# July 25

**Problem 1.**

Assume that $\mu : \mathcal{B}_{\mathbb{R}} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is such that

(i) $\mu(\varnothing) = 0$

(ii) $\mu\left(\bigsqcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$ for any pairwise disjoint sequence $E_1, E_2, \ldots$ in $\mathcal{P}(\mathbb{R})$.

(iii) $\mu([0, 1]) = 1$.

(iv) $\mu(x + E) = \mu(E)$ for any $E \subseteq \mathbb{R}$ and $x \in \mathbb{R}$.

Prove the following:

(a) If $E_1 \subseteq E_2$ then $\mu(E_1) \leq \mu(E_2)$.

(b) Prove that any countable set is Borel and has measure 0.

(c) If $E_1 \subseteq E_2 \subseteq \ldots$ is an increasing sequence of Borel sets, then

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \sup_{n \in \mathbb{N}} \mu(E_n).$$

(d) Prove that $\mu([a, b)) = b - a$ for any $b \geq a$.

(e) Prove that $\mu((a, b)) = \mu([a, b]) = b - a$ for any $b \geq a$.

**Problem 2.**

Let $(X, \mathcal{M})$ be a measurable space. Show the following properties of measurable functions $X \to \mathbb{R}$. (Here we are using the Borel $\sigma$-algebra on $\mathbb{R}$.) You can use the fact that a function $f : X \to \mathbb{R}$ is measurable if and only if $f^{-1}((-\infty, t])$ is measurable for all $t \in \mathbb{R}$.

(a) If $f$ is measurable and $a \geq 0$ is a scalar, then $af$ is measurable.

(b) If $f, g$ are measurable, then $f + g$ is measurable.

(c) If $f$ is measurable and $g : \mathbb{R} \to \mathbb{R}$ is continuous, then $g \circ f$ is measurable.

**Problem 3.**

Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove the *dominated convergence theorem*: if $(f_n)$ is a sequence of measurable functions $X \to \mathbb{R}$ converging to a measurable function $f$, and there exists an integrable function $g : X \to \mathbb{R}_{\geq 0}$ such that $|f_n| \leq g$ for all $n$, then $f$ is integrable and

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0.$$

In particular, $\lim_n \int_X f_n \, d\mu = \int_X f \, d\mu$.