Problem 1. Consider the ellipse parametrized by \( c(t) = (a \cos t, b \sin t) \) with \( a \neq b \). Find the vertices of the ellipse, i.e. the local extrema of the function \( \kappa(t) \).

Problem 2. The cycloid is the curve traced out by a fixed point on a circle as the circle rolls along a straight line (figure above).

1. Find a parametrization of the cycloid.
2. Compute the curvature of the cycloid.

Problem 3. Let \( \gamma : I \to \mathbb{R}^3 \) be a regular curve parametrized by arc length. Show that for any point \( s_0 \in s \), the curvature of \( \gamma \) at \( s_0 \) is equal to the curvature of the projection of \( \pi \circ \gamma : I \to \mathbb{R}^2 \) at \( s_0 \) where \( \pi \) is the projection onto the osculating plane.

Problem 4. Consider a plane curve given in polar coordinates \((r, \theta)\) by the equation \( r = r(\theta) \) and denote by \( r' = \frac{dr}{d\theta} \).

1. Show that the arc length from \( \theta_1 \) to \( \theta_2 \) can be calculated as \( \int_{\theta_1}^{\theta_2} \sqrt{r'^2 + r^2} d\theta \).
2. Show that the curvature is given by \( \kappa(\theta) = \frac{2r^2 - rr'' + r^2}{(r'^2 + r^2)^{3/2}} \).
3. Calculate the curvature for the Archimedean spiral given by \( r(\theta) = a\theta \).

Problem 5. Let \( c : I \to \mathbb{R}^3 \) be a Frenet curve with nonzero torsion \( \tau \) and consider the unit normal vector \( e_2(s) \), called the principal normal vector. We say that \( c \) is a Bertrand curve if there exists a scalar function \( r \) such that the curve \( \bar{c}(s) := c(s) + r(s)e_2(s) \) has the same principal normal vector as \( c(s) \), namely \( e_2(s) \). In this case, we say \( c \) and \( \bar{c} \) are a Bertrand pair. Suppose \( c \) and \( \bar{c} \) are a Bertrand pair.

1. Show that \( r(s) \) is constant. Conclude that the distance between \( c \) and \( \bar{c} \) is also constant.
2. Show that the angle between the tangent vectors of \( c \) and \( \bar{c} \) is constant.
3. Show that there exist constants \( a \) and \( b \) such that \( ak + b\tau \equiv 1 \) where \( \kappa \) and \( \tau \) are the curvature and torsion of \( c \).
4. Give an example of a Bertrand pair.