
- We define for $1 \leq k \leq n$, the following elements in $\mathbb{Q}(t, x_1, \ldots, x_n)$:
$$\mathfrak{p}_k := \frac{t^{-n+1}(1-t)}{x_k - t^{-n+2}}, \qquad \text{and} \qquad \mathfrak{q}_k := \frac{(1-t)x_k}{x_k - t^{-n+2}}.$$
- We assume that $0 < t < 1$ and that $x_i > t^{-n+1}$ for $1 \leq i \leq n$. Under these hypotheses on the parameters, the quantities $\mathfrak{p}_k$ and $\mathfrak{q}_k$ are probabilities.

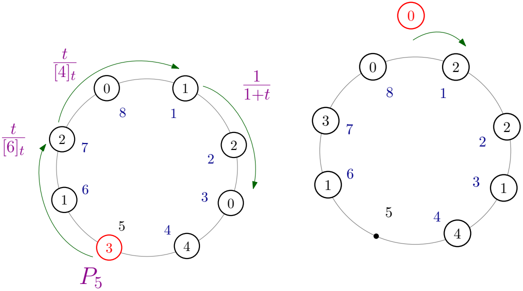
The Interpolation PushTASEP model

Fix a partition $\lambda = (\lambda_1, \dots, \lambda_n)$. The interpolation *PushTASEP* with content λ is a Markov chain such that each state is indexed by a composition $\mu \in S_n(\lambda)$. We interpret μ as n particles on a ring, labeled $\mu_1, \dots, \mu_n$.

The dynamics are as follows:

Step 0 We choose the particle in the j-th position with probability P_j proportional to $\prod_{k<j}\left(x_k - \frac{1}{t^{n-2}}\right)\prod_{k>j}\left(x_k - \frac{1}{t^{n-1}}\right)$.

Step 1 The particle at position j, say with label a, is activated, and starts traveling clockwise according to the rules of the (classical) t-Push TASEP.



Step 1 of the interpolation t-PushTASEP

The Interpolation PushTASEP model

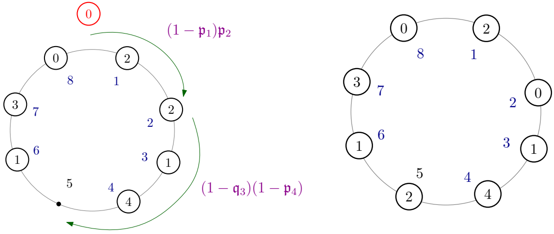
Fix a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$. The interpolation *PushTASEP* with content λ is a Markov chain such that each state is indexed by a composition $\mu \in S_n(\lambda)$. We interpret μ as n particles on a ring, labeled $\mu_1, \ldots, \mu_n$.

The dynamics are as follows:

Step 2 The last activated particle in Step 1 labeled $a := 0$, now goes to position 1 and starts traveling clockwise. When the activated particle labeled a gets to site k for $1 \leq k \leq j-1$ containing a particle with label $b \geq 0$, it settles at that site (displacing and activating the site's particle) with probability

$$\begin{cases} \mathfrak{p}_k & \text{if } b > a, \\ \mathfrak{q}_k & \text{if } b < a \end{cases}$$

The activated particle always settles at position j with probability 1.



Step 2 of the interpolation t-PushTASEP

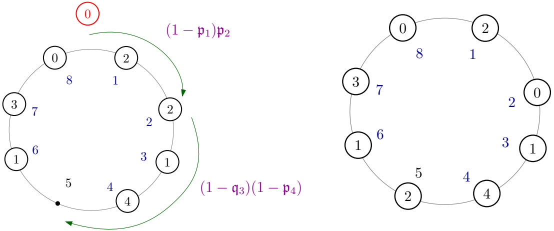
The interpolation PushTASEP model

Theorem (B.D–Williams '25)

The stationary distribution of the interpolation PushTASEP with content λ is proportional to the interpolation ASEP polynomials evaluated at $q = 1$. Equivalently,

$$\pi^*_\lambda(\mu) = \frac{f^*_\mu(x_1, \ldots, x_n; q = 1, t)}{P_\lambda(x_1, \ldots, x_n; q = 1, t)}.$$

The Interpolation PushTASEP model



Step 2 of the interpolation t-PushTASEP

Remarks:

- Unlike in the classical PushTASEP, this interpolation model is not invariant under rotation.
- In Step 2, the active particle can push both weaker and stronger particles.
- When $x_i \gg 1$, we recover the classical PushTASEP:

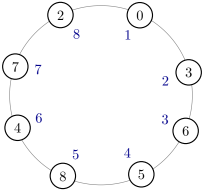
$$\mathfrak{p}_k := \frac{t^{-n+1}(1-t)}{x_k - t^{-n+2}} \longrightarrow 0,$$

the active particle cannot push a stronger particle and Step 2 is trivial.

Correspondence between signed multiline queues and transitions in the interpolation PushTASEP

- We can encode the transitions in Step 1 of the interpolation PushTASEP using generalized two-line queues.
- We can encode the transitions in Step 2 of the interpolation PushTASEP using signed generalized two-line queues:

`Step_2_restricted.pdf`



A transition in Step 2 of the interpolation PushTASEP