HW 1

You may use any results from the warm-up problem sets or that were proved in class (including any problems we've done), except where otherwise indicated. In particular, you'll want to use the spectral theorem for problems 3-5. Problem 6 is optional.
Recall that the operator norm of an operator L on a normed space V is

$$\|L\| = \sup\{\, \|L\mathbf{v}\| \mid \|\mathbf{v}\| \leq 1\,\}.$$

Definition 1.

Let L be an operator on a complex normed space V. The *resolvent set* of L, denoted $\rho(L)$, is the set

$$\rho(L) \doteq \{\, \lambda \in \mathbb{C} \mid (T - \lambda I) \text{ is an invertible map and } \|(T - \lambda I)^{-1}\| < \infty\,\}.$$

The *spectrum* of L, denoted $\sigma(L)$, is the set

$$\sigma(L) \doteq \{\, \lambda \in \mathbb{C} \mid (T - \lambda I) \text{ is not invertible, or } \|(T - \lambda I)^{-1}\| = \infty\,\},$$

which is the complement of the resolvent set of L.

Problem 1.

Show that the spectrum of an operator L on $\mathbb{C}^n$ is the same as the set of eigenvalues of L.

Remark 2. In an infinite dimensional space, the spectrum can fail to consist of the eigenvalues of L. For instance, if V has a basis indexed by $\mathbb{N}$, then the operator L which acts via

$$\mathbf{e}_i \mapsto \mathbf{e}_{i+1}$$

does not have any eigenvalues. But 0 must be in $\sigma(L)$, since L is not invertible.

Problem 2.

Show (without using the spectral theorem) that the eigenspaces of a normal operator L on $V = \mathbb{C}^n$ satisfy

$$V_\lambda \perp V_{\lambda'}$$

whenever $\lambda \neq \lambda'$.

Problem 3.

Let L be a linear operator on $\mathbb{C}^n$. Prove the following.

(a) L is self-adjoint if and only if L is normal and $\sigma(L) \subseteq \mathbb{R}$.

(b) L is an orthogonal projection if and only if L is normal and

$$\sigma(L) \subseteq \{0, 1\}.$$

Problem 4.

Let U be an operator on $\mathbb{C}^n$. Show that the following statements are equivalent.

(a) U is unitary

(b) $U^*U = I = UU^*$

(c) U is normal and $\sigma(U) \subseteq S^1$.

Here S^1 denotes the set of complex numbers with norm 1.

Problem 5.

Let's prove the most general form of the spectral theorem for $V = \mathbb{C}^n$.

(a) Let L be an operator with eigenspaces $\{V_\lambda\}_{\lambda \in \mathbb{C}}$. Let L' be an operator which commutes with L. Show that L' has an eigenvector in V_λ whenever $V_\lambda \neq 0$.

(b) Let S be a set of normal operators, any two of which commute. Then there exists an orthogonal decomposition

$$\bigoplus_{f : S \to \mathbb{C}} V_f,$$

where the sum is over all functions $f : S \to \mathbb{C}$, and

$$V_f \doteq \{\, \mathbf{v} \in V \mid \forall L \in S,\ L\mathbf{v} = f(L)\mathbf{v} \,\}.$$

(c) Let S be a set of normal operators, any two of which commute. Then there exists an orthonormal basis for V such that every basis vector is an eigenvector for every element of S.

We call the decomposition in part (b) the *simultaneous eigendecomposition* or *weight decomposition* of S. The space V_f is a *simultaneous eigenspace* or a *weight space* for S, with *weight* f. The basis in part (c) is a *simultaneous eigenbasis* for S.

Problem 6. o

This problem is optional. Here we recast the results in Problem 5 in a way that will generalize directly to infinite dimensional Hilbert spaces. The goal will be to state a version of the spectral theorem (in part (c)) which does not use any "eigen-" words. Set $V = \mathbb{C}^n$ and let S be a set of commuting normal operators on V. Let $\{V_f\}_{f:S\to\mathbb{C}}$ be the weight decomposition of S constructed above.

(a) Assume that S is a subspace of $\mathrm{End}(V)$ containing the identity operator, and that the product of any two operators in S is also in S. Show that if V_f is nonzero, then

$$\begin{aligned} f(L_1 + L_2) &= f(L_1) + f(L_2) \\ f(\lambda L) &= \lambda f(L) \\ f(L_1 L_2) &= f(L_1) f(L_2) \end{aligned}$$

for any $L, L_1, L_2 \in S$ and any scalar λ.

(b) Continue from the setup in part (a), but add the assumptions that the identity operator is in S and that V_f is one-dimensional whenever $V_f \neq 0$. Let $f_1, \ldots, f_n : S \to \mathbb{C}$ be the distinct functions such that $V_{f_i} \neq 0$. (So that

$$V = \bigoplus_{j=1}^{n} V_{f_i}$$

is an orthogonal decomposition.) Show that, for any scalars $\lambda_1, \ldots, \lambda_n$, there is a unique operator $L \in S$ such that $f_j(L) = \lambda_j$ for all j.

(c) Continue from the setup in part (b). Recall that $[n]$ denotes the set $\{1, \ldots, n\}$. Let $L^2([n])$ denote the inner product space of all functions $\psi : [n] \to \mathbb{C}$. Given a function $f : [n] \to \mathbb{C}$, we define a linear operator

$$\begin{aligned} L_f : L^2([n]) &\to L^2([n]) \\ \psi &\mapsto f\psi. \end{aligned}$$

Now we define a set of operators

$$L^\infty([n]) \doteq \{\, L_f \mid f : [n] \to \mathbb{C} \,\}.$$

Show that there exists a unitary map U from V to $L^2([n])$, such that

$$L \mapsto U L U^{-1}$$

maps operators in S bijectively onto operators in $L^\infty([n])$.

To see that the assumptions in parts (a) and (b) are not restrictive, we will show that any set of commuting normal operators is contained in a set S satisfying parts (a) and (b).

(d) Let S be a *maximal* set of commuting normal operators in $\mathrm{End}(V)$. That is, assume that S consists of commuting normal operators, and if L is any normal operator which is not in S, then there is an operator $L' \in S$ which does not commute with L. Prove the following properties of S.

(i) S contains the identity operator.

(ii) S is a subspace of $\mathrm{End}(V)$.

(iii) If $L_1, L_2 \in S$, then $L_1 L_2 \in S$.

(iv) There are n distinct nonzero weight spaces of S. (Equivalently, every nonzero weight space is one-dimensional.)