Problem 1. Let \( c : I \to \mathbb{R}^n \) be an arc-length parametrized Frenet curve. Show that
\[
\det(c', c'', \ldots, c^{(n)}) = \prod_{i=1}^{n-1} (\kappa_i)^{n-i}.
\]

Problem 2. Consider a line segment \( L \) between points \( A \) and \( B \) and let \( l > \text{length}(L) \). Let \( C \) be a curve of length \( l \) passing through \( A \) and \( B \) such that \( L \cup C \) is a simple closed curve (see the picture above). Show that the curve \( C \) such that \( L \cup C \) bounds the largest possible area is an arc of a circle.

Problem 3. Recall Green’s theorem: let \( C \) be a positively oriented simple closed curve bounding a region \( D \) in the plane and let \( L, M \) be two \( C^1 \) functions, then
\[
\oint_C L \, dx + M \, dy = \iint_D (M_x - L_y) \, dxdy.
\]
Use this to show that
\[
A = \frac{1}{2} \int_a^b (xy' - yx') \, dt
\]
where \( \alpha : [a,b] \to \mathbb{R}^2, \alpha(t) = (x(t), y(t)) \) is a parametrization of \( C \) and \( A \) is the area of \( D \).

Problem 4. Let \( f : U \to \mathbb{R}^n \) be a regular parametrized hypersurface. Show that the matrix \( (g_{ij}) \) of the first fundamental form can be written as
\[
(g_{ij}) = (Jf)^T Jf
\]
where \( Jf \) is the Jacobian matrix of \( f \).

Problem 5. A surface of revolution is obtained by rotating a regular curve in the \( x - z \) plane around the \( z \) axis; it can be parametrized by
\[
f(t, \theta) = (r(t) \cos \theta, r(t) \sin \theta, h(t))
\]
where the curve in the \( x - z \) plane is given by \( (r(t), h(t)) \). Compute the first fundamental form of \( f \).