The tangent compact $K_t = \| T_t \|^2$ is the \textit{geodesic curvature}.

Note $K^2 > K_n^2$ with equality at $p \iff e_2 \parallel N$ at $p$.

$\iff K_t = 0 \iff$ the osculating plane of $c$ at $p$ contains $N(p)$.

\textbf{Remark:} Since $K_n = \| T(X, X) \|$ where $X = \frac{d}{dt}$, $K_n$ depends only on the tangent vector of a curve, not on the curve itself!

\textbf{Theorem (known as Menelaus's)}

$C_1$ & $C_2$ have the same tangent vector.

So they have the same normal curvature.

$C_1$ has nonzero geodesic curvature but $C_2 = S \cap \text{Span}(N, v)$ has zero geodesic curvature & $X = K_n$.

For example, $K_n$ varies between 1 if $X = t e_2$ & 0 if $X = t e_1$.

\[ \| T(X, X) \| = \langle x_1 e_1 + x_2 e_2, L(x_1 e_1 + x_2 e_2) \rangle = x_2^2 \]

$L_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ if $X$ is a unit vector, then $K_n$ varies between 1 if $X = t e_2$, & 0 if $X = t e_1$. 
Linear Algebra:

1) The eigenvalues of \( L \) are real.

**Proof:** Let \( v \) be an eigenvector with eigenvalue \( \lambda \), then
\[
\langle Av, Av \rangle = \langle v, \lambda^2 v \rangle = \lambda^2 \langle v, v \rangle = \lambda^2 \|v\|^2
\]
\[
\Rightarrow \lambda = \frac{\langle Av, Av \rangle}{\|v\|^2} \in \mathbb{R}_{\geq 0}
\]
\[
\Rightarrow \lambda \in \mathbb{R}
\]

If \( \lambda_1 \neq \lambda_2 \), then the eigenvectors \( e_1, e_2 \) are orthogonal,

\[
\lambda_1 \langle e_1, e_2 \rangle = \langle Le_1, e_2 \rangle = \langle e_1, Le_2 \rangle = \langle e_1, e_2 \rangle \lambda_2
\]
\[
\Rightarrow \langle e_1, e_2 \rangle = 0 \quad \text{since} \quad \lambda_1 \neq \lambda_2
\]

**Def - Prop (Olindo - Rodriguez)**

Let \( X \in T_x S \) be a unit tangent vector (\( I(X, X) = 1 \)), then the following are equivalent:

1) \( X \) is a stationary point for \( II(X, X) \) under the constraint \( I(X, X) = 1 \)

2) \( X \) is an eigenvector of \( L \)

In this case, \( X \) is a principal curvature direction

\( \lambda \) (the eigenvalue) is a principal curvature.

\( II(X, X) = I(X, LX) = \lambda I(X, X) = \lambda \) so \( \lambda \) is normal curvature in \( X \) direction

- Recall stationary = derivative is zero. This is a Lagrange multiplier problem.

Note that the gradient of \( II(X, X) \) is \( LX \) and gradient of \( I(X, X) = X \)
So \( X \) is stationary \( \Leftrightarrow \exists \lambda \) Lagrange multiplier with constraint \( \mathcal{I}(x,x)=1 \)

\[ \Rightarrow \begin{align*}
X \text{ eigenvec } \chi \quad \& \quad \text{eigenvalue } \lambda \quad \text{ s.t.} \\
\text{grad } \mathcal{I} &= \lambda \text{grad } x \\
LX &= \lambda XX
\end{align*} \]

---

**Rmk:** We may pick orthonormal eigenvectors \( e_1, e_2 \) for \( T_pS \) with eigenvalues \( \lambda_1, \lambda_2 \).

Then any unit vector \( X = a e_1 + b e_2 \), so

\[ K_n = \mathcal{I}(X,x) = \mathcal{I}(LX,x) = \mathcal{I}(\lambda_1 \cos\theta e_1 + \lambda_2 \sin \theta e_2, \cos \theta e_1 + \sin \theta e_2) \]

\[ = \lambda_1 \cos^2 \theta + \lambda_2 \sin^2 \theta = \mathbf{K}_n \]

---

**Rmk:** (Signs) Everything we've done here is independent of choices except for the choice of orientation. There are two choices of \( N \) which differ by sign \( \Rightarrow \); \( L \) differs by a sign but \( \mathcal{I} \) does not.

\( \lambda_1, \lambda_2 \) differ by \( \lambda_1 \lambda_2 = \det(L) \), etc...

(sometimes we'll add \( \chi, \mathbf{K}_1 \) for curvature)

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**Def:** 1) The **Gaussian Curvature** \( K = K_1 K_2 = \det(L) \)

2) The **mean curvature** \( H = \frac{1}{2} \text{Tr}(L) = \frac{1}{2}(K_1 + K_2) \)

---

**Rmk:** 1) \( H^2 - K = \frac{1}{4}(k_1 - k_2)^2 \geq 0 \) with equality \( \Leftrightarrow \) \( K_1 = K_2 \)

Recall then since \( K_i \) are the max & min of normal curvature then this happens \( \Leftrightarrow \) the normal curvature is constant.
Fix a chart \( S : U \rightarrow S \)
\[
L = \begin{pmatrix} h_1 & \cdots & h_n \end{pmatrix} = (h_{ab}) (g^{cd})
\]

\[
h_{i} = \sum_{k} h_{ik} g^{k;i} \implies K = \det(L) = \frac{h_{11} h_{22} - h_{12} h_{21}}{g_{11} g_{22} - g_{12} g_{21}}
\]

**Recall:** \( dA = \sqrt{EG - F^2} \) \( da_1 da_2 \)

\[
\ell = \text{parametrization of } S^2
\]
\[
\text{via Nof: } U \rightarrow S^2
\]

with \( \text{1st fundamental form} \ (h_{ij}) \)

Thus, \( K \) is measuring the difference or "distortion" between the area form on \( S \) at the area form on \( S^2 \) via Gauss map.

\[
H = \frac{1}{2} Tr(L) = \frac{1}{2} (h'_1 + h'_2) = \frac{1}{2} \sum_{i,j} h_{i;j} g^{i;j}
\]

\[
= \frac{1}{2 \det(g_{ij})} \left( h_{11} g_{22} - 2 h_{12} g_{12} + h_{22} g_{11} \right)
\]

If a point \( p \) on \( S \) is called

- **Elliptic** if \( K(p) > 0 \)
- **Hyperbolic** if \( K(p) < 0 \)
- **Parabolic** if \( K(p) = 0 \) but \( H(p) \neq 0 \)
- **Umbilical** if \( K_1(p) = K_2(p) \)

Strictly umbilical if \( K(p) \neq 0 \) \( \implies K \neq 0 \)

Umbilic \& \( K \neq 0 \)

Level point if \( K_1 = K_2 = 0 \)
Ex: 1) Sphere or radius \( r \) and every point is umbilic

\[ L = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \]

\( k_1 = 0 < k_2 = 1 \)

every point is parabolic

\[ K = 0 \]

2) Cylinder \( L = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \]

\( k_1 = 0 < k_2 = 1 \)

every point is parabolic

3) Elliptic point

\[ a^2 x^2 + b^2 y^2 + c^2 z^2 = 1 \]

\( k_1 \) and \( k_2 \) have the same sign

4) Hyperbolic point

\[ x^2 + y^2 - z^2 = 1 \]

"Saddle point"

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Def 1) a line of curvature is a regular curve \( c: I \rightarrow S \)

such \( c' \) is a principal curvature direction \( c(t) \) for all \( t \in I \)

2) an asymptotic direction is a direction in which the normal curvature is zero

3) an asymptotic curve is a curve for which \( c(t) \) is an asymptotic direction for all \( t \in I \).