Gauss–Bonnet Theorem

**Def:** A **geodesic triangle** is a closed, continuous, piecewise regular curve \( \alpha : [a, b] \to S \) such that there exist exactly 2 times \( t_1, t_2 \in [a, b] \) where \( \alpha \) is not regular & \( \alpha_1 = \alpha \mid_{[t_1, t_2]} \) & \( \alpha_2 = \alpha_3 \) are geodesics. More generally, a **geodesic polygon** or **n-gon** is the same definition with \( t_1, ..., t_n \in (a, b) \) times where \( \alpha \) is not regular.

Equivalently, it is a collection of \( n \) geodesics \( \alpha_1, ..., \alpha_n \) such that:

\[
\alpha_i : [t_{i-1}, t_i], \quad \alpha_i(t_i) = \alpha_{i+1}(t_{i+1}) \quad \alpha_i(t_0) = \alpha_n(t_n)
\]

**Def:** The **interior angle** \( \psi_i \) & **exterior angle** \( \theta_i \) are as in the picture:

\( \theta_i \in (-\pi, \pi) \) \( \psi_i \in (0, 2\pi) \)

\( \psi_i = \pi - \theta_i \)
Thm (Gauss–Bonnet I)

Let $T \subseteq S$ be the interior of a geodesic triangle with interior angles $\psi_1, \psi_2, \psi_3$.

Then $\psi_1 + \psi_2 + \psi_3 - \pi = \iint_T K \, dA$

$k = \text{Gauss curvature}$

Remark: To make this precise, we need to assume $T$ is contained in the image of a chart $\nu: U \to S$.

Example: don't consider curves like this.

Thm (Gauss–Bonnet II)

Let $\mathcal{P} \subseteq S$ be the interior of a geodesic $n$-gon with exterior angles $\Theta_1, \ldots, \Theta_n$. Then

$$\iint_{\mathcal{P}} K \, dA = 2\pi - \sum_{i=1}^{n} \Theta_i$$

Cor: 1) if $K > 0$, $\Rightarrow \sum \psi_i > \pi$

2) if $K = 0$, $\Rightarrow \sum \psi_i = 0$

3) if $K < 0$, $\Rightarrow \sum \psi_i < 0$

Example: no bigons if $K < 0$
More generally, we can consider piecewise regular simple closed curves \( \alpha \colon [a,b] \to S \) which are not regular at corners \( t_i \in [a,b] \), so some picture as before except the arcs \( \alpha_i \) don't have to be geodesics.

We have interior and exterior angles \( \psi_i, \theta_i \) as before.

(Gauss-Bonnet III, local GB)

Thm: Let \( \alpha \) be a piecewise regular closed simple curve with exterior angles \( \theta_i \) and interior angles \( \psi_i \) contained in a chart of \( S \). Then

\[
\int K dA + \int_{\alpha} K_g ds + \sum \theta_i = 2\pi
\]

\( K_g = \) geodesic curvature

Note GB III \( \Rightarrow \) GB II \( \Rightarrow \) GB I

Indeed, \( \alpha \) is a geodesic quadrilateral for all \( t \neq t_0, \ldots, t_n \).

Cor. For a simple closed plane curve \( \alpha \colon I \to \mathbb{R}^2 \), we have

\[
\int K dA + \sum \theta_i = 2\pi
\]
Cor 2] If \( \alpha \) is a regular simple closed curve with no corners, then \( \iint k \, dA + \int k_g \, ds = 2\pi \)

Cor 3] For a smooth regular simple closed planar curve, then \( \int k_g \, ds = 2\pi \)

Global version: What if we integrate \( K \) on all of \( S \)?

To make sense of this, we need \( S \) to be compact, so the integral converges.

Compact closed & bounded

For simplicity, we also want to avoid \( S \) having a boundary: \( S \) is no boundary good.

So we assume \( S \) does not have a boundary.

"\( S \) closes in on itself" or "\( S \) is closed"

This closed is a different meaning than above.
Theorem (Gauss-Bonnet IV, global version)

Let $S \subseteq \mathbb{R}^3$ be a compact oriented regular surface without boundary. Then

$$\int_S K \, dA = 2\pi \chi(M)$$

Where $\chi(M) \in \mathbb{Z}$ is the topological Euler characteristic.