

SYZ mirror symmetry for del Pezzo surfaces and rational elliptic surfaces

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Joint w/ A. Jacob and Y.-S. Lin

Outline

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- 6 SYZ mirror symmetry for del Pezzo surfaces, rational elliptic surfaces and Hodge numbers.

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- For degree $k \neq 8$ there is one deformation family of del Pezzo surfaces.
- For degree 8 there are two families given by the first Hirzebruch surface $\text{Bl}_p \mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

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- For degree $k \neq 8$ there is one deformation family of del Pezzo surfaces.
- For degree 8 there are two families given by the first Hirzebruch surface $\text{Bl}_p \mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.
- All del Pezzo surfaces admit a smooth divisor $D \in | -K_Y |$.

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del Pezzo surfaces and rational elliptic surfaces as Calabi-Yau pairs

- Let Y be a RES or a del Pezzo surface, and $D \in |-K_Y|$ a divisor. If $s \in H^0(Y, -K_Y)$ is a holomorphic section with $\{s = 0\} = D$, then $\frac{1}{s}$ is a holomorphic $(2, 0)$ form on $Y \setminus D$ with a simple pole on D .

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- Therefore, $Y \setminus D$ is a natural *non-compact* Calabi-Yau manifold.
- The existence of a Ricci-flat Kähler metric does not follow from Yau's theorem, since $Y \setminus D$ is non-compact.

SYZ mirror symmetry

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- The notion makes sense for ω *not* the Calabi-Yau symplectic form, but in this case they are no longer volume minimizing.

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- The T -dual fibrations exchange complex and symplectic affine structures on B .

Mirror symmetry beyond CYs

- If Y is not a compact Calabi-Yau manifold, then the mirror to be a (partial compactification of) a *Landau-Ginzburg* model: ie. a non-compact Kähler manifold M together with a holomorphic function $W : M \rightarrow \mathbb{C}$.

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- If Y is compact Kähler and $D \in | -K_Y |$ is a divisor, Auroux laid out a general picture for constructing the mirror to Y by applying SYZ mirror symmetry to the non-compact CY manifold $X = Y \setminus D$.

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- Lunts-Przyjalkowski proved mirror symmetry for “Hodge numbers”, as proposed by Katzarkov-Kontsevich-Pantev.
- Doran-Thompson studied del Pezzo \leftrightarrow RES mirror symmetry at a lattice theoretic level.

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- Explain a proof of SYZ mirror symmetry for del Pezzo surfaces of degree k and RES with an I_k fiber. Compare what we find with other approaches.
- Explain mirror symmetry for Hodge numbers in terms of moduli of complete CY metrics.
- Describe applications to existence of some new CY metrics, a question of Yau, etc.

del Pezzo surfaces: Complete Calabi-Yau metrics

The first ingredient we need is a fundamental result of Tian-Yau, which in our case gives

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and such that ω_{TY} is asymptotic to the Calabi model (with estimates...)

Remark

The Tian-Yau theorem holds in all dimensions

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$$\mathcal{C} := \{\xi \in E : 0 < |\xi|_h < 1\}, \quad \Omega_{\mathcal{C}} := \Omega_D \wedge \frac{dw}{w}.$$
$$\omega_{\mathcal{C}} = \frac{2}{3} \sqrt{-1} \partial \bar{\partial} \left(-\log |\xi|_h^2 \right)^{\frac{3}{2}}.$$

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Then $(\mathcal{C}, \Omega_{\mathcal{C}}, \omega_{\mathcal{C}})$ is Calabi-Yau, and furthermore complete at $0 \subset E$.

del Pezzo surfaces: Complete Calabi-Yau metrics

- The Riemannian geometry of $(\mathcal{C}, \Omega_{\mathcal{C}}, \omega_{\mathcal{C}})$ can be visualized by considering the level sets

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are S^1 bundles over D , fibering \mathcal{C} .

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SYZ fibrations on del Pezzo surfaces

Using this model geometry we prove:

Theorem (C.-Jacob-Lin)

Let Y be a del Pezzo surface, D a smooth elliptic curve, and equip $X = Y \setminus D$ with the Tian-Yau metric ω_{TY} .

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Remark

In fact, X admits countably many distinct special Lagrangian fibrations, one for each choice of simple closed loop $\gamma \in H_1(D, \mathbb{Z})$.

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- **technical issue:** The geometry of the Lagrangian fibers degenerates near ∞ : they are collapsing circle bundles with fixed volume and unbounded diameter.
 - **key point:** The degeneration of the geometry is *polynomial* in the distance to a fixed point, while the mean curvature decays *exponentially*.

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- 7 Now the main issue is to extend the fibration into the interior of X .

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- 11 In every compact set $K \subset X$ there can be only countably many singular holomorphic curves obtained as limits of smooth fibers. From this it follows easily that the fibration extends to a global fibration.

Picture time!

Applications of SYZ fibrations on del Pezzo surfaces

Theorem (C.-Jacob-Lin)

Let Y be del Pezzo of degree k , $D \in |-K_Y|$ smooth, and $X = Y \setminus D$.
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Question (Yau \sim 80s): What is the symplectic form for the HK rotated Tian-Yau metric?

Some Remarks

- The Lagrangian mean curvature flow argument works in all dimensions and applies to Fano manifolds Y , with $D \in |-K_Y|$ assuming D has a special Lagrangian submanifold with respect to the CY metric in $c_1(Y)|_D$.

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- We also prove analogous results for Y a rational elliptic surface with $D \in |-K_Y|$ a smooth fiber. These have complete CY metrics due to Tian-Yau and we show they admit SYZ fibrations and their HK rotations can be compactified to another RES by adding a smooth fiber.
- If $Y = \mathbb{P}^2$ or a generic rational elliptic surface we can identify the singular fibers of the SYZ fibration as the appropriate number of nodal special Lagrangian spheres verifying some conjectures of Auroux.

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The model geometry: Let $\Delta^* = \{z \in \mathbb{Z} : 0 < |z| < 1\}$, consider

$$X_{mod} = (z, x) \in \Delta^* \times \mathbb{C} / (\mathbb{Z} + \frac{k}{2\pi\sqrt{-1}} \log(z) \cdot \mathbb{Z})$$

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$$\Omega = \frac{\kappa(z) dx \wedge dz}{z} \quad \kappa(z) : \Delta \rightarrow \mathbb{C} \text{ hol'c}, \quad \kappa(0) \neq 0$$

Let's assume: $\kappa = 1$ for simplicity.

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- If $\pi : X \rightarrow \Delta$ is an elliptic fibration with $\pi^{-1}(0)$ an I_k fiber, then for any choice of section $\sigma : \Delta^* \rightarrow X \setminus \pi^{-1}(0)$, we get $F_\sigma : X \rightarrow X_{mod}$.

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$$\begin{aligned} \omega_{sf,\varepsilon} &= \sqrt{-1} \frac{k |\log |z||}{2\pi\varepsilon} \frac{dz \wedge d\bar{z}}{|z|^2} \\ &+ \frac{\sqrt{-1}}{2} \frac{2\pi\varepsilon}{k |\log |z||} (dx + B(x,z)dz) \wedge \overline{(dx + B(x,z)dz)} \end{aligned}$$

where $B(x,z) = -\frac{\text{Im}(x)}{\sqrt{-1}z|\log |z||}$, $\varepsilon = \text{volume of the fibers}$.

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- The metric $\omega_{sf,\sigma,\varepsilon} = F_\sigma^* \omega_{sf,\varepsilon}$ is Ricci-flat and complete near the I_k fiber. **Standard semi-flat metric with respect to σ .**

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For all $\alpha > \alpha_0$ there exists a CY metric in the Bott-Chern cohomology class of ω_0 converging exponentially fast to $\alpha \omega_{sf,\sigma,\frac{\varepsilon}{\alpha}}$ at infinity (with very precise estimates to all orders)

This applies, for example, to Kähler metrics restricted from Y .

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Main problem: If we want to understand the Kähler moduli (to do mirror symmetry, define Hodge numbers etc.) on a RES pair (Y, D) in terms of moduli of Calabi-Yau metrics, then we need a parameter space.

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Outstanding questions from Hein's theorem:

- $H_{dR}^2(X, \mathbb{R})$ has dimension $11 - k$.

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- Recall that Bott-Chern cohomology is a refinement of de Rham cohomology given by

$$H_{BC}^{p,q} := \frac{\{\text{Ker } d : \Lambda^{p,q} \rightarrow \Lambda^{p+1,q} \oplus \Lambda^{p,q+1}\}}{\{\text{Im}(\sqrt{-1}\partial\bar{\partial} : \Lambda^{p-1,q-1} \rightarrow \Lambda^{p,q})\}}$$

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Hein's theorem depends on a construction which leaves open the possibility of (infinitely many) distinct CY metrics *even in a fixed Bott-Chern class*.

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- By Leray spectral sequence calculations one can show that

$$H_{BC}^{1,1}(X) \sim H_{dR}^2(X) \times H^0(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$$

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- Lunts-Przyjalkowski computed these Hodge numbers for del Pezzos of degree k and RES with an I_k fiber, and obtained $10 - k$ on both sides (proving mirror symmetry at the level of KKP Hodge numbers).

A brief digression on sections

- a section $\sigma : \Delta^* \rightarrow X_{mod}$ can be written as

$$\sigma(z) = h(z) + \frac{a}{2\pi\sqrt{-1}} \log z + \frac{b}{(2\pi\sqrt{-1})^2} (\log(z))^2$$

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- **Key point:** Pulling back the semi-flat metric by a *multivalued* section still yields a well-defined, semi-flat, Calabi-Yau metric.

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- **Key point:** Pulling back the semi-flat metric by a *multivalued* section still yields a well-defined, semi-flat, Calabi-Yau metric. We call these **non-standard** semi-flat metrics and say they are **quasi-regular** in the \mathbb{Q} case, and **irregular** in the \mathbb{R} case.

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- As we will see, the quasi-regular metrics are important for mirror symmetry.

an Application to a question of Yau

Theorem (C.-Jacob-Lin)

Let (Y, D) be a del Pezzo pair, $\gamma \in H_1(D, \mathbb{Z})$, and $(X, g_{TY}, \omega_{TY}, J)$ be the Tian-Yau Ricci-flat Kähler structure on X .

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- ω_γ is the symplectic form of the Ricci-flat metric produced by the previous theorem.
- ω_γ is asymptotic to a non-standard semi-flat metric unless D is the torus with fundamental domain determined by the lattice $\mathbb{Z} + \sqrt{-1}\lambda\mathbb{Z}$ for $\lambda \in \mathbb{R}_{>0}$, and γ is one of the cycles generating the lattice. Generically, ω_γ is irregular.

More applications of the uniqueness result

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Suppose Y is a RES, and D an I_k fiber. Suppose ω_1, ω_2 are Calabi-Yau metrics with $\omega_1^2 = \omega_2^2$, $[\omega_1]_{dR} = [\omega_2]_{dR}$. Suppose that ω_1, ω_2 converge (at a polynomial rate) to the same semi-flat metric $\omega_{sf, \epsilon}$ at ∞ . Then there exists a fiber preserving map $F : X \rightarrow X$, which is homotopic to the identity such that $F^* \omega_1 = \omega_2$.

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If we define the Kähler moduli to be

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\Rightarrow we have a hope to define the Hodge numbers in terms of moduli of Calabi-Yau metrics.

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- (ii) after HK rotating so that the fibration becomes holomorphic $\pi : X \rightarrow \mathbb{C}$, X can be compactified to another rational elliptic surface by adding an I_k fiber.

- argument is the same as that for del Pezzo surfaces using the "bad cycle" of the quasi-regular semi-flat metric as the model.

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- For $k \neq 8$ there is a unique deformation family of RES with an I_k fiber. For $k = 8$ there are two families, but these can be distinguished topologically.
- Using the Torelli theorem of Gross-Hacking-Keel there is a distinguished pair (Y_e, D_e) in the deformation family (Y', D') with trivial periods. This pair was also used by Hacking-Keating to study HMS for log CY surfaces.

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- Moreover, $(X_e, \check{\omega}_{CY})$ admits an SYZ fibration with fibers in the class α .
- By direct calculation, the SYZ fibrations on $Y \setminus D$ and $Y_e \setminus D_e$ are dual, in the sense that they exchange the complex and symplectic affine structures on the base \mathbb{R}^2 , and that their volumes are inverse to one another.

Comparison to Gross-Hacking-Keel

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- We can show that the complex affine structure of the SYZ fibration on the RES is *asymptotic* to the tropical affine structure of GHK.
- This follows from the fact that the LMCF flows the model fibration to the SYZ fibration near ∞ , and a calculation of Pascaleff.

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- On the complex side, the Torelli theorem of McMullen also gives a $10 - k$ dimensional complex moduli of degree k del Pezzo pairs (Y, D) . These geometric Hodge numbers agree with the KKP Hodge numbers.
- Comparing the *complex* moduli of rational elliptic surfaces with the *symplectic* moduli of del Pezzos is an interesting question for future work.

The End