The Higher Dimensional Tropical Vertex

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Introduction

- (X, D) a log Calabi-Yau pair: X n-dimensional smooth projective variety over $\mathbb{C}$ and D a reduced simple normal crossings divisor in X with $K_X + D = 0$.
- $\{D_i\}_{i \in I}$ the irreducible components of D.
- Assume that D is maximally degenerate, i.e. that there exists a set $\{D_j\}_{j \in J}$ of components of D such that $\cap_{j \in J} D_j$ is non-empty and 0-dimensional.
- (Trivial) example: X a smooth projective toric variety and D the union of toric boundary divisors.
- More interesting example: start with X_Σ a smooth projective toric variety and choose general hypersurfaces $H_1, ..., H_s$ inside some toric divisors of X_Σ. Let X be the blow-up of X_Σ along $H = H_1 \cup ... \cup H_s$, and D be the strict transform of the union of toric boundary divisors of X_Σ. Then (X, D) is a maximally degenerate log Calabi-Yau pair.

Introduction

- Given (X, D) a maximally degenerate log Calabi-Yau pair, the complement U of D in X is a non-compact Calabi-Yau variety.
- SYZ mirror symmetry predicts that U should admit a Lagrangian torus fibration and that the mirror of U should be constructed by dualizing this torus fibration and by correcting the resulting complex structure using counts of Maslov index 0 holomorphic disks with boundary on the SYZ torus fibers.
- Algebro-geometric realization of this idea: replace counts of Maslov index 0 disks by counts of punctured log maps. Use these counts to construct the *canonical scattering diagram*, and reconstruct the mirror from this scattering diagram (Gross-Siebert, see Mark's talk).

Introduction

- For (X, D) toric, $U = (\mathbb{C}^*)^n$, which admits a smooth Lagrangian torus fibration. There are no Maslov index 0 holomorphic disks, the canonical scattering diagram is essentially trivial and the mirror is just the dual $(\mathbb{C}^*)^n$.
- For (X, D) obtained from a toric pair by blow-up of hypersurfaces in its toric boundary, U is no longer $(\mathbb{C}^*)^n$. The Lagrangian torus fibration necessarily has singularities, there are non-trivial Maslov index 0 disks and the canonical scattering diagram can be very non-trivial.

Theorem (A-Gross, 2007.08347)

For (X, D) obtained from a toric pair by blow-up of hypersurfaces in its toric boundary, the canonical scattering diagram admits a completely algorithmic description.

Introduction

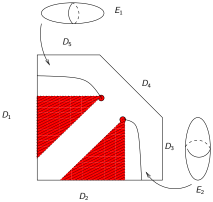
- Review of the two-dimensional case
  - The canonical scattering diagram $\mathfrak{D}$: Gross-Hacking-Keel
  - Enumerative interpretation of $\mathfrak{D}$ using log maps: Gross-Pandharipande-Siebert (The tropical vertex)
- The higher dimensional case
  - The canonical scattering diagram $\mathfrak{D}$: Gross-Siebert
  - Enumerative interpretation of $\mathfrak{D}$ using punctured log maps: A. Gross (The higher dimensional tropical vertex)

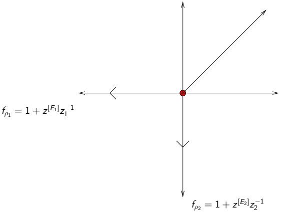
The two-dimensional case

- (X, D): log Calabi-Yau surface obtained from a toric surface $(\bar{X}, \bar{D})$ by blow-up of finitely many smooth points of $\bar{D}$.
- Example: $(\bar{X}, \bar{D})$ toric blow-up of $\mathbb{P}^2$ in two points (dP_7). Let (X, D) obtained by blowing up two smooth points on distinct components of $\bar{D}$ (dP_5).

The canonical scattering diagram $\mathfrak{D}$ attached to (X, D) is is a union of pairs (ρ, f_ρ), where

- ρ are **walls**, i.e. codimension one cells of $(B, \mathcal{P})$.
- f_ρ are **wall-crossing functions** attached to walls ρ which instruct us how to transform monomials across the wall.





Walls and wall-crossing functions

- B: dual intersection complex of (X, D), union of two-dimensional cones corresponding to the 0-dimensional strata of D. Denote by $\mathscr{P}$ the corresponding cone decomposition of B.
- Define an integral affine structure on $B - \{0\}$ from the intersection pattern of D (using as model the toric case).
- Walls of $\mathfrak{D}$ are rays in B coming out from the origin $0 \in B$.
- For every σ a cone in $\mathscr{P}$, v an integral tangent vector to σ whose direction is contained in σ, and $\beta \in NE(X)$ a curve class, let $N_{v,\beta}$ be the genus 0 log Gromov-Witten invariant counting rational curves in X of class β intersecting D at a single point with contact order v.
- Include the wall

$$(\mathbb{R}_{\geq 0} v, \exp(k N_{v,\beta} t^{\beta} z^{-v}))$$

  in the canonical scattering diagram $\mathfrak{D}$, where $k \in \mathbb{Z}_{>0}$ is the divisibility of v.
- The function attached to the wall is an element in $\mathbb{Q}[[NE(X)]][[z^{-v}]]$.

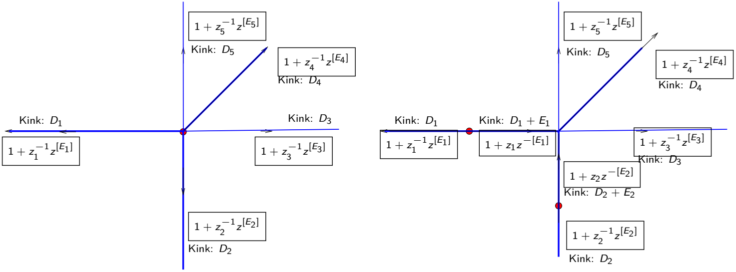


Figure: The canonical scattering diagram $\mathfrak{D}$ associated to (dP_5, D) and its reduction $\mathfrak{D}^{GS}$ to the Gross–Siebert locus

The mirror to $dP_5 \setminus D$

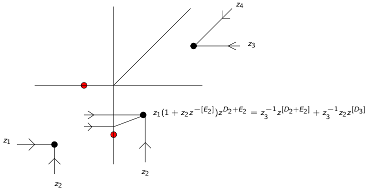


Figure: Broken lines in $\mathfrak{D}^{GS}$

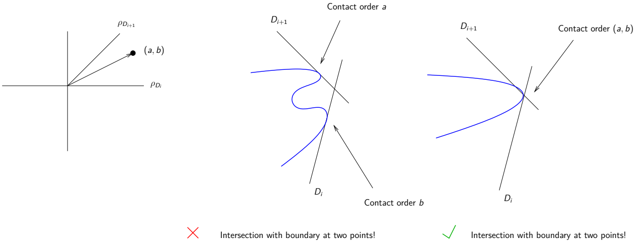
$$
\begin{aligned}
\vartheta_1 &= \ldots \\
\vartheta_2 &= z_4^{-1} z_3 (1 + z_3^{-1} z^{[E_3]}) z^{[D_3]} \\
\vartheta_3 &= z_3 \\
\vartheta_4 &= z_4 \\
\vartheta_5 &= \ldots
\end{aligned}
$$

$$
\boxed{(dP_5 \setminus D)^\vee = \operatorname{Spec} \mathbb{C}[\vartheta_1, \ldots, \vartheta_5] / (\vartheta_{i-1}\vartheta_{i+1} = z^{[D_i]}(\vartheta_i + z^{[E_i]}))\,.}
$$

The (tropical) enumerative problem in dimension two

The canonical scattering diagram $\mathfrak{D}$ is defined using enumerative information of (X, D). This **enumerative problem in dimension two** is as follows:

- Fix $\beta \in H_2(X, \mathbb{Z})$ and a two dimensional cone, with rays with primitive generators corresponding to the divisor D_i and D_{i+1} and fix $v = am_i + bm_{i+1}$ where a and b are nonnegative integers.
- Question: What is the number of genus zero curves in Y of class β which intersect D at a single point with multiplicity a, b along respectively D_i and D_{i+1}?



The two-dimensional "toric" scattering diagram

- Fix a realization of (X, D) as the blow-up of a toric surface X_Σ at a set H of finitely many smooth points of the toric boundary.
- Then we can define algorithmically, without using enumerative geometry, a scattering diagram $\mathfrak{D}_{X_\Sigma,H}$ in $\mathbb{R}^2$.
- Start with the fan Σ of X_Σ.
- For every ray ρ_i of Σ, let $m_i \in \mathbb{Z}^2$ be the primitive generator of ρ_i, and let $H_i = \{H_{ij}\} \subset H$ the set of points on the divisor D_{ρ_i} that will be blown-up to obtain X.
- Let $\mathfrak{D}^{in}_{X_\Sigma,H}$ be the scattering diagram in $\mathbb{R}^2$ consisting of the walls

$$(\rho_i, \prod_j (1 + t_{ij} z^{m_i}))\,.$$

- The scattering diagram $\mathfrak{D}^{in}_{X_\Sigma,H}$ is not consistent (the composition around $0 \in \mathbb{R}^2$ of the automorphisms defined by the walls is not the identity).

The two-dimensional "toric" scattering diagram

- Define $\mathfrak{D}_{X_\Sigma,H}$ as the minimal consistent scattering diagram containing $\mathfrak{D}^{in}_{X_\Sigma,H}$. The construction of $\mathfrak{D}_{X_\Sigma,H}$ from $\mathfrak{D}^{in}_{X_\Sigma,H}$ is entirely algebraic (Kontsevich-Soibelman, Gross-Siebert).

Theorem

The canonical scattering diagram $\mathfrak{D}$ and the "toric" scattering diagram $\mathfrak{D}_{X_\Sigma,H}$ determine each other.

- Need to translate the variables t_{ij} appearing in $\mathfrak{D}_{X_\Sigma,H}$ into curve classes on X (t_{ij} related to the exceptional divisor E_{ij} in X obtained by blowing-up the point H_{ij} on the toric divisor D_{ρ_i}).
- This theorem gives an efficient way to compute the log Gromov-Witten invariants $N_{v,\beta}$ (algebro-geometric version of counts of Maslov index 0 disks) and so the mirror geometry.

The higher dimensional case

- (X, D): n-dimensional log Calabi-Yau pair obtained from a toric variety $(\bar{X}, \bar{D})$ by blow-up of finitely many general hypersurfaces contained in the irreducible components of $\bar{D}$.
- Example: start with $\bar{X} = \mathbb{P}^3$ and $\bar{D} = \bar{D}_1 + \bar{D}_2 + \bar{D}_3 + \bar{D}_4$ its toric boundary. Take H to be the union of a line ℓ_1 in $\bar{D}_1$ and of a line ℓ_2 in $\bar{D}_2$. Assume that ℓ_1 and ℓ_2 are chosen generally, so that they do not intersect. Then $X = Bl_{\ell_1 \cup \ell_2}(\mathbb{P}^3)$ is an interesting 3-dimensional Fano variety.
- B: dual intersection of (X, D), with the cone decomposition $\mathscr{P}$, whose cones are in one-to-one correspondence with the strata of D. In particular, 0-dimensional strata of D correspond to n-dimensional cones of $\mathscr{P}$.
- Define an integral structure on B away from cones of $\mathscr{P}$ of codimension ≥ 2. Codimension 1 cones correspond to curves C_i contained in D and the integral affine structure across these codimension 1 cones is defined in terms of the intersection numbers $C_i \cdot D_j$.

The higher dimensional canonical scattering diagram

- $\mathfrak{D}$: the canonical scattering diagram attached to (X, D).
- Walls of $\mathfrak{D}$ are cones contained in codimension 1 integral affine subspaces of B.
- For every σ a cone in $\mathscr{P}$, v an integral tangent vector to σ and $\beta \in NE(X)$, consider genus 0 punctured log maps in X of class β intersecting D at a point with contact order v.
- If the direction of v is contained in σ, v only involves non-negative contact orders, and punctured log maps are just usual stable log maps in this case.
- If the direction of v is not contained in σ, then v involves negative contact orders and the full theory of punctured log maps is necessary.

The higher dimensional canonical scattering diagram

- Punctured log maps in X of class β intersecting D at a point with contact order v have tropicalizations, which are family of tropical curves in B with one leg, corresponding to the marked point where the contact condition with D is imposed.
- The $(n-1)$-dimensional walls of $\mathfrak{D}$ are formed by the union of legs of $(n-2)$-dimensional families of punctured log maps.
- More precisely, for every τ a type of tropical curve having a $(n-2)$-dimensional space of deformation, there is a corresponding moduli space of class β punctured log maps of virtual dimension 0 and we consider the corresponding invariant $N_{\tau,\beta} \in \mathbb{Q}$.
- Include the wall

$$(\mathfrak{d}_\tau, \exp(k_\tau N_{\tau,\beta} t^\beta z^{-v}))$$

in the canonical scattering diagram $\mathfrak{D}$, where k_τ is some multiplicity (tropical curves are really tropical maps and k_τ is some lattice index).

The higher dimensional canonical scattering diagram

- In dimension $n = 2$, the only type τ of tropical curves with 0-dimensional moduli space of deformation are the rays $\mathbb{R}_{\geq 0} v$ with the direction of v contained in σ. It is why the two-dimensional canonical scattering diagram only involves log (and not punctured log) invariants and so is much simpler than the higher dimensional case.
- In dimension $n \geq 3$, there can exist tropical curves with $(n-2)$-dimensional moduli space of deformation and with the direction of v not contained in σ: the corresponding leg will be bounded (either hitting the boundary of σ or another wall), corresponding to the tropicalization of a punctured point.

The higher dimensional “toric" scattering diagram

- Fix a realization of (X, D) as the blow-up of a toric variety X_Σ at a set H of finitely many general hypersurfaces in irreducible components of the toric divisor $\bar{D}$.
- Then we can define algorithmically, without using enumerative geometry, a scattering diagram $\mathfrak{D}_{X_\Sigma, H}$ in $\mathbb{R}^n$.
- Start with the fan Σ of X_Σ.
- For every ray ρ_i of Σ, let $m_i \in \mathbb{Z}^n$ be the primitive generator of ρ_i and let $H_i = \{H_{ij}\} \subset H$ be the set of connected hypersurfaces H_{ij} contained in the divisor D_{ρ_i} that will be blown-up to obtain X.
- Let $\mathfrak{D}^{in}_{X_{\Sigma,H}}$ be the scattering diagram in $\mathbb{R}^n$ consisting of the walls

$$(W_{ij}, 1 + t_{ij} z^{m_i})$$

where W_{ij} are piecewise-linear codimension 1 suspaces of $\mathbb{R}^n$ obtained by translating in the direction m_i the tropicalization of H_{ij}. In dimension $n = 3$, W_{ij} is a “widget".

Example

Let X be obtained by blowing-up $\mathbb{P}^3$ with centre two disjoint lines ℓ_1, ℓ_2 Two widgets W_1 and W_2 formed by $\mathrm{Trop}(\ell_1)$ and $\mathrm{Trop}(\ell_1)$ are obtained by moving $\mathrm{Trop}(\ell_1)$ and $\mathrm{Trop}(\ell_2)$ along the rays $e_1 = (1,0,0)$ and $e_2 = (0,1,0)$.

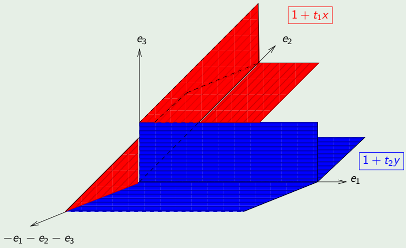


Figure: Two widgets.

The higher dimensional "toric" scattering diagram

- The scattering diagram $\mathfrak{D}^{in}_{X_\Sigma,H}$ is not consistent.
- Define $\mathfrak{D}_{X_\Sigma,H}$ as the minimal consistent scattering diagram containing $\mathfrak{D}^{in}_{X_\Sigma,H}$. The construction of $\mathfrak{D}_{X_\Sigma,H}$ from $\mathfrak{D}^{in}_{X_\Sigma,H}$ is entirely algebraic (Kontsevich-Soibelman, Gross-Siebert).

Theorem (A-Gross, 2020)

The canonical scattering diagram $\mathfrak{D}$ and the "toric" scattering diagram $\mathfrak{D}_{X_\Sigma,H}$ determine each other.

- Need to translate the variables t_{ij} appearing in $\mathfrak{D}_{X_\Sigma,H}$ into curve classes on X (t_{ij} related to the exceptional curves E_{ij} in X fibers of the exceptional divisor resulting of the blow-up of H_{ij}).
- This theorem gives an efficient way to compute the punctured log Gromov-Witten invariants $N_{\tau,\beta}$ (algebro-geometric version of counts of Maslov index 0 disks) and so the mirror geometry (explicit descriptions for some Fano 3-folds).

Idea of the proof

- Consider a degeneration over $\mathbb{A}^1$ whose general fiber is X and whose special fiber is the union of X_Σ and of blown-up $\mathbb{P}^1$-bundles.
- Consider the canonical scattering diagram attached to the total space of the degeneration: it interpolates between the canonical scattering diagram $\mathfrak{D}$ in B and the "toric" scattering diagram $\mathfrak{D}_{X_\Sigma,H}$ in $\mathbb{R}^n$ by "pushing the singularities to infinity".

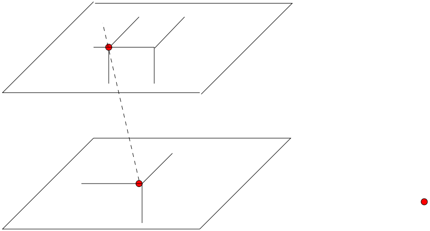
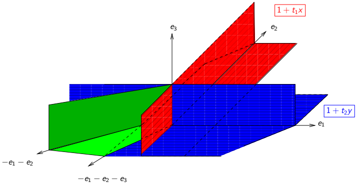


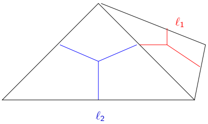
Figure: Degeneration to the normal cone.

The toric scattering diagram associated to $\mathbb{P}^3$ with two lines ℓ_1 and ℓ_2

$\mathfrak{d}$	$f_{\mathfrak{d}}$
$\langle e_1, e_2\rangle, \langle e_1, e_3\rangle, \langle e_1, e_4\rangle$	$1 + t_1 x$
$\langle e_2, e_1\rangle, \langle e_2, e_3\rangle, \langle e_2, e_4\rangle$	$1 + t_2 y$
$\langle e_3, -e_1\rangle, \langle e_4, -e_1\rangle$	$1 + t_1 x$
$\langle e_3, -e_2\rangle, \langle e_4, -e_2\rangle$	$1 + t_2 y$
$\langle -e_2, -e_1 - e_2\rangle, \langle -e_1, -e_1 - e_2\rangle, \langle e_3, -e_1 - e_2\rangle, \langle e_4, -e_1 - e_2\rangle$	$1 + t_1 t_2 xy$
$\langle e_1, -e_2\rangle$	$1 + t_2 y + t_1 t_2 xy$
$\langle e_2, -e_1\rangle$	$1 + t_1 x + t_1 t_2 xy$

Table: Walls of $\mathfrak{D}_{(\mathbb{P}^3, \ell_1 \cup \ell_2)}$





$\mathfrak{d}$	$f_{\mathfrak{d}}$
$\langle e_1, e_2\rangle, \langle e_1, e_3\rangle, \langle e_1, e_4\rangle$	$1 + t^{E_1}x^{-1}$
$\langle e_2, e_1\rangle, \langle e_2, e_3\rangle, \langle e_2, e_4\rangle$	$1 + t^{E_2}y^{-1}$
$\langle e_3, -e_1\rangle, \langle e_4, -e_1\rangle$	$1 + t^{L-E_1}x$
$\langle e_3, -e_2\rangle, \langle e_4, -e_2\rangle,$	$1 + t^{L-E_2}y$
$\langle -e_1, -e_1 - e_2\rangle, \langle -e_2, -e_1 - e_2\rangle, \langle e_3, -e_1 - e_2\rangle, \langle e_4, -e_1 - e_2\rangle$	$1 + t^{L-E_1-E_2}xy$
$\langle e_1, -e_2\rangle$	$1 + t^{L-E_2}y + t^{L-E_1-E_2}xy$
$\langle e_2, -e_1\rangle$	$1 + t^{L-E_1}x + t^{L-E_1-E_2}xy$

Table: Walls of $\mathfrak{D}_{\mathrm{Bl}_{\ell_1 \cup \ell_2}(\mathbb{P}^3), \tilde{D})}$

Here L is the strict transform of a general line in 3, and E_i are exceptional fibers over general points on ℓ_i. Recall from Mark's talk these walls of the canonical scattering diagram are of the form

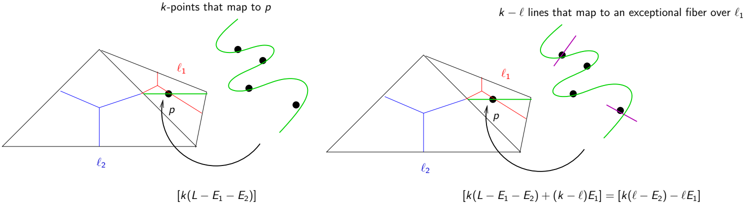
$$\big(h(\tau_{\mathrm{out}}), \exp(k_\tau N_{\tilde{\tau}} t^{\beta} z^{-u})\big).$$

where β is a curve class in $\mathrm{Bl}_{\ell_1 \cup \ell_2}(\mathbb{P}^3)$ and $\tilde{\tau} = (\tau, \underline{\beta})$ records the type τ of a punctured map $f : C^\circ/W \to \mathrm{Bl}_{\ell_1 \cup \ell_2}(\mathbb{P}^3)$, with one punctured point, and $N_{\tilde{\tau}}$ is the corresponding punctured invariant, and k_τ is a constant lattice index.

For $(\mathfrak{d}, f_{\mathfrak{d}}) = (\langle e_1, -e_2\rangle, 1 + t^{L-E_2}y + t^{L-E_1-E_2}xy)$ we have

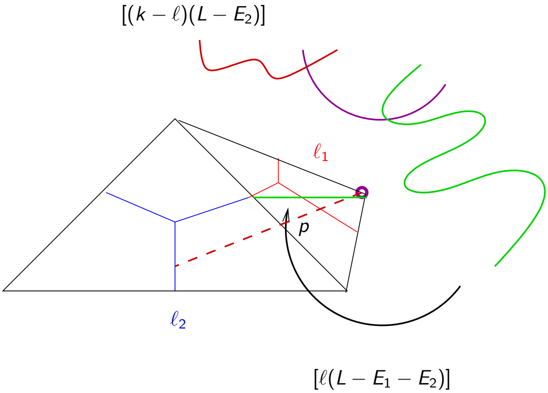
$$\log(f_{\mathfrak{d}}) = \sum_{k\geq 1}\sum_{\ell=0}^{k} k\frac{(-1)^{k+1}}{k^2}\binom{k}{\ell} t^{k(L-E_2)-\ell E_1}x^{\ell}y^{k}.$$

There are two one-dimensional families of punctured maps of class $k(L - E_2 - \ell E_1)$. The first family:



Example

The second family of punctured maps of class $k(L - E_1 - E_2) + (k - \ell)E_1$.



The two tropical families.

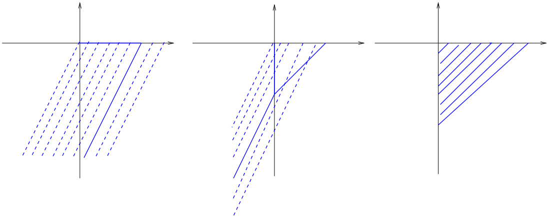


Figure: The two tropical families and their cancellation with support $\langle e_1, -e_2 \rangle$