Skein algebras from higher genus mirror symmetry

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Floer-theoretic and algebro-geometric aspects of SYZ mirror symmetry
October 1st, 2020

Introduction

- Talk based on arXiv:2009.02266.
- Extended application of my "quantum mirror" paper 1808.07336.
- Mirror symmetry: given a log Calabi-Yau variety (Y, D), construct a commutative associative algebra from genus 0 log Gromov-Witten invariants of (Y, D). It is the algebra of regular functions on the mirror of (Y, D). (Dimension 2: Gross-Hacking-Keel, higher dimension: Gross-Siebert, Keel-Yu).
- "Quantum mirror", given a log Calabi-Yau surface (Y, D), construct an associative algebra from higher genus log Gromov-Witten invariants of (Y, D), in general non-commutative: deformation quantization of the Poisson algebra of regular functions on the Gross-Hacking-Keel mirror (B.).

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- Sometimes, the "quantum mirror" construction (based on higher genus Gromov-Witten theory) provides new insights.
- Will focus on the case where Y is a smooth projective cubic surface and D is a triangle of lines on Y. Mirror: Gross-Hacking-Keel-Siebert.
- Main result: the "quantum mirror" of (Y, D) is the skein algebra of the 4-punctured sphere, an associative algebra originally defined in the context of low-dimensional topology.
- Application: proofs of results on the skein algebra of the 4-punctured sphere previously conjectured by topologists, such as the q-positivity of the bracelets basis (D. Thurston, 2013, refined by Bakshi, Mukherjee, Przytycki, Silvero, Wang, 2018).

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- Skein algebras of topological surfaces.
- Conjectural positive bases of skein algebras.
- Results for the skein algebra of the 4-punctured sphere (and of the 1-punctured torus).
- Scattering diagrams and broken lines for the skein algebra of the 4-punctured sphere.
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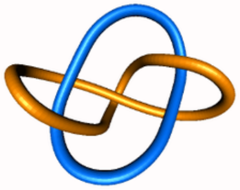
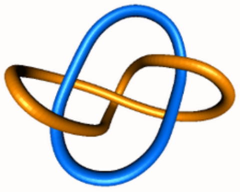


- Knot in a manifold: a connected compact embedded 1-dimensional submanifold.
- Link in a manifold: the disjoint union of finitely many knots.
- Framing of a link: a choice of nowhere vanishing section of its normal bundle, that is the choice of realization of the link as the union of boundary components of some annuli.

Knots, links and framing



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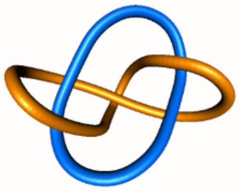

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- The Kauffman bracket skein module (Przytycki, Turaev, 1988) of an oriented 3-manifold $\mathbb{M}$ is the $\mathbb{Z}[A^{\pm}]$-module generated by isotopy classes of framed links in $\mathbb{M}$ satisfying the skein relations

- The diagrams in each relation indicate framed links that can be isotoped to identical embeddings except within the neighborhood shown, where the framing is vertical.
- The skein module of $\mathbb{M} = \mathbb{R}^3$ is $\mathbb{Z}[A^{\pm}]$ (generated by the empty link). The class of a framed link $L \subset \mathbb{R}^3$ in $\mathbb{Z}[A^{\pm}]$ is the Kauffman bracket polynomial of L (equivalent to the Jones polynomial).

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Skein modules of 3-manifolds

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- Given an oriented 2-manifold $\mathbb{S}$, one can define a natural algebra structure on the Kauffmann bracket skein module of the 3-manifold $\mathbb{M} := \mathbb{S} \times (-1,1)$: given two framed links L_1 and L_2 in $\mathbb{S} \times (-1,1)$, and viewing the interval $(-1,1)$ as a vertical direction, the product $L_1 L_2$ is defined by placing L_1 on top of L_2.
- We denote by $\mathrm{Sk}_A(\mathbb{S})$ the resulting associative $\mathbb{Z}[A^{\pm}]$-algebra with unit. The skein algebra $\mathrm{Sk}_A(\mathbb{S})$ is in general non-commutative.

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Skein algebras of surfaces

- We consider the case where $\mathbb{S}$ is the complement $\mathbb{S}_{g,\ell}$ of a finite number ℓ of points in a compact oriented 2-manifold of genus g.
- A multicurve on $\mathbb{S}_{g,\ell}$ is the union of finitely many disjoint compact connected embedded 1-dimensional submanifolds of $\mathbb{S}_{g,\ell}$ such that none of them bounds a disc in $\mathbb{S}_{g,\ell}$. Identifying $\mathbb{S}_{g,\ell}$ with $\mathbb{S}_{g,\ell} \times \{0\} \subset \mathbb{S}_{g,\ell} \times (-1,1)$, a multicurve on $\mathbb{S}_{g,\ell}$ endowed with the vertical framing naturally defined a framed link in $\mathbb{S}_{g,\ell} \times (-1,1)$.

Theorem (Przytycki)

Isotopy classes of multicurves form a basis of $\mathrm{Sk}_A(\mathbb{S}_{g,\ell})$ as $\mathbb{Z}[A^{\pm}]$-module.

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The skein algebra as deformation quantization of the SL_2 character variety

- $\mathrm{Ch}_{SL_2}(\mathbb{S}_{g,\ell})$: the SL_2-character variety of the $\mathbb{S}_{g,\ell}$, affine variety of finite type over $\mathbb{Z}$ obtained as affine GIT quotient by the SL_2 conjugation action of the affine variety of representations of the fundamental group $\pi_1(\mathbb{S}_{g,\ell})$ into SL_2.
- For every γ multicurve on $\mathbb{S}_{g,\ell}$ with connected components $\gamma_1, \cdots, \gamma_r$, the map sending a representation $\rho \colon \pi_1(\mathbb{S}_{g,\ell}) \to SL_2$ to $\prod_{j=1}^r (-\mathrm{tr}(\rho(\gamma_j)))$ defines a regular function f_γ on $\mathrm{Ch}_{SL_2}(\mathbb{S}_{g,\ell})$.

Theorem (Bullock, Przytycki-Sikora, Charles-Marché)

The map $\gamma \mapsto f_\gamma$ defines a ring isomorphism between the specialization $\mathrm{Sk}_{-1}(\mathbb{S}_{g,\ell})$ of the skein algebra at $A = -1$ and the ring of regular functions of $\mathrm{Ch}_{SL_2}(\mathbb{S}_{g,\ell})$.

Classical limit of the skein relation: for every $M, N \in SL_2$,

$$\mathrm{tr}(M)\,\mathrm{tr}(N) = \mathrm{tr}(MN) + \mathrm{tr}(M^{-1}N)\,.$$

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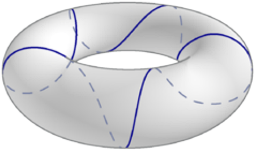
Example: closed torus



- On the closed torus $\mathbb{S}_{1,0}$, isotopy classes of multicurves are in bijection with

$$B(\mathbb{Z}) := \mathbb{Z}^2/\langle \pm id \rangle \simeq \{(m,n) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} \mid m \geq 0 \text{ if } n = 0\}.$$

- For every $p = (m,n) \in B(\mathbb{Z})$, denote by γ_p the corresponding isotopy class of multicurves.
- γ_p has $\gcd(m,n)$ connected components.
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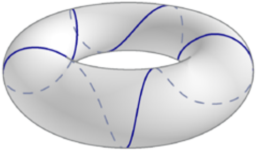

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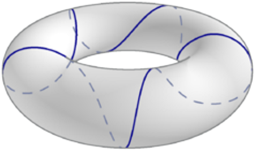

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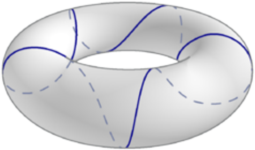

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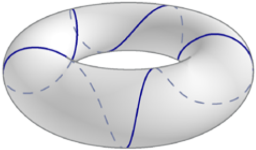

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Structure constants and positive bases

- The structure constants $C^l_{j,k} \in R$ of a basis $\{e_j\}_{j\in J}$ of an algebra $\mathcal{A}$ over a ring R are defined by

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- For the skein algebra, $R = \mathbb{Z}[A^{\pm}]$.

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A basis $\{e_j\}_{j\in J}$ of the skein algebra $\mathrm{Sk}_A(\mathbb{S}_{g,\ell})$ is called *positive* if its structure constants belong to $\mathbb{Z}_{\geq 0}[A^{\pm}]$, i.e. are Laurent polynomials in A with positive coefficients.

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A basis $\{e_j\}_{j \in J}$ of the skein algebra $\mathrm{Sk}_A(\mathbb{S}_{g,\ell})$ is called *positive* if its structure constants belong to $\mathbb{Z}_{\geq 0}[A^{\pm}]$, i.e. are Laurent polynomials in A with positive coefficients.

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Structure constants and positive bases

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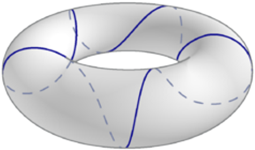

- One crossing to resolve:

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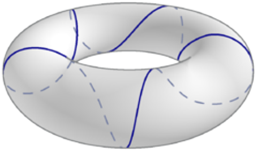

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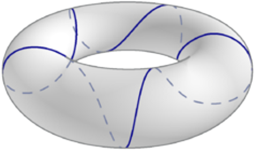

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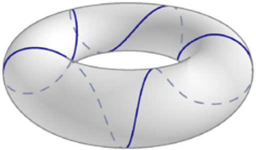

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The bracelets basis

- Let $T_n(x)$ be the Chebyshev polynomials defined by

$$T_0(x) = 1\,,\; T_1(x) = x\,,\; T_2(x) = x^2 - 2\,,$$

and for every $n \geq 2$,

$$T_{n+1}(x) = xT_n(x) - T_{n-1}(x)\,.$$

Writing $x = \lambda + \lambda^{-1}$, we have $T_n(x) = \lambda^n + \lambda^{-n}$ for every $n \geq 1$.

- Given an isotopy class γ of multicurve on $\mathbb{S}_{g,\ell}$, one can uniquely write γ in $\mathrm{Sk}_A(\mathbb{S}_{g,\ell})$ as $\gamma = \gamma_1^{n_1} \cdots \gamma_r^{n_r}$ where $\gamma_1, \cdots, \gamma_r$ are all distinct isotopy classes of connected multicurves and $n_j \in \mathbb{Z}_{>0}$, and we define

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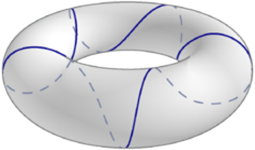

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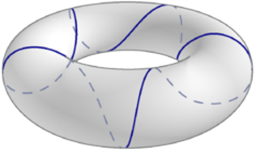

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Conjectural positivity of the bracelets basis

Conjecture (Dylan Thurston, 2013)

For every g and ℓ, the bracelets basis $\{\mathbf{T}(\gamma)\}_\gamma$ of $\mathrm{Sk}_A(\mathbb{S}_{g,\ell})$ is positive.

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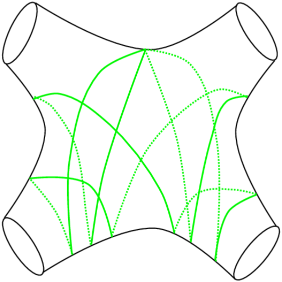
Main result

Theorem (B, 2020)

The bracelets bases $\{\mathbf{T}(\gamma)\}_\gamma$ of the skein algebras $\mathrm{Sk}_A(\mathbb{S}_{0,4})$ and $\mathrm{Sk}_A(\mathbb{S}_{1,1})$ of the 4-punctured sphere and the 1-punctured torus are positive.

Unlike the case of the closed torus $\mathbb{S}_{1,0}$, there does not seem to exist a simple closed formula for the structure constants of the bracelets basis of $\mathrm{Sk}_A(\mathbb{S}_{0,4})$ and $\mathrm{Sk}_A(\mathbb{S}_{1,1})$.



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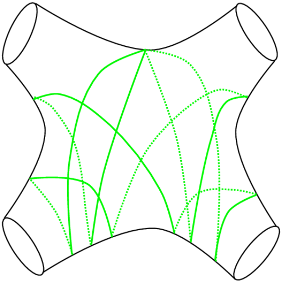

- Focus on the case of the 4-punctured sphere $\mathbb{S}_{0,4}$.
- Peripheral curves a_1, a_2, a_3, a_4, in the center of $\mathrm{Sk}_A(\mathbb{S}_{0,4})$, so we can view $\mathrm{Sk}_A(\mathbb{S}_{0,4})$ as a R-module, where $R = \mathbb{Z}[A^{\pm}][a_1, a_2, a_3, a_4]$.
- Isotopy classes of multicurves in $\mathbb{S}_{0,4}$ without peripheral connected components are in bjection with

$$B(\mathbb{Z}) := \mathbb{Z}^2/\langle \pm id \rangle \simeq \{(m,n) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} \mid m \geq 0 \text{ if } n = 0\}.$$

(View $\mathbb{S}_{0,4}$ as a $\mathbb{Z}/2\mathbb{Z}$-quotient of a 4-punctured torus)

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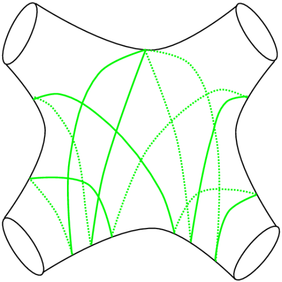

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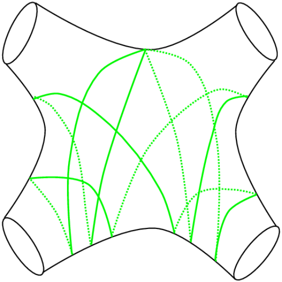

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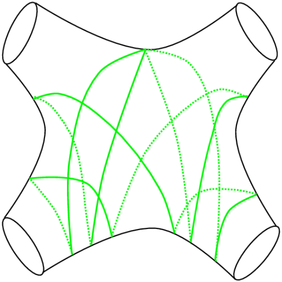

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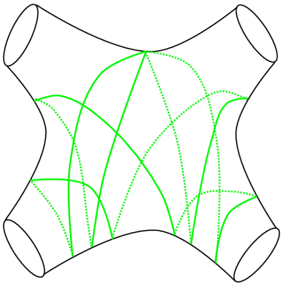

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Example

- $$\mathbf{T}(\gamma_{(1,0)})\mathbf{T}(\gamma_{(0,1)}) = A^2\mathbf{T}(\gamma_{(1,1)}) + A^{-2}\mathbf{T}(\gamma_{(-1,1)}) + R_{1,1}\,.$$
- $$\mathbf{T}(\gamma_{(1,1)})\mathbf{T}(\gamma_{(-1,1)}) = A^{-4}\mathbf{T}(\gamma_{(2,0)}) + A^4\mathbf{T}(\gamma_{(0,2)}) + A^{-2}R_{1,0} + A^2R_{0,1} + y\,.$$

Theorem (Bullock-Przytycki, 2000)

$\mathrm{Sk}_A(\mathbb{S}_{0,4})$ is generated as R-algebra by $\gamma_{v_1} := \gamma_{(1,0)}$, $\gamma_{v_2} := \gamma_{(0,1)}$, $\gamma_{v_3} = \gamma_{(-1,1)}$, with the relations

$$A^{-2}\gamma_{v_1}\gamma_{v_2} - A^2\gamma_{v_2}\gamma_{v_1} = (A^{-4} - A^4)\gamma_{v_3} - (A^2 - A^{-2})R_{1,1}\,,$$

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$$+A^{-2}R_{1,1}\gamma_{v_3} + y - 2(A^4 + A^{-4})\,.$$

Generators and relations

Setting $A = -1$, we get that the ring of regular fuctions on the SL_2 character variety $\mathrm{Ch}_{SL_2}(\mathbb{S}_{0,4})$ is generated as $\mathbb{Z}[a_1, a_2, a_3, a_4]$-algebra by $\gamma_{v_1} := \gamma_{(1,0)}$, $\gamma_{v_2} := \gamma_{(0,1)}$, $\gamma_{v_3} = \gamma_{(-1,1)}$, with the cubic relation

$$\gamma_{v_1}\gamma_{v_2}\gamma_{v_3} = \gamma_{v_1}^2 + \gamma_{v_2}^2 + \gamma_{v_3}^2 + R_{1,0}\gamma_{v_1} + R_{0,1}\gamma_{v_2} + R_{1,1}\gamma_{v_3} + y - 4\,.$$

In other words, $\mathrm{Ch}_{SL_2}(\mathbb{S}_{0,4})$ is a 4-parameters families of affine cubic surfaces (complement of a triangle of lines in a projective cubic surface)(known at the end of the 19th century: Vogt, Fricke-Klein). Think about the skein algebra $\mathrm{Sk}_A(\mathbb{S}_{0,4})$ as a family of non-commutative cubic surfaces.

$PSL_2(\mathbb{Z})$ action

- The mapping class group of $\mathbb{S}_{g,\ell}$ (group of isotopy classes of orientation-preserving diffeomorphisms) acts on $\mathrm{Sk}_A(\mathbb{S}_{g,\ell}$. As the mapping class group of $\mathbb{S}_{0,4}$ contains a copy of $PSL_2(\mathbb{Z})$, we get an action on $PSL_2(\mathbb{Z})$ on $\mathrm{Sk}_A(\mathbb{S}_{0,4})$ by R-algebra automorphisms.
- For every $p \in B(\mathbb{Z}) = \mathbb{Z}^2/\langle \pm id \rangle$ and $M \in PSL_2(\mathbb{Z})$, we have $M.\gamma_p = \gamma_{M.p}$.
- Generators of $PSL_2(\mathbb{Z})$:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} .$$

- Action of S and T on generators of $\mathrm{Sk}_A(\mathbb{S}_{0,4})$:

$$S\gamma_{v_1} = \gamma_{v_2} \,, S\gamma_{v_2} = \gamma_{v_3} \,, S\gamma_{v_3} = \gamma_{v_1} \,,$$

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Structure of the proof

- Algorithm computing the structure constants $C^p_{p_1,p_2} \in R$ defined by

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Scattering diagram

We have $B(\mathbb{Z}) \subset B$, where

$$B := \mathbb{R}^2/\langle \pm id \rangle \simeq \{(x,y) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} \; | \, x \geq 0 \; \text{if} \; y = 0\}\,.$$

"Scattering diagram": attach a power series to every ray in B with rational slope of primitive direction $(m,n) \in B(\mathbb{Z})$

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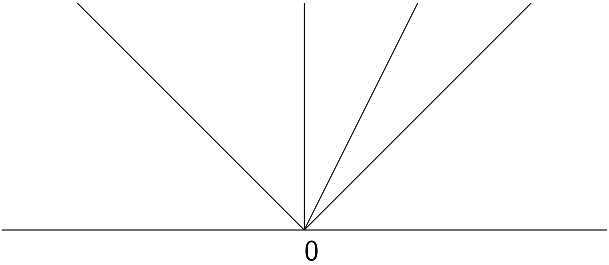

Broken lines

- γ broken line in $\mathfrak{D}$ of asymptotic direction $p \in B(\mathbb{Z})$ and endpoint Q
- Continuous piecewise integral affine line, bending along rays of rational slopes, decorated by monomials.
- Monomial attached to the linearity domain L of the form $c_L z^{p_L}$, where $c_L \in R$, and $-p_L \in \mathbb{Z}^2$ parallel to the direction of L.
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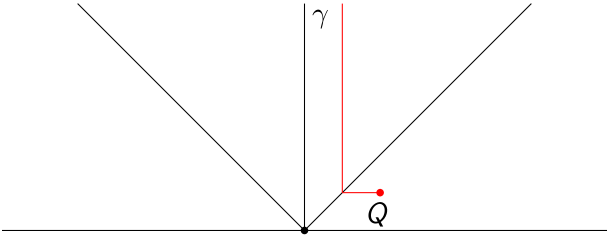

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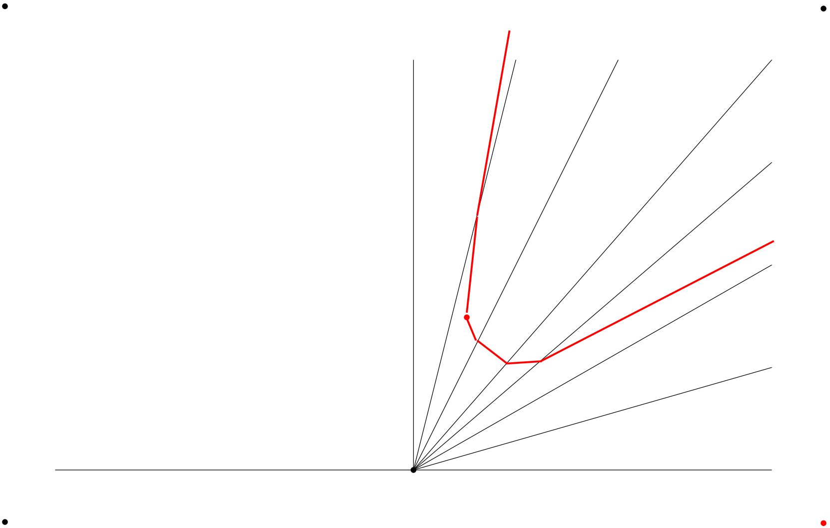

Broken lines

- Bending formula for a broken line passing between the domains of linearity L and L' by bending along $\mathbb{R}_{\geq 0}(m,n)$.
- Write $m_L = c_L z^{p_L}$, $m_{L'} = c_{L'} z^{p_{L'}}$, $N = |\det((m,n),p_L)|$, and $f_{m,n} = \sum_{k\geq 0} c_k z^{-(km,kn)}$, then there exists a monomial $d_\ell z^{-\ell(m,n)}$ in the series

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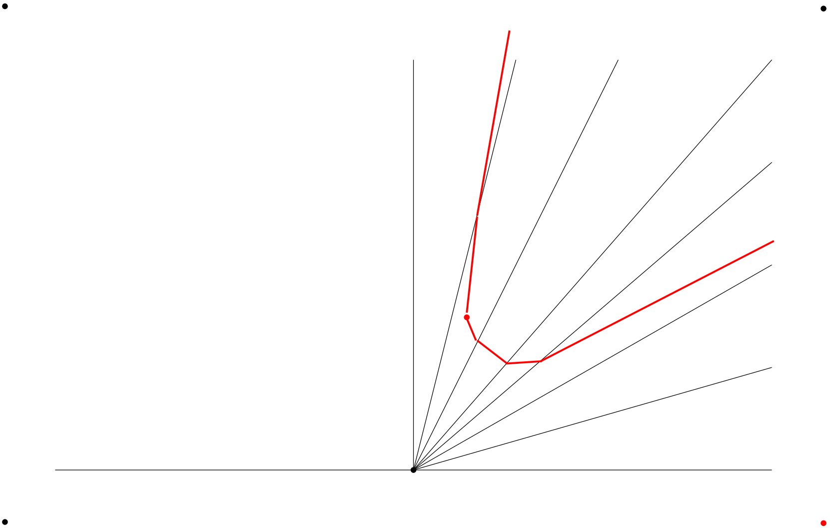

Broken lines

For every $p_1, p_2, p \in B(\mathbb{Z})$ and $Q \in B$ generic close to p, define

$$C^{\mathfrak{D},p}_{p_1,p_2}(Q) := \sum_{(\gamma_1,\gamma_2)} c(\gamma_1)c(\gamma_2)A^{2\det(s(\gamma_1),s(\gamma_2))} \in R\,,$$

where the sum is over pairs (γ_1, γ_2) of quantum broken lines for $\mathfrak{D}$ with charges p_1,p_2 and common endpoint Q, such that writing $c(\gamma_1)z^{s(\gamma_1)}$ and $c(\gamma_2)z^{s(\gamma_2)}$ the final monomials, we have $s(\gamma_1) + s(\gamma_2) = p$.

Associative algebras from consistent scattering diagrams

A scattering diagram $\mathfrak{D}$ is *consistent* if for every $p_1, p_2, p \in B(\mathbb{Z})$, $C^{\mathfrak{D},p}_{p_1,p_2}(Q)$ does not depend on the choice of the point Q, and the product on the free R-module

$$\mathcal{A}_{\mathfrak{D}} := \bigoplus_{p \in B(\mathbb{Z})} R\,\vartheta_p$$

defined by

$$\vartheta_{p_1}\vartheta_{p_2} = \sum_{p \in B(\mathbb{Z})} C^{\mathfrak{D},p}_{p_1,p_2}\vartheta_p$$

is associative.

Proof

Strategy: construct a consistent scattering diagram $\mathfrak{D}$ and an isomorphism

$$\varphi \colon \mathcal{A}_{\mathfrak{D}} \to \mathrm{Sk}_A(\mathbb{S}_{0,4})$$

such that

$$\varphi(\vartheta_p) = \mathbf{T}(\gamma_p)$$

for every $p \in B(\mathbb{Z})$.

Scattering diagram

- Notations:

$$F(r,s,y,x) := 1 + \frac{rx(1+x^2)}{(1-A^{-4}x^2)(1-A^4x^2)} + \frac{yx^2}{(1-A^{-4}x^2)(1-A^4x^2)} + \frac{sx^3(1+sx+x^2)}{(1-A^{-4}x^2)(1-x^2)^2(1-A^4x^2)} \,.$$

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Scattering diagram

- Notations:

$$F(r,s,y,x) := 1 + \frac{rx(1+x^2)}{(1-A^{-4}x^2)(1-A^4x^2)} + \frac{yx^2}{(1-A^{-4}x^2)(1-A^4x^2)}$$
$$+ \frac{sx^3(1+sx+x^2)}{(1-A^{-4}x^2)(1-x^2)^2(1-A^4x^2)}\,.$$

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$$R_{1,0} := a_1a_2 + a_3a_4\,,\quad R_{0,1} := a_1a_3 + a_2a_4\,,\quad R_{1,1} := a_1a_4 + a_2a_3\,,$$
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Results

Theorem 1 (B, 2020)

The scattering diagram $\mathfrak{D}$ is consistent.

Theorem 2 (B, 2020)

There exists an isomorphism

$$\varphi \colon \mathcal{A}_{\mathfrak{D}} \to \mathrm{Sk}_A(\mathbb{S}_{0,4})$$

such that

$$\varphi(\vartheta_p) = \mathbf{T}(\gamma_p)$$

for every $p \in B(\mathbb{Z})$.

Theorem 2 is proved by computations (comparison of presentations by generators and relations and $PSL_2(\mathbb{Z})$-symmetries). The proof of Theorem 1 relies on enumerative algebraic geometry.

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Mirror symmetry

- $\text{Sk}_A(\mathbb{S}_{0,4})$ is a 4-parameters family of non-commutative cubic surfaces, deformation quantization of a 4-parameters family of affine cubic surfaces.
- This 4-parameters family of affine cubic surfaces is the mirror family of the log Calabi-Yau surface (Y, D), where Y is a smooth projective surface in $\mathbb{P}^3$ over $\mathbb{C}$ and D is a triangle of lines on Y. (Gross-Hacking-Keel (2011), Gross-Hacking-Keel-Siebert (2019)). Construction in terms of scattering diagrams constructed in terms of genus 0 log Gromov-Witten invariants of (Y, D).
- "Quantum mirror" (B. (2018)): construction of a deformation quantization of the mirror family of Gross-Hacking-Keel using higher genus log Gromov-Witten invariants of (Y, D).
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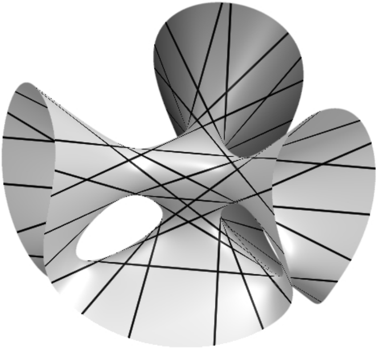
Cubic surfaces

Y: smooth projective cubic surface in $\mathbb{P}^3$ over $\mathbb{C}$. It is known since Cayley and Salmon (1849) that Y contains 27 lines and that they are 45 triangle configurations of lines. We fix $D = D_1 \cup D_2 \cup D_3$ one of such triangle. The divisor D is anticanonical, and so the pair (Y, D) is log Calabi-Yau.



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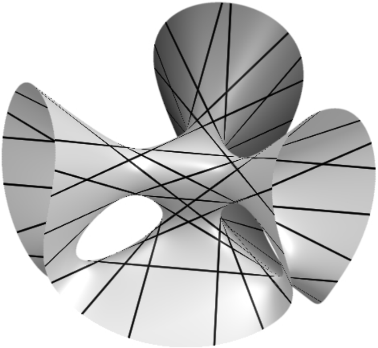
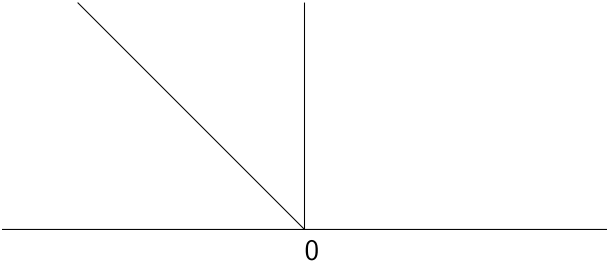


$B := \mathbb{R}^2/\langle \pm id \rangle$ is the dual intersection complex of (Y, D). The set $B(\mathbb{Z})$ of integral points define tangency conditions for curves in (Y, D).



Log Gromov-Witten invariants

- $g \geq 0$, $\beta \in NE(Y)$, $v \in B(\mathbb{Z})$: $N_{g,v}^{\beta} \in \mathbb{Q}$, log Gromov-Witten invariant counting genus g stable maps to (Y, D) of class β with contact order v with D, and insertion of $(-1)^g \lambda_g$. "counts" of genus g holomorphic curves in Y of class β intersecting D in a single point (logarithmic Gromov-Witten invariants).
- Define a scattering diagram $\mathfrak{D}_{(Y,D)}$ on B over $\mathbb{Q}[[\hbar]][NE(Y)]$ by attaching to the ray of direction (m, n) the generating series

$$f_{m,n} := \exp\left(\sum_{k\geq 1}\sum_{\beta\in NE(Y)}\sum_{g\geq 0} 2\sin\left(\frac{k\hbar}{2}\right) N_{g,k(m,n)}^{\beta} \hbar^{2g-1} t^{\beta} z^{-k(m,n)}\right).$$

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Log Gromov-Witten invariants

Theorem (B., 2018)

The scattering diagram $\mathfrak{D}_{(Y,D)}$ is consistent.

The genus 0 limit was previously known by Gross-Hacking-Keel (proof relying on the "tropical vertex" paper of Gross-Pandharipande-Siebert).

Theorem (B., 2018)

After explicit specializations and identifications of variables, we have $\mathfrak{D}_{(Y,D)} \simeq \mathfrak{D}$.

Genus 0 case treated by Gross-Hacking-Keel-Siebert (2019).

Log Gromov-Witten invariants

- $f_{1,0}$: look at curves intersecting D_1 in one point, and not intersecting D_2 and D_3. The are such $8 = (27-3)/3$ lines L_m, intersecting D_1 with contact order 1. There are also two such curves of class $D_2 + D_3$, intersecting D_1 with contact order 2 (strict transforms of the 2 conics passing through 4 given points and tangent to one given line in $\mathbb{P}^2$).

$$f_{(1,0)} =$$

$$\frac{\prod_{m=1}^{8}(1+t^{L_m}z^{-(1,0)})}{(1-A^{-4}t^{D_2+D_3}z^{-(2,0)})(1-t^{D_2+D_3}z^{-(2,0)})^2(1-A^4t^{D_2+D_3}z^{-(2,0)})}$$

where $A^4 = q = e^{i\hbar}$.

- The numerator comes from multicovers of the 8 lines L_m and the denominator comes from multicovers of the 2 curves of class $D_2 + D_3$.
- Multicover contributions of lines are "easy" (quiver moduli: point). Multicover contributions of curves of class $D_2 + D_3$ are more sublte (quiver moduli: $\mathbb{P}^1$).

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- $PSL_2(\mathbb{Z})$ of log birational automorphisms of (Y, D): S acts by cyclic permutation of D_1, D_2, D_3, and T acts by blow-up of $D_1 \cap D_2$ and blow-down of D_3.
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End

Thank you for your attention!