July 27

Problem 1.

Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove the *monotone convergence theorem*: if (f_n) is a sequence of measurable functions $X \to [0, \infty]$ converging to a measurable function f, and the sequence satisfies $f_n(x) \leq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in X$, then

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Problem 2.

Let V be a normed space.

(a) Prove that if a Cauchy sequence $(\mathbf{v}_n)_{n \in \mathbb{N}}$ has a convergent subsequence $(\mathbf{v}_{n_k})_{k \in \mathbb{N}}$, then $(\mathbf{v}_n)$ converges.

(b) Assume that every absolutely convergent series in V is convergent. Prove that V is Banach.

Problem 3.

Let X be a subset of $\mathbb{Z}$ and set $\mathcal{M} = \mathcal{P}(X)$. Let μ be the counting measure on $(X, \mathcal{M})$. Describe $L^2(X)$ and prove it is Banach.