July 29

Problem 1.

Let $f : X \to \mathbb{C}$ be measurable. Recall that

$$\|f\|_\infty \doteq \inf\{\, C \geq 0 \mid |f(x)| \leq C \text{ almost everywhere.} \,\}.$$

Show that f is essentially bounded iff $\|f\|_\infty < \infty$. If f is essentially bounded, verify also the following:

(a) Let $N = \{\, x \mid |f(x)| > \|f\|_\infty \,\}$. Then $\mu(N) = 0$.

(b) If $C < \|f\|_\infty$, then $E_C = \{\, x \mid |f(x)| > C \,\}$ has positive measure.

(c) Let N be as in (a). Show that

$$\sup_{x \in X \setminus N} |f(x)| = \|f\|_\infty.$$

(d) Let $\psi \in L^2(X)$. Show that $\|f\psi\|_2 \leq \|f\|_\infty \|\psi\|_2$.

Problem 2.

We work in the Hilbert space $L^2(X)$. Let $f : X \to \mathbb{C}$ be an essentially bounded function.

(a) Show that $L_f^* = L_{f^*}$. Conclude that L_f is normal.

(b) Let Π_E^X (or just Π_E) denote the operator $L_{\mathbb{1}_E}$, where $\mathbb{1}_E$ is the indicator function of E. Show that Π_E^X is an orthogonal projection. What are its image and kernel?

(c) When is L_f unitary?

(d) What is the spectrum of L_f? What if $X = \mathbb{R}$ and f is continuous?