July 6

Problem 1.

Let $|s\rangle, |\omega\rangle$ be two nonzero vectors in a quantum system V. Say the current state of the system is (the span of) $|s\rangle$. Let Π be the orthogonal projection onto $\mathbb{C}|\omega\rangle$. Let $|s'\rangle \doteq \Pi^{\perp}|s\rangle$. Let W be the 2-dimensional subspace spanned by $|s\rangle$ and $|\omega\rangle$, which has orthonormal basis $|\omega\rangle, |s'\rangle$. A depiction of W is shown in Figure 1. Let θ denote the angle between $|s\rangle$ and $|s'\rangle$; assume that $0 < \theta < \frac{\pi}{4}$.

Assume U_ω, U_s are unitary operators which preserve W. Assume also that if we restrict these operators to W, then U_ω acts by reflecting over the $|s'\rangle$ axis, and U_s acts by reflecting over the $|s\rangle$ axis.

(a) Check that $U_s U_\omega$ is a rotation operator (when restricted to W)

(b) Find its angle of rotation in terms of θ

(c) Determine the minimum number of times that we need to

`Apply` $U_s U_\omega$ `to the current state`

to make it so that there is at least a 50% chance that the answer to

`Is the current state contained in` $\mathbb{C}|\omega\rangle$`?`

is `yes`.

(d) Verify that the operators U_ω and U_s defined in class, with the states $|\omega\rangle$ and $|s\rangle$ used in Grover's algorithm, satisfy the hypotheses of this problem. If you have time, find θ in terms of N.

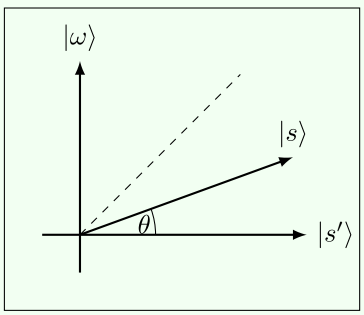

Figure 1:

Problem 2.

Let L be a linear operator on $\mathbb{C}^n$. Show that there is a unique linear operator L^* such that

$$\langle L\mathbf{v} \mid \mathbf{w}\rangle = \langle \mathbf{v} \mid L^*\mathbf{w}\rangle$$

for all $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$.

Problem 3.

Check the following properties of the "taking adjoints" operation.

(a) $L^{**} = L$

(b) $(L_1 L_2)^* = L_2^* L_1^*$

(c) $(\lambda L)^* = \lambda^* L$

(d) $(|\mathbf{v}\rangle \langle \mathbf{w}|)^* = |\mathbf{w}\rangle \langle \mathbf{v}|$.

(e) $(L_1 + L_2)^* = L_1^* + L_2^*$.