July 8

Problem 1.

Prove the following properties of operators on $\mathbb{C}^n$.

(a) If L is a linear operator, then $\ker(L^*) = (\operatorname{im} L)^\perp$ and $\operatorname{im}(L^*) = (\ker L)^\perp$.

(b) A projection operator Π is an orthogonal projection if and only if Π is a self-adjoint operator.

Problem 2.

Let L be a linear operator on $V = \mathbb{C}^n$ with eigenspaces $\{V_\lambda\}_{\lambda\in\mathbb{C}}$. Check that the following statements are equivalent:

(a) $V = \bigoplus_{\lambda\in\mathbb{C}} V_\lambda$ is an orthogonal decomposition.

(b) There exists an orthonormal basis for V consisting of eigenvectors of L.

(c) There exists a unitary operator U such that the matrix representation of ULU^{-1} relative to the standard basis is a diagonal matrix.

If you have time, figure out the analogous list of statements for (non-unitarily) diagonalizable operators.

Problem 3.

Prove the following about commuting operators L, L' on $\mathbb{C}^n$, with eigenspaces $\{V_\lambda\}_{\lambda\in\mathbb{C}}$ and $\{V'_\lambda\}_{\lambda\in\mathbb{C}}$ respectively.

(a) For all $\lambda \in \mathbb{C}$, L' preserves V_λ.

(b) L and L' have a simultaneous eigenvector.

Problem 4.

Prove that if U is a unitarily diagonalizable operator on $\mathbb{C}^n$, then U is normal.

Problem 5.

Prove that if X, Y are subspaces of $\mathbb{C}^n$, then $L(X) \subseteq Y$ if and only if $L^*(Y^\perp) \subseteq X^\perp$.
An important corollary of this is that if L preserves a subspace X, then L^* preserves $X^\perp$.