July 18

Problem 1.

If S is a subspace of a normed space V, then $\overline{S}$ is also a subspace of V.

Problem 2.

Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of real numbers defined by

$$b_1 = -1, \qquad b_{n+1} = b_n - b_n^2.$$

Let $V = \bigoplus_{k=0}^{\infty} \mathbb{C}$ be an inner product space with (linear) orthonormal basis $\{|k\rangle\}_{k=0}^{\infty}$. Define a sequence of vectors

$$\begin{aligned}
\mathbf{v}_0 &= |0\rangle \\
\mathbf{v}_1 &= |0\rangle + |1\rangle \\
\mathbf{v}_2 &= |0\rangle + b_1 |1\rangle + |2\rangle \\
\mathbf{v}_3 &= |0\rangle + b_1 |1\rangle + b_2 |2\rangle + |3\rangle \\
&\vdots \\
\mathbf{v}_k &= |0\rangle + b_1 |1\rangle + \ldots + b_{k-1} |k-1\rangle + |k\rangle .
\end{aligned}$$

Define subspaces $M = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5, \ldots\}$ and $N = \operatorname{span}\{\mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_6, \ldots\}$ of V. Show that $M^\perp = N$ and $N^\perp = M$, but $\mathbf{v}_0 \notin M + N$.

Problem 3.

Show that $|0\rangle, |1\rangle, \ldots$ is a linear basis for a dense subspace of $\ell^2(\mathbb{N})$.