July 15

Problem 1.

Prove that if V_1 and V_2 are Banach spaces, then the (external) direct sum

$$V_1 \oplus V_2,$$

is itself a Banach space, using the norm defined by

$$\|\mathbf{v}_1 + \mathbf{v}_2\| = \max\{\, \|\mathbf{v}_1\|_{V_1}\ ,\ \|\mathbf{v}_2\|_{V_2}\,\}$$

for $\mathbf{v}_1 \in V_1$ and $\mathbf{v}_2 \in V_2$.

Deduce, using this and results from the analysis warm-up, that $\mathbb{R}^n$ is a Banach space with any norm.

Problem 2.

Let V be a normed space and L_1, L_2 two bounded operators on V. Show that

$$\|L_1 L_2\| \leq \|L_1\|\|L_2\|.$$

Problem 3.

Let V and W be normed spaces. Show that the following properties of a linear map $L : V \to W$ are equivalent.

(a) L is bounded.

(b) L is continuous at the point $\mathbf{0} \in V$.

(c) L is continuous.