Genericity vs Symmetry

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Perturbing the moduli spaces (of pseudo holomorphic curves) while keeping symmetry as much as possible.
The kinds of symmetry we want to study.

1) Exterior group action. 
\[ G \subseteq (X, L) \]

- \( G \) is a compact Lie group
- \( X \) is a symplectic manifold
- \( L \) is a Lagrangian submanifold

2) Forgetful map
Cyclic symmetry
Two different ways to study these problems.

1) Homotopy method

Replace exact symmetry by ‘symmetry up to (infinity) homotopy’ and work (hard) in homological algebra.

2) Direct Geometric method

Work (hard) to achieve symmetry as much as possible.
1) Exterior group action.

\[ G \subseteq (X, L) \]

A possible goal:
Obtain an equivariant Lagrangian Floer homology
and an equivariant A infinity category

\[ HF_G(L) \]
\[ HF_G(L_1, L_2) \]
1) Homotopy method

(Seidel-Smith, Hendricks-Lipschitz-Sarkar)

$L_1, L_2 \subset X$

data to perturb the moduli space to define boundary operator: Hamiltonian, almost complex structure etc.

$\Xi$ is not $G$ equivariant.

$g \in G$

$CF(L_1, L_2; \Xi) \xrightarrow{g^*} CF(L_1, L_2; g\Xi)$

$\varphi_g \xRightarrow[\text{homotopy equivalence}]{\sim} CF(L_1, L_2; \Xi)$

Floer’s chain complex

$CF(L_1, L_2; \Xi)$
\[ g \mapsto \varphi_g \text{ is not a } G \text{ action.} \quad \varphi_{g_1} \varphi_{g_2} \neq \varphi_{g_1 g_2} \]

However there exists a chain homotopy

\[ h_{g_1, g_2} \quad \varphi_{g_1} \varphi_{g_2} \sim \varphi_{g_1 g_2} \]

\[ g_1, g_2, g_3 \in G \]

\[ \varphi_{g_1} \varphi_{g_2} \varphi_{g_3} \quad h_{g_1, g_2} \varphi_{g_3} \quad \varphi_{g_1} h_{g_2, g_3} \quad \varphi_{g_1} \varphi_{g_2 g_3} \quad H_{g_1, g_2, g_3} \quad h_{g_1, g_2 g_3} \quad \varphi_{g_1 g_2 g_3} \]

\[ CF(L_1, L_2; \Xi) \]
Go up to infinity homotopy.
We can replace by ‘this infinite homotopy action’ the actual $G$ action.

This kinds of idea is worked out in case when Lagrangian submanifold satisfies (rather restrictive) assumption of monotonicity.

Maybe using VFC it works in complete generality.

Con: The homological algebra becomes more and more complicated.
2) Direct geometric method.

\[ \mathcal{M}_{k+1}(L; \beta) \]

has natural \( G \) action.

Can we perturb this space in a \( G \) equivariant way?

If we can perform ‘finite dimensional reduction’ in a \( G \)-equivariant way, then we can use various techniques of equivariant cohomology to achieve our goal.
Finite dimensional reduction: \( \mathcal{M}_{k+1}(L; \beta) \)

Replace \( \bar{\partial}u = 0 \) by \( \bar{\partial}u \equiv 0 \mod E(u) \), a finite dimensional subspace of \( C^\infty(\Sigma, u^*TX) \).

A difficulty \((\Sigma, u)\) is stable but \(\Sigma\) may be unstable.

Need to kill \(\text{Aut}(\Sigma)\) to obtain \(E(u)\).
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Usual method: Add (marked) points so that $\Sigma$ becomes stable.

This breaks symmetry. ($G$ equivariance.)

Solution:

Define universal deformation of unstable curve using differential Geometric Analogue of Artin Stack.
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Define universal deformation of unstable curve using differential Geometric Analogue of Artin Stack.

By this method I mostly finish writing the detailed construction of equivariant A infinity category.
(20-30 pages are to be written. 120 pages of the first paper are done (3 years ago) and 80 pages are written for the second.)
2) Forgetful map

\[ \mathcal{M}_{k+1}(L; \beta) \]

\[ \beta \in H_2(X, L) \]

\[ u : (\Sigma, \partial \Sigma) \to (X, L) \]

Forgetful map: \( \mathcal{M}_{k+1}(L; \beta) \to \mathcal{M}_k(L; \beta) \)

\[ ((\Sigma, (z_0, \ldots, z_k), u) \mapsto ((\Sigma, (z_0, \ldots, z_{k-1}), u) \]
We want to perturb $\mathcal{M}_{k+1}(L; \beta)$ $\mathcal{M}_k(L; \beta)$ so that it is compatible with $\mathcal{M}_{k+1}(L; \beta) \rightarrow \mathcal{M}_k(L; \beta)$

Why

1) Unitality
2) Divisor axiom
1) **Unitality**

A infinity algebra or category is unital if there exists $e$ such that

$$m_{k+1}(x_1, \ldots, e, \ldots, x_k) = 0$$

except $\pm m_2(x, e) = \pm m_2(e, x) = x$

Useful for

a) Weak bounding cochain

b) Yoneda embedding
a) Weak bounding cochain

\[ b \in CF^{\text{odd}}(L) \] is a bounding cochain (or Maurer Cartan element)

if \[ \sum_{k} m_{k}(b, \ldots, b) = 0 \]

Weak bounding cochain relax this condition to

\[ \sum_{k} m_{k}(b, \ldots, b) = ce \]

Important in case Mirror is LG model.

In fact \( c = W(b) \) is the LG potential
b) Yoneda embedding

An A infinity category $\mathcal{C}$

Yoneda embedding functor

Category of A infinity modules over $\mathcal{C}^{\text{op}}$

We need unit to prove that this is an embedding (up to homotopy equivalence.)
2) Divisor axiom

\[ b \in H^1(L) \]

\[ m_{k+1,\beta}(b, x_1, \ldots, x_k) + \cdots + m_{k+1,\beta}(x_1, b, x_2, \ldots, x_k) \]

\[ + \cdots + m_{k+1,\beta}(x_1, x_2, \ldots, x_k, b) \]

\[ = (\partial \beta \cap b)m_{k,\beta}(x_1, \ldots, x_k) \]

\[ m_{k,\beta}(x_1, \ldots, x_k) \] is the contribution of phc of homology class

\[ \beta \in H_2(X; L) \]
Divisor axiom is important to study family Floer homology and relation to rigid analytic geometry.
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\[
\text{Tate torus } \quad \Lambda_*/\mathbb{Z} \quad \Lambda \sim \mathbb{C}[T^{-1}, T]] \\
\Lambda_* = \Lambda \setminus \{0\} \\
y \mapsto Ty \quad \mathbb{Z} \text{ action} \\
\frac{\Lambda}{\mathbb{Z} \oplus \mathbb{Z}} \quad \Lambda_*/\mathbb{Z} \\
x \quad y = e^x \\
y \sim Ty \quad \leftrightarrow \quad x \sim x + \log T \quad \text{However } \log T \notin \Lambda
The coordinate change \( y = e^x \)

is important in Family Floer homology

\( L \subset X \)  \hspace{1cm} \text{Lagrangian submanifold}

\( H^1(L) \)  \hspace{1cm} \text{is the deformation parameter}

\[
H^1(L) \quad \quad \quad \exp H^1(L) \\
(x_1, \ldots, x_m) \quad \quad \quad \quad \quad (y_1, \ldots, y_m) = (e^{x_1}, \ldots, e^{x_m})
\]

\( m = \text{rank} H^1(L) \)
\[ H^1(L) \quad \text{exp} \quad H^1(L) \]

\[ (x_1, \ldots, x_m) \quad \mapsto \quad (y_1, \ldots, y_m) = (e^{x_1}, \ldots, e^{x_m}) \]

This change of variable is important to study family Floer homology

\[ (a_1, \ldots, a_m) \in \mathbb{R}^n \cong H^1(L; \mathbb{R}) \]

\[ L(a_1, \ldots, a_m) \quad \text{deformation of} \quad L \quad \text{by} \quad (a_1, \ldots, a_m) \]

This corresponds to \( y_i \mapsto T^{a_i} y_i \) in family Floer homology.

\[ x_i \mapsto x_i + a_i \log T \quad \text{However} \quad \log T \notin \Lambda \]
\[ H^1(L) \quad \text{exp} \quad H^1(L) \]
\[
(x_1, \ldots, x_m) \quad \mapsto \quad (y_1, \ldots, y_m) = (e^{x_1}, \ldots, e^{x_m})
\]

This change of variable is possible when divisor axiom is satisfied.

\[
m^b_k(z_1, \ldots, z_k) = m(e^b, z_1, e^b, \ldots, e^b, z_k, e^b) \quad \text{(definition)}
\]

If \( b \in H^1(L) \) and if divisor axiom is satisfied

\[
m_{*,\beta}^b(e^b, z_1, e^b, \ldots, e^b, z_k, e^b) = e^{\partial \beta \cap b} m_k,\beta(z_1, \ldots, z_k)
\]

\( b = (x_1, \ldots, x_m) \quad y_i = e^{x_i} \quad \text{and} \quad \partial \beta = (\beta_1, \ldots, \beta_m) \in H^1(L) \equiv \mathbb{Z}^m
\]

\[ e^{\partial \beta \cap b} = y_1^{\beta_1} \ldots y_m^{\beta_m} \]
3) Cyclic symmetry
4) Symmetry between input and output.
3) Cyclic symmetry

\[ \langle \cdot, \cdot \rangle \quad \text{Poincare duality on } L \]

Cyclic symmetry

\[ \langle m_k(x_1, \ldots, x_k), x_0 \rangle = \pm \langle m_k(x_0, \ldots, x_{k-1}), x_k \rangle \]

Geometrically this follows from the cyclic permutation of the marked points.
4) Symmetry between input and output.

\[ q : H^*(X) \rightarrow HH^*(HF(L)) \quad \text{closed open map.} \]

\[ p : HH_*(HF(L)) \rightarrow H_*(X) \quad \text{open closed map.} \]

\[ HH_*(HF(L)) \leftrightarrow HH^*(HF(L)) \quad \text{dual via Poincare duality of } L \]

\[ H_*(X) \leftrightarrow H^*(X) \quad \text{dual via Poincare duality of } X \]

Closed open map is dual to open closed map by these duality
4) Symmetry between input and output. Geometrically trivial but nontrivial to prove.
Geometric method

We can realize all of

1) Unitality
2) Divisor axiom
3) Cyclic symmetry
4) Closed open - Open closed duality

in de-Rham model
in case we have single embedded Lagrangian $L$. 
Homotopy method

Unitality ➔ homotopy unitality

OK for Yoneda lemma
weak bounding chain
it might work but story becomes a bit complicated when we work with homotopy unit

Divisor axiom ➔ ? (I will go back.)

Cyclic symmetry ➔ Calabi-Yau category

CO-OC duality ➔ ?
To realize this kinds of symmetry by homotopy method it seems useful to lift the story to loop space

\[ \mathcal{L}(L) \quad \text{free loop space of} \quad L \]

\[
\begin{array}{ccc}
C(\mathcal{L}(L)) & \xrightarrow{\Psi} & CH(C(L), C(L)) \\
\text{chain model of homology of} & & \text{Hochschild cochain complex} \\
\mathcal{L}(L) & & \\
\end{array}
\]

if we can lift here divisor axiom and unitarity is automatic
\[ C(\mathcal{L}(L)) \rightarrow CH(C(L), C(L)) \]

Chain model of homology of \( \mathcal{L}(L) \)

Hochschild cochain complex

\[ \Psi \]

\[ \mathcal{M}_{k+1}(L; \beta) \]

If we can lift here divisor axiom and unitarity is automatic

It seems likely we can do it using Irie's model of loop space homology.

Cyclic symmetry we need to use equivariant loop space homology

I think appropriate chain model for this purpose is not known yet.
Going back to Geometric method

In case $L$ is immersed or several $L$’s (category case), it seems impossible to realize all of

1) Unitality
2) Divisor axiom
3) Cyclic symmetry
4) Closed open - Open closed duality

as exact symmetry at the same time. (Lino Amorin)

1) + 2) or 3) + 4) is possible.
In case $L$ is immersed or several $L$’s (category case), it seems impossible to realize all of 4 symmetry at the same time.

The reason: Constant map!

$$\beta = \beta_0 = 0 \in H_2(X, L)$$

$$\mathcal{M}_{2+1}(L; \beta_0) = L$$ is transversal and

$$\text{ev}_0 : \mathcal{M}_{2+1}(L; \beta_0) = L \to L$$ is a submersion.

This is the Key point to realize 4 symmetry at the same time.
In case \( L \) is immersed or several \( L \)'s (category case)

\[
\begin{align*}
\text{This moduli is one point and transversal.} \\
\text{But} \\
\text{is not a submersion.}
\end{align*}
\]
The problem is very much similar to a problem to perturb moduli space of bordered curve of higher genus.

Constant map is NOT transversal (obstructed)!

$H^1$ is non-zero.
Solution:

Distinguish forgetful and unforgetful marked points.

This looks very much technical but I do not know other way to resolve this problem at this point.
Note:

It might be **impossible** to resolve the problem about higher genus bordered curve by **homotopy method**.

The count of constant maps from higher genus bordered curve should give perturbative Chern-Simons invariant, that is **not** invariant of homotopy type.

Note chain level Poincare duality seems to be beyond algebraic topology but is related differential topology such as surgery theory.