10/2 Nick Sheridan

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The Gamma and Strominger-Yau-Zaslow conjectures

(joint with Abouzaid, Ganatra, Iritani)

Introduction: Gamma conjectures in the CY case

Defn: A $\mathbb{C}$-VHS/$\mathcal{M}$ consists of

- holomorphic vec. bun. $\mathcal{V} \longrightarrow \mathcal{M}$
- filtration $F^{\geq *}\mathcal{V}$
- flat connection $\nabla : \mathcal{V} \to \Omega^1_{\mathcal{M}} \otimes \mathcal{V}$

satisfying Griffiths transversality:

$$\nabla(F^{\geq p}\mathcal{V}) \subset \Omega^1_{\mathcal{M}} \otimes F^{\geq p-1}\mathcal{V}.$$

A polarization of a $\mathbb{C}$-VHS is a pairing

$$(\cdot,\cdot) : \mathcal{V}^{\otimes 2} \longrightarrow \mathcal{O},$$

satisfying:

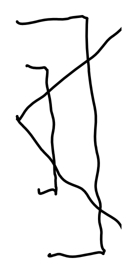
- $(\cdot,\cdot)$ is covariantly constant
- $(F^{\geq p}\mathcal{V}, F^{\geq q}\mathcal{V}) = 0$ if $p+q>0$
- $(\cdot,\cdot) : Gr^p_F \mathcal{V} \otimes Gr^{-p}_F \mathcal{V} \to \mathcal{O}$ non-deg.

E.g. $\mathcal{Y} \longrightarrow \mathcal{M}$ fam. of smooth compact CY k

manifolds $\rightsquigarrow$ $\mathbb{C}$-VHS$/_{\mathcal{M}}$ $\mathcal{V}^B(\mathcal{Y})$,

$$V_m = H^*(Y_m ; \mathbb{C})$$

$$F^{\geq k} V_m = \bigoplus_{p-q \geq 2k} H^{p,q}$$



$$\nabla = \text{"Gauss-Manin" conn.}$$

$$(\alpha,\beta) = i^{|\alpha|}\int \alpha \cup \beta$$

$\boxed{\bullet\ (u,v) = (-1)^n (v,u) \text{ for some } n \in \mathbb{Z}/2}$

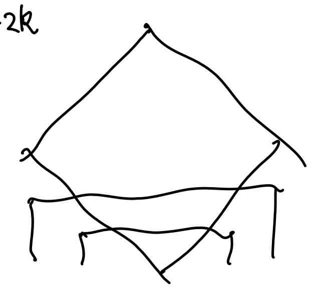
E.g. X = sm. cpct. CY, ω = Kähler

$\rightsquigarrow$ $\mathbb{C}$-VHS$/\Delta^*$ $\quad \mathcal{V}^A(X,\omega)$:

$\Delta^* = \{t \in \mathbb{C}^* : |t| < \delta\} \quad T = t^{-1}$

$V_T = H^\bullet(X;\mathbb{C})$

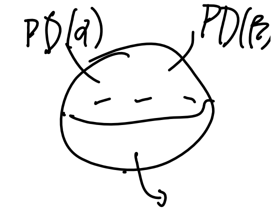
$F^{\geq k} V_T = \bigoplus_{p \leq n-2k} H^p(X;\mathbb{C})$



$\nabla^{Dub/Giv}_{\frac{\partial}{\partial T}}(\alpha) = \frac{\partial \alpha}{\partial T} + T^{-1}[\omega] \star$

$(\alpha,\beta) = i^{n(n+2)-|\alpha|} \int \alpha \cup \beta$

$PD(\alpha \star_T \beta) = \sum_{u \text{ hol.}}$



$Y \xleftrightarrow{\text{mirror}} (X,\omega)$

fam. of CYs over Δ^*_t; $t \to 0$ "large complex structure limit"

fam. of symp. CYs over Δ^*_t (think: $\omega_t = -\log t \cdot \omega$); $t \to 0$ "large volume limit"

$\Downarrow$

$$\begin{array}{ccc} \mathcal{V}^B(Y) & \underset{\mathbb{C}\text{-VHS}}{\cong} & \mathcal{V}^A(X,\omega) \\ \downarrow & & \downarrow \\ \Delta^* & \underset{\text{"mirror map"}}{\xleftarrow{\sim}} & \Delta^* \end{array}$$

Γ conj.: refine to iso of $\mathbb{Z}$-

Defn: $\mathbb{Z}$-VHS := $\mathbb{C}$-VHS +

Lattice of flat sections $V_{\mathbb{Z}} \subset$

(i.e. $\mathbb{Z}$ loc. sys., $V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \mathcal{V}_{fla}$

s.t. $F^{\geq *}$ complementary $\overline{F^{\geq -*}}$

(+ positivity for polarization...)

$\rightsquigarrow$ can study periods

E.g. $\bigoplus_p (2\pi i)^{\frac{p}{2}} H^p(Y_m;\mathbb{Z}) \subset \mathcal{V}^B($

E.g. $\left\{ \begin{array}{l} \text{flat sections of } \mathcal{V}^A(X) \text{ asymptotic, as } T \to \infty, \text{ to} \\ \hat{\Gamma}_X \cup \underset{\uparrow \ \exp(\log T \cdot [\omega])}{T^{\omega}} \cup \bigoplus_p (2\pi i)^{\frac{p}{2}} H^p(X;\mathbb{Z}) \end{array} \right\}$

where $\hat{\Gamma}_X := \prod_i \Gamma(1+\delta_i) = \exp\left(\sum_{k\geq 2} (-1)^k \zeta(k) \cdot (k-1)! \, ch_k(TX)\right) \in H^*(X$

Chern roots of TX (δ_i)

$$= 1 + \zeta(2)\, ch_2(TX) - 2\zeta(3)\, ch_3(TX$$

$$= \hat{\Gamma}_0 + \hat{\Gamma}_2 + \hat{\Gamma}_3$$

Gamma Conjectures (build on work of Horja, Libgober, Gelfand-Kapranov-Zele
Golyshev.

① (Iritani, Katzarkov-Kontsevich-Pantev)

$$\mathcal{V}^A(X) \underset{\mathbb{Q}\text{-VHS}}{\cong} \mathcal{V}^B(Y)$$

② (Hosono)

If $L_T \subset Y_T$ Lag. mirr. to $E \in D^b(X)$

$$\begin{array}{ccc} PD(L_T) & \longleftrightarrow & \text{flat. sec. asymp. to } \hat{\Gamma}_X \cup T^{\omega} \\ \cap & & \cap \\ \mathcal{V}^B(Y) & \cong & \mathcal{V}^A(X) \\ \Psi & & \Psi \\ \Omega_T & \longleftrightarrow & 1 \end{array}$$

hol. vol. form

Γ Conj 2 $\Rightarrow$ $$\boxed{\int_{L_T} \Omega_T \simeq \int_X \hat{\Gamma}_X \cup T^{\omega} \cup (2\pi i)^{\frac{\deg}{2}} ch(E) + (}$$

<u>Thm</u> (Barannikov-Kontsevich, Costello, Ganatra-Perutz-S.):

$$DFuk(X) \simeq DCoh(Y) \quad \Rightarrow \quad \mathcal{V}^A(X) \underset{\mathbb{C}\text{-VHS}}{\cong} \mathcal{V}^B(Y)$$

$\mathbb{Z}$? $\mathbb{Q}$?

Hope: upgrade to $\mathbb{Z}$- or $\mathbb{Q}$-VHS.

Note: suffices to match the lattices asymptotically as $T \to \infty$; hence our formulati

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<u>Thm</u> (Abouzaid-Ganatra-Iritani-S.):
Γ Conj 2 holds if (X, Y) are Batyrev mirrors, E = 'ambient' line bundle.

(due to Iritani; give new proof based on SYZ).

Focus on case $E = \mathcal{O}_X \Rightarrow ch(E) = 1$.

$$\rightsquigarrow \int_{L_T} \Omega_T + O(T^{-\varepsilon}) = \int_X \hat{\Gamma}_X \cup T^{\omega}$$

$$= \sum_i \int_X \hat{\Gamma}_i \cup \frac{(\log T \cdot \omega)^{n-i}}{(n-i)!}$$

SYZ: $X \longrightarrow B \longleftarrow Y_T$, $B \supset B^{sing}$ "

Local models:

$X \longrightarrow B \setminus B^{sing}$ looks like moment m
T^n-action.

$Y_T \longrightarrow B \setminus B^{sing}$ looks like
$(\mathbb{C}^*)^n \xrightarrow{\log_T |\cdot|} \mathbb{R}^n$;
"correct" by holom. discs emana
from B^{sing}, weighted by $T^{-\text{area}}$.

Ω_T looks like $d\log z_1 \wedge \dots \wedge d\log z_n$

L_T looks like $\mathbb{R}^n_+ \subset (\mathbb{C}^*)^n$

$$\rightsquigarrow \int_{L_T} \Omega_T = (\log T)^n \cdot vol(B) + l.o.t.$$

$i > 0$ terms come from 'corre
modulo $O(T^{-\varepsilon})$, contribut
come from <u>constant</u> discs
B^{sing}.
$\Rightarrow$ <u>Local</u> in B.

$$= (\log T)^n \underbrace{\int_X \frac{\omega^n}{n!}}_{i=0 \text{ term}} + \text{l.o.t. (Duistermaat-Heckman)}$$

| $\hat{\Gamma}_i$ arises from codim-i

E.g. $X = K3$, $B = S^2$, $B^{sing} = \{24 \text{ pts}\}$

$$\int_X T^\omega \cup \hat{\Gamma}_X = \int_X \left(1 + \log T \cdot \omega + \tfrac{1}{2} (\log T)^2 \omega^2\right) \cup \left(1 - 24\zeta(2) \cdot pt\right)$$

$$= (\log T)^2 \int_X \frac{\omega^2}{2} - 24\zeta(2)$$

$$= \underset{(DH)}{(\log T)^2 \cdot \mathrm{vol}(B)} - 24\zeta(2)$$

↖ contribution of B^{sing} to asymptotics of $\int_{L_T} \Omega_T$.

$$= \int_{L_T} \Omega_T + O(T^{-\varepsilon})$$

E.g. $X = CY3$, $B = S^3$, B^{sing} = trivalent graph

$$\int_X T^\omega \cup \hat{\Gamma}_X = (\log T)^3 \int_X \frac{\omega^3}{3!} - (\log T) \cdot \zeta(2) \cdot \int_X c_2(TX) \cup \omega - \zeta(3) \cdot \int_X$$

$$= \underset{\text{codim-0}}{(\log T)^3 \mathrm{vol}(B)} - \underset{\text{codim-2}}{(\log T) \cdot \zeta(2) \cdot \mathrm{length}(B^{sing})} - \underset{\text{codim}}{\zeta(3) \cdot \mathrm{i}}$$

$$= \int_{L_T} \Omega_T + O(T^{-\varepsilon})$$

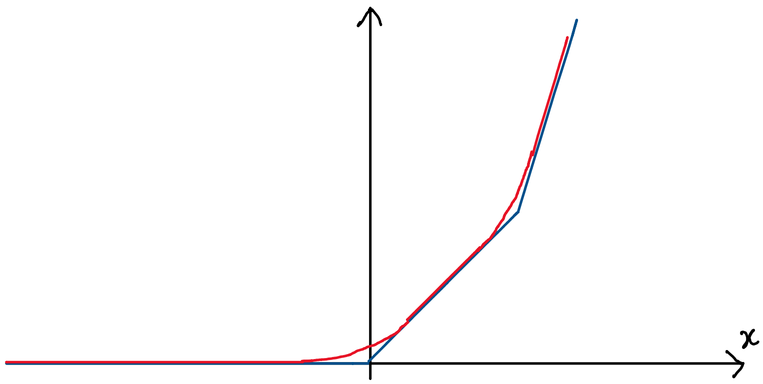
Local computations

$$\lim_{T\to\infty} \log_T\left(T^x + T^y\right) = \max(x,y) \left(+\, O\left(T^{-\varepsilon}\right) \quad |x-y|>\varepsilon\right)$$

alg. geom. $\longleftrightarrow$ trop. geom.

E.g. $p_T(a) = 1 + a + T^{-1}a^2$

$$\lim_{T\to\infty} \log_T\left(p_T(T^x)\right) = \max(0, x, 2x-1)$$



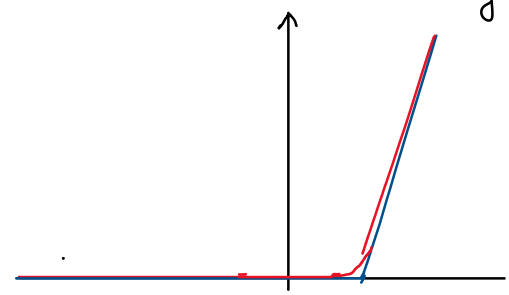
$$\int_{-B} \text{LHS}\, dx = \int_{-B} \text{RHS}\, dx + \frac{2\zeta(2}{\log T}$$

$$\left(\zeta(2) = \int_{-\infty}^{\infty} \operatorname{Log}(1+e^x) - \max(0,$$

E.g. $p_T(a) = 1 + T^{-2}a + T^{-1}a^2$

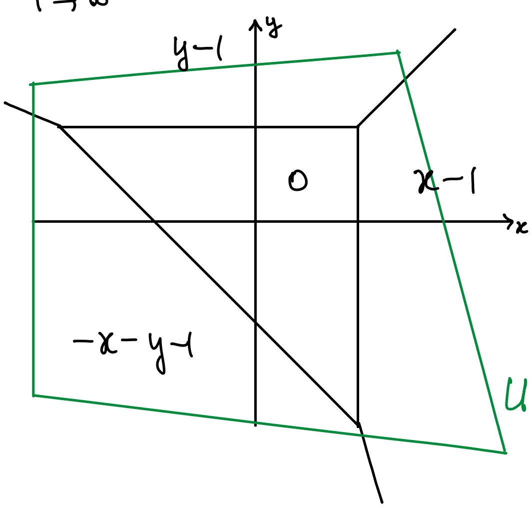
$$\lim_{T\to\infty} \log_T\left(p_T(T^x)\right) = \max(0, x-$$

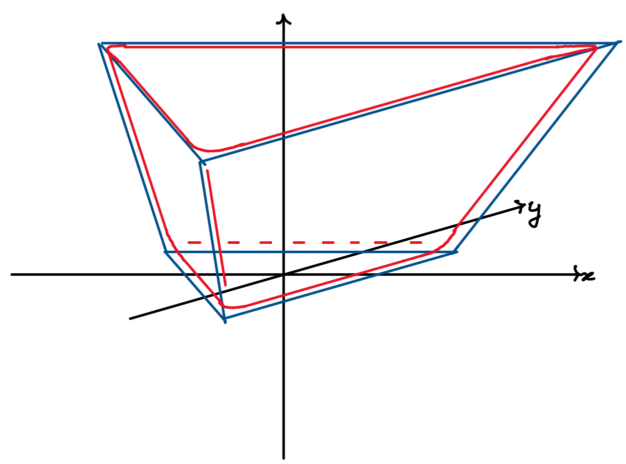
$$\int_{-B}^{B} \text{LHS}\, dx = \int_{-B}^{B} \text{RHS}\, dx + \frac{\frac{1}{2}\zeta}{\log}$$



E.g. $p_T(a,b) = -1 + T^{-1}\left(a + b + \frac{1}{ab}\right)$

$$\lim_{T\to\infty} \log_T\left(p_T(T^x, T^y)\right) = \max(0, x-1, y-1, -x-y-1)$$





$$\iint_U \text{LHS}\, dA = \iint_U \text{RHS}\, dA + L \cdot \frac{\zeta(2)}{\log T} + 3\,\frac{\zeta(3)}{(\log T)^2} + O(T$$

"length inside U

E.g. Focus-focus singularity $\{xy = 1+z\} \subset \mathbb{C}^2 \times \mathbb{C}^*$, $\Omega_T = d\log z$

$= -d\log$

$a = \log|x| \quad b = \log|y| \quad c = \log|z|$

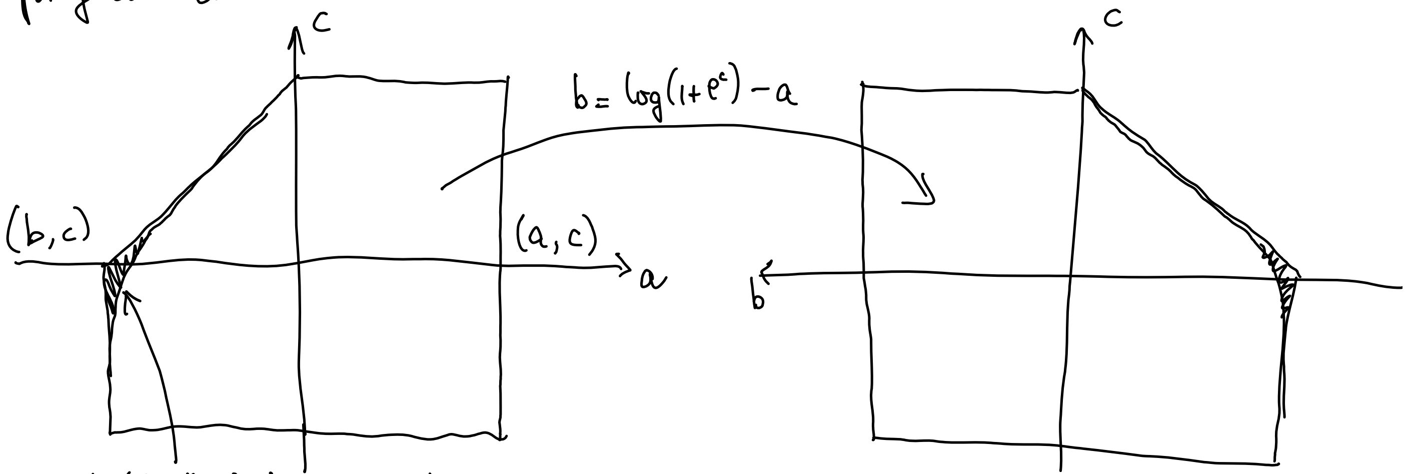
region $\{c \in [-K, K],\ a \leq K,\ b \leq K\} \subset L_T$

projects to

L_T = pos. r

$\Omega_T|_{L_T} = d\iota$

$= -$



$\zeta(2)$ "missing" from linear approx.