Consider the 2nd order Taylor expansion

\[ c(s) = c(0) + sc'(0) + \frac{s^2}{2} c''(0) + \frac{s^3}{6} c'''(0) + o(s^3) \]

This can be rewritten as

\[ c(s) = c(0) + \alpha(s) e_1(0) + \beta(s) e_2(0) + \gamma(s) e_3(0) \]

Using the Frenet eqns. (Hint: Write \( c', c'', \) and \( c''' \) as functions of \( \kappa_i \)).

Osculating sphere

Let \( c \) be a Frenet curve in \( \mathbb{R}^3 \) with \( z(s_0) = 0 \). Then there exists a unique sphere centered at \( z(s_0) = c(s_0) + \frac{1}{\kappa(s_0)} e_2(s_0) - \frac{k'(s_0)}{z(s_0) k^2(s_0)} e_3(s_0) \)

which is tangent to \( c \) at \( c(s_0) \) to order 3.

Write

\[ z(s) = c(s) + \alpha e_1(0) + \beta e_2(0) + \gamma e_3(0) \]

Consider the function

\[ \Gamma(s) = \langle z(s) - c(s), z(s) - c(s) \rangle \]

The sphere we're looking for is given by \( \Gamma(s) = \) constant. \( \Gamma \) being tangent to order 3 means

\[ \Gamma'(s_0) = \Gamma''(s_0) = \Gamma'''(s_0) = 0 \]
\[
\Gamma'(s_0) = \frac{d}{ds} \left< z(s_0) - c(s), z(s_0) - c(s) \right>_{s=s_0} = 0
\]

\[
\Gamma''(s_0) = 2 \left< c'(s), c'(s) \right>_{s=s_0} - 2 \left< z(s_0) - c(s), c''(s) \right>_{s=s_0} = 0
\]

\[
1 - \left< \alpha e_1(s_0) + \beta e_2(s_0) + \gamma e_3(s_0), e_2(s_0) \right>_{s=s_0} = 0
\]

\[
1 - K \beta = 0 \quad \Rightarrow \quad \beta = \frac{1}{K}
\]

\[
\Gamma'''(s_0) = 2 \frac{d}{ds} \left< c'(s), c'(s) \right>_{s=s_0} + 2 \left< c'(s), c''(s) \right>_{s=s_0} - 2 \left< z(s_0) - c(s), c'''(s) \right>_{s=s_0} = 0
\]

\[
\left< \alpha e_1(s_0) + \beta e_2(s_0) + \gamma e_3(s_0), K'(s_0) e_2(s_0) \right>_{s=s_0} = 0
\]

\[K'(s_0) \beta = K(s_0) \alpha + K(s_0) \tau(s_0) \gamma = 0\]

\[
\frac{K'(s_0)}{K(s_0)} + K(s_0) \tau(s_0) \gamma = 0
\]

\[
C''(s) = K' e_2 + K(-\kappa e_1 + \tau e_3) \quad \text{by Freeet}
\]

\[
\frac{-K'(s_0)}{K^2(s_0) \alpha(s_0)} = \gamma
\]
So the sphere centered at

\[ \mathbf{z}(s_0) = \mathbf{c}(s_0) + \frac{1}{k(s_0)} \mathbf{e}_2(s_0) + \frac{-k'(s_0)}{k^2(s_0)} \mathbf{e}_3(s_0) \]

Passing through \( \mathbf{c}(s_0) \) works, if its

unique \( \forall \mathbf{c} \) we solved uniquely

for the coefficients.

As \( s \) varies, the center of the osculating

sphere varies as

\[ \mathbf{z}(s) = \mathbf{c} + \frac{\mathbf{e}_2}{k} - \frac{k'}{k^2} \mathbf{e}_3 \]

with radius given by \( \| \mathbf{z}(s) - \mathbf{c}(s) \| \).

**Thm. (Spherical Curves)**

Let \( \mathbf{c} \) be a Frenet curve in \( \mathbb{R}^3 \)

which is of class \( C^4 \) and suppose \( \kappa \neq 0 \).

Then, \( \mathbf{c} \) lies on a sphere iff

\[ \frac{\kappa}{K} = \frac{d}{ds} \left( \frac{k'}{2kk'} \right) \]

**Proof**

\( \implies \) \( \mathbf{c} \) lies on a sphere

the osculating sphere is constant.
\[ z'(s) = 0 \quad \& \quad r'(s) = 0 \]

where \( z(s) = \) center of the osculating sphere

\[ r(s) = \langle z(s) - c(s), z(s) - c(s) \rangle = \text{radius}^2 \]

\[ z'(s) = c' - \frac{k'}{k^2} e_2 + \frac{e_2}{k} - \frac{1}{ds} \left( \frac{k'}{k^2} \right) e_3 - \frac{k'}{2k^3} e_3 \]

\[ = c' - \frac{k'}{k^2} e_2 - \frac{k e_1}{k} + \tau e_3 - \frac{1}{ds} \left( \frac{k'}{k^2} \right) e_3 + \frac{k'}{2k^3} \tau e_2 \]

\[ = c' - \frac{k'}{k^2} e_2 - \frac{k e_1}{k} + \tau e_3 - \frac{1}{ds} \left( \frac{k'}{k^2} \right) e_3 + \frac{k'}{2k^3} \tau e_2 \]

\[ = \left[ -\frac{\tau}{k} - \frac{1}{ds} \left( \frac{k'}{k^2} \right) e_3 \right] = 0 \quad \iff \quad \frac{\tau}{k} - \frac{1}{ds} \left( \frac{k'}{k^2} \right) = 0 \]

Thus, \( z'(s) = 0 \) so \( \frac{\tau}{k} - \frac{1}{ds} \left( \frac{k'}{k^2} \right) = 0 \) lies on a sphere. Then

\[ z'(s) = 0 \quad \text{so} \quad \frac{\tau}{k} = \frac{1}{ds} \left( \frac{k'}{k^2} \right) = 0 \]

Conversely, suppose this eqn holds, then \( z'(s) = 0 \). Now consider \( r'(s) = \frac{d}{ds} \left( \langle z(s) - c(s), z(s) - c(s) \rangle \right) \)

Using \( z'(s) = 0 \), we compute

\[ r'(s) = -2 \langle z(s) - c(s), c'(s) \rangle \]

\[ = -2 \left( \frac{1}{k} e_2 - \frac{k'}{k^2} e_3 \right) \cdot e_1(s) = 0 \]

So the center \( r \) and radius are constant & \( c \) lies on a sphere.
Curves on a sphere

Consider $S^2 \subset \mathbb{R}^3$

The sphere contains lesser circles with $\tau = 0$, $K > 1$, $\ell$

Great circles with $\tau = 0$, $K = 1$.

How do we characterize these circles?

Equator = Great circle
$\tau = 0$
$K = 1$

$C$, $C'$, $C \times C'$ are an orthonormal basis

$C'' = \langle C'', C' \rangle C + \langle C'', C' \rangle C' + \langle C'', C \times C' \rangle \times C C'$

Compute that $\langle C'', C' \rangle = 0$, $\langle C'', C \rangle = -\langle C', C' \rangle = -1$

$C'' = -C + \langle C \times C', C'' \rangle \frac{C \times C'}{\det \langle C, C', C'' \rangle} = \frac{-C + J \, C \times C'}{\det \langle C, C', C'' \rangle} = \frac{-C + J \, C \times C'}{J}$

$K^2 = \|C''\|^2 = 1 + J^2$

$e_1 = C'$
$e_2 = \frac{C''}{K}$
$e_3 = e_1 \times e_2 = \frac{C \times C''}{K}$

$K = \sqrt{1 + J^2}$
\[
\mathcal{I} = - \left( e_2' e_2 \right) = - \left( \frac{d}{ds} \frac{c \times c''}{K} \right) \frac{c''}{K}
\]

Frechet

\[
= -\frac{1}{K^2} \left( \frac{d}{ds} \frac{c k c''}{K} \right) \frac{c''}{K} + \frac{K^1}{K^3} \left( c \times c'' \right) \times \left( c \times c'' \right)
\]

\[
= -\frac{1}{K^2} \left( c \times c'' \right) - c + J \Omega c
\]

\[
\left( c''', c \right) = -\left( c'', c \right) = 0 \quad \text{so} \quad c''' \perp c
\]

b/c \quad \left( c'', c \right) = -1 \quad \text{is constant}

\[\Rightarrow c \times c''' \perp c \times c \]

\[\Rightarrow \left( c \times c''' \right) \left( c \times c \right) = 0
\]

\[= \frac{1}{K^2} \left( c \times c''' \right) c = \frac{1}{K^2} \det(c, c', c''')
\]

\[\frac{d}{ds} \det(c, c', c'') = \det(c, c', c'') + \det(c, c', c'') + \det(c, c', c''')
\]

\[J = \text{geodesic curvature} \quad (\text{we'll learn about this later})
\]

\[J\] measures the curvature of \( c \) relative to that of the sphere.

\[J = 0 \iff c \text{ is a great circle} \implies c \text{ is as flat as possible on the sphere}
\]

\[J \text{ constant} \iff c \text{ is a lesser circle} \implies \text{somewhat less flat}
\]