Polytopes, wall crossings, and cluster varieties

Man Wai, Mandy, Cheung
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Current Advances in Mirror Symmetry, 2020
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Today: how to describe the compactification
Gross-Hacking-Keel mirror

\((X, D)\) log Calabi-Yau pair, where \(X\) is a smooth complex projective surface and \(D \in | - K_X|\) reduced, normal crossing, singular, cycle of rational curves.
Gross-Hacking-Keel mirror

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$\sim$ tropicalization (dual intersection complex) $(B, \Sigma)$, where $B$ is an integral affine manifold with singularities, and $\Sigma$ cone decomposition of $B$. 
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⇝ tropicalization (dual intersection complex) (B, Σ), where B is an integral affine manifold with singularities, and Σ cone decomposition of B.
Canonical scattering diagram (dim 2)

Wall $(\mathfrak{d}, f_{\mathfrak{d}})$, where

- $\mathfrak{d} \subset B$ is a ray enumerating from the origin with rational slope $v$.
- $f_{\mathfrak{d}} = 1 + \sum_{k \geq 1} a_k z^{kv}$, $a_k$ come from some curve counting invariants.

[ Gross-Hacking-Keel ] The canonical scattering diagrams are consistent.
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Associate each chambers with an algebraic torus $\mathbb{G}_m^2$ and glue the tori by the wall crossing

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Define global $\theta_q \in \Gamma(X^0_D, \mathcal{O}_{X^0_D})$
GHK mirror construction

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$\sim \ X^o_{\mathcal{D}}$

Define global $\theta_q \in \Gamma(X^o_{\mathcal{D}}, \mathcal{O}_{X^o_{\mathcal{D}}})$

Construct partial compactification $X_{\mathcal{D}} = \text{Spec} \ \Gamma(X^o_{\mathcal{D}}, \mathcal{O}_{X^o_{\mathcal{D}}})$
<table>
<thead>
<tr>
<th>family Floer SYZ</th>
<th>Gross-Hacking-Keel-Siebert mirror</th>
</tr>
</thead>
<tbody>
<tr>
<td>large complex structure limit</td>
<td>toric degeneration</td>
</tr>
<tr>
<td>base of SYZ fibration with complex affine structure</td>
<td>dual intersection complex of the toric degeneration</td>
</tr>
<tr>
<td>loci of SYZ fibres bounding holomorphic discs</td>
<td>rays in scattering diagram</td>
</tr>
<tr>
<td>homology of boundary of a holomorphic disc</td>
<td>direction of the ray</td>
</tr>
<tr>
<td>exp of generating function of open Gromov-Witten invariants of Maslov index zero</td>
<td>slab functions attached to the ray</td>
</tr>
<tr>
<td>isomorphisms of Maurer-Cartan spaces induced by pseudo isotopies</td>
<td>wall crossing transformation</td>
</tr>
<tr>
<td>family Floer mirror</td>
<td>GS/GHK mirror</td>
</tr>
<tr>
<td>family Floer mirror</td>
<td>GHKS mirror</td>
</tr>
<tr>
<td>---------------------</td>
<td>-------------</td>
</tr>
<tr>
<td>Tate algebra</td>
<td>$\mathbb{C}[L]$</td>
</tr>
<tr>
<td>rational domain</td>
<td>$\mathbb{G}_m^n$</td>
</tr>
<tr>
<td>analytic torus $\mathbb{G}_\text{an}^n$</td>
<td></td>
</tr>
<tr>
<td>gluing (Wall crossing and GAGA)</td>
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</tr>
</tbody>
</table>
The family Floer mirror of the hyperKahler rotation of complement of a II*, III*, IV* fibres in a rational elliptic surfaces have the compactifications which are the analytification of dP5 and dP6, and cluster varieties of type $G_2$ respectively.
Note that the resulting schemes are not compact, e.g. $\mathbb{C}^*$, so we want to compactify it

$$\mathbb{C}^* \subset \mathbb{C} \subset \mathbb{CP}^1.$$
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**Projective toric varieties:**

Example: \( \mathbb{CP}^2 \)

Homogeneous coordinate ring of \( \mathbb{CP}^2 \) is the graded ring \( \mathbb{C}[z_0, z_1, z_2] \).

The grading can be described by a convex polytope.
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$$\mathbb{C}^* \subset \mathbb{C} \subset \mathbb{C}\mathbb{P}^1.$$ 

**Projective toric varieties:**

Example: $\mathbb{C}\mathbb{P}^2$

Homogeneous coordinate ring of $\mathbb{C}\mathbb{P}^2$ is the graded ring $\mathbb{C}[z_0, z_1, z_2]$. The grading can be described by a convex polytope.
Compactification

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Projective toric varieties:

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Projective toric varieties
Projective toric varieties
Polytope construction:
Consider a convex lattice polytope $\Delta$ in $\mathbb{R}^n$.

\[ S_\Delta = \langle z^m \rangle_{m \in k\Delta}. \]

Projective toric geometry $\mathbb{P}_\Delta = \text{Proj}(S_\Delta)$. 
Cluster varieties can be defined by gluing tori by birational maps given by cluster transformation, e.g. \( x \mapsto x(1 + y) \).
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The gluing can be described by (cluster) scattering diagrams.
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The gluing can be described by (cluster) \textbf{scattering diagrams}.

Fix a lattice \( N \cong \mathbb{Z}^n, M = \text{Hom}_\mathbb{Z}(M, \mathbb{Z}) \). \( N_\mathbb{R} = N \otimes \mathbb{R}, M_\mathbb{R} = M \otimes \mathbb{R} \).

\textbf{Cluster scattering diagram} \( \mathcal{D} = \) collection of walls with finiteness and consistent condition

\textbf{Wall} : \((\mathcal{d}, f_\mathcal{d})\)

- \( \mathcal{d} \subseteq M_\mathbb{R} \) support of walls - convex rational polyhedral cone of codim 1, contained in \( n^\perp \in N \).
- \( f_\mathcal{d} = 1 + \sum c_k z^{k\mathcal{v}}, \text{ where } \mathcal{v} \in n^\perp \).
Example: $A_2$
Path-ordered product (wall crossing transformation):
Consider a path $\gamma$ passing a wall $\mathcal{d}$, we define a map

$$p_\gamma : z^m \mapsto z^m f^\pm_{\mathcal{d}} \langle n_0, m \rangle,$$

where $n_0$ is the primitive normal of the wall $\mathcal{d}$. 
Path-ordered product (wall crossing transformation):
Consider a path $\gamma$ passing a wall $\partial$, we define a map

$$p_{\gamma} : z^m \mapsto z^m f_\partial^{\pm \langle n_0, m \rangle},$$

where $n_0$ is the primitive normal of the wall $\partial$.

\[
\begin{align*}
1 + z^{(0,1)} & = 1 + A_2 \\
1 + z^{(-1,0)} & = 1 + A_1^{-1} \\
1 + z^{(-1,1)} & = 1 + A_1^{-1}A_2 \\
\end{align*}
\]

$$z^{(-1,0)} \mapsto z^{(-1,0)}(1 + z^{(0,1)}).$$
Scattering diagram as fan
Scattering diagram as fan

$f_0 \rightsquigarrow \text{wall crossing} \rightsquigarrow \text{gluing the } \mathcal{G}_m^2 \text{'s.}$

$\rightsquigarrow \mathcal{A}-\text{cluster variety of type } A_2$

Similar construction hold for general setting
Theta functions

Motivating example - $(\mathbb{C}^*)^2: H^0((\mathbb{C}^*)^2, \mathcal{O}) = \bigoplus_{m_1, m_2 \in \mathbb{Z}} \mathbb{C} z_1^{m_1} z_2^{m_2}$. 
Theta functions

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• To each point $m \in M^\circ \setminus \{0\}$, associate a **theta function** $\vartheta_m$
Theta functions

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• To each point \(m \in M^0 \setminus \{0\}\), associate a **theta function** \(\vartheta_m\).
• Theta function \(\vartheta_m\) is defined from a collection of **broken lines** with initial slope \(m\) and endpoint \(Q\) (in the positive chamber).
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Example: initial slope \((-1, 0)\) (← go opposite direction!!):

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\vartheta_{Q,(-1,0)} = z^{(-1,0)} + z^{(-1,1)}.
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- To each point $m \in M^o \setminus \{0\}$, associate a **theta function** $\vartheta_m$
- Theta function $\vartheta_m$ is defined from a collection of *broken lines* with initial slope $m$ and endpoint $Q$ (in the positive chamber)

Example: initial slope $(-1, 0)$ (← go opposite direction!!):

![Diagram showing theta function example](image)
Theta functions

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[Gross-Hacking-Keel-Konsevich]

\[ \vartheta_p \cdot \vartheta_q = \sum_{r \in L} \alpha^r_{pq} \vartheta_r, \]

where \( L = M^\circ \) or \( N \), \( \alpha^r_{pq} \) structure constant.

\* gives **algebra structure** to the vector space generated by theta functions.
Algebra structure

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where \( L = M^o \) or \( N \), \( \alpha_{pq}^r \) structure constant.

\( \star \) gives algebra structure to the vector space generated by theta functions.

\( \alpha_{pq}^r \) are expressed in terms of broken lines:

\[ \alpha_{pq}^r := \sum c(\gamma^{(1)}) c(\gamma^{(2)}), \]

where summing over pairs of broken lines \( (\gamma^{(1)}, \gamma^{(2)}) \) such that \( l(\gamma^{(1)}) = p, \ l(\gamma^{(2)}) = q, \ \gamma^{(1)}(0) = \gamma^{(2)}(0) = r, \ F(\gamma^{(1)}) + F(\gamma^{(2)}) = r \)
Example:

\[ \vartheta_{(-1,0)} \cdot \vartheta_{(2,1)} = \vartheta_{(1,1)} + \vartheta_{(1,2)}. \]
## Toric v.s. Cluster

<table>
<thead>
<tr>
<th>Toric</th>
<th>Cluster</th>
</tr>
</thead>
<tbody>
<tr>
<td>fan</td>
<td>scattering diagram</td>
</tr>
<tr>
<td>toric monomials</td>
<td>theta functions</td>
</tr>
<tr>
<td>convex polytope</td>
<td>positive polytope</td>
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\( \vartheta_p \cdot \vartheta_q = \sum_{r \in L} \alpha^r_{pq} \vartheta_r, \)

**Definition**
A closed subset \( S \subseteq L_{\mathbb{R}} \) is positive if
for every \( a, b \in \mathbb{Z}_{\geq 0}, p \in aS(\mathbb{Z}), q \in bS(\mathbb{Z}), \) and \( r \in L \) with \( \alpha^r_{pq} \neq 0, \)
\( \Rightarrow r \in (a + b)S. \)

Notation: \( L = M^\circ \) or \( N, dS(\mathbb{Z}) \) is the cone of \( S \) at the ‘d’th-level.
Positive polytope

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<td>line</td>
<td>broken line</td>
</tr>
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</tr>
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Definition (C-Magee-Nájera Chávez)
A closed subset $S$ is called *broken line convex* if for any $x, y \in S(\mathbb{Q})$, every broken line segment connecting $x$ and $y$ is entirely contained in $S$. 
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Idea: The structure constant $\alpha_{pq}^r$ in GHKK were expressed as two broken lines with initial slope $p$ and $q$.

$\star$ [C-Magee-Nájera Chávez] construct the correspondence of those two broken lines with broken line segments with (scaling of) the endpoints $p, q$ and $r$. 
Compactification

Result:

⇝ get graded ring $R$

⇝ get compactification $\text{Proj}R$
Compactification

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Example:

Type $A_2$:

[Gross-Hacking-Keel-Kontsevich ] del Pezzo surface of degree 5
Compactification

Type $B_2$:

[C-Magee] del Pezzo surface of degree 6
Compactification

Type $B_2$:

[C-Magee] del Pezzo surface of degree 6

Type $G_2$

**non-integral** point coming from bending of broken line!
Any evidence?  
Why we care?
Grassmannian $\text{Gr}(k, n)$ is the space that parameterizes all $k$-dimensional linear subspaces of the $n$-dimensional vector space $\mathbb{C}^n$.

[Scott] Coordinate rings of (affine) Grassmannians carry cluster structure.
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[ Rietsch-Williams] for Grassmannian $\text{Gr}_k(\mathbb{C}^n)$

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\[
\text{Newton Okounkov body} \quad \rightarrow \quad \text{Tropicalize the superpotential of } A
\]

[Bossinger-C-Magee-Nájera Chávez] = positive polytopes

[Rietsch-Williams] NO body is a rational polytope.
Non-integral example from NO body calculation: $\text{Gr}_3(\mathbb{C}^6)$.

[Bossinger-C-Magee-Nájera Chávez] Get the non-integral point from broken line convexity!
Non-integral example from NO body calculation: $\Gr_3(\mathbb{C}^6)$.

[Bossinger-C-Magee-Nájera Chávez] Get the non-integral point from broken line convexity!

**Figure 1:** Part of the scattering diagram of $\Gr_3(\mathbb{C}^6)$.

$$\frac{\nu(f)}{2} = \left( \frac{1}{2}, 1, \frac{3}{2}, 1, 1, \frac{1}{2}, 1, 1, \frac{3}{2}, 1 \right)$$
[Kaveh-Khovanskii, Lazarsfeld-Mustata, Rietsch-Williams]

\( X = \text{Gr}_{n-k}(\mathbb{C}^n) \), with anticanonical divisor \( D_{ac} = D_1 + \cdots + D_n \).

\( X^\circ = X \setminus D_{ac} \).

Consider ample divisor \( D = r_1D_1 + \ldots + r_nD_n \), and the valuation \( \text{val} : \mathbb{C}(X) \setminus \{0\} \rightarrow \mathbb{Z}^K \).

Then the NO body for the divisor \( D \) and \( \text{val} \) is

\[
\Delta(D) = \text{ConvexHull} \left( \bigcup_r \frac{1}{r} \text{val}(H^0(X, \mathcal{O}(rD))) \right)
\]
Intrinsic NO body

Define a map $\nu_\theta$: $\nu_\theta(\vartheta_p) = p$, where $p \in L$ (set of tropical points)
Intrinsic NO body

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**Definition (Bossinger-C-Magee-Nájera Chávez)**
Given a regular function $f = \sum_{p \in L} a_p \vartheta_p$, we define the $\vartheta$-function analogue of the Newton polytope of $f$ to be

$$\text{Newt}_\vartheta(f) := \text{conv}_{BL} \{ p \in L : a_p \neq 0 \}.$$
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Theorem (Bossinger-C-Magee-Nájera Chávez)

\[ \Delta_{BL}(L) = \text{conv}_{BL} \left( \bigcup_{j \geq 1} \left( \bigcup_{f \in R_j(L)} \frac{1}{j} \text{Newt}_\vartheta(f) \right) \right) \]
Theorem (Bossinger-C-Magee-Nájera Chávez)

\[ \Delta_{BL}(\mathcal{L}) = \text{conv}_{BL} \left( \bigcup_{j \geq 1} \left( \bigcup_{f \in R_j(\mathcal{L})} \frac{1}{j} \text{Newt}_{\vartheta}(f) \right) \right) \]

Grassmannian: Let \( \mathcal{L} \rightarrow \text{Gr}_k(\mathbb{C}^n) \) be the pullback of \( \mathcal{O}(1) \) under the Plücker embedding. (By definition of this embedding, the Plücker coordinates are a basis for \( \Gamma(\text{Gr}_k(\mathbb{C}^n), \mathcal{L}). \))

Theorem (Bossinger-C-Magee-Nájera Chávez)

\[ \Delta_{BL}(\mathcal{L}) = \text{conv}_{BL} \left( \left\{ \vartheta(p_J) : J \in \binom{[n]}{k} \right\} \right) \]
**Landau Ginzburg mirror**

[C-Magee]

<table>
<thead>
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</tr>
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<tbody>
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<td>$D_{n_i} \sim z^{n_i}$ Landau Ginzburg mirror $W = \sum_i z^{n_i} : T^\vee \to \mathbb{C}$</td>
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<td></td>
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<td>$Y := \text{TV} (\text{Newt}(W))$</td>
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**Landau Ginzburg mirror**

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<td>$Y := \text{TV}(\text{Newt}(W))$</td>
<td>Newt$_g(W) := \text{conv}(v_i)$</td>
</tr>
<tr>
<td>Sections of $\mathcal{O}_X(D)$ and $\mathcal{O}_Y(D')$ are mirror</td>
<td>?</td>
</tr>
</tbody>
</table>
Mutation of polytopes

Cluster mutation of (X-)scattering diagram / polytope

The underlying scheme is not changing
Another mutation

Scattering diagram with monodromy

$$\begin{align*}
1 + Z^{e_1} &\quad \rightarrow &\quad 1 + Z^{e_2} \\
1 + Z^{e_1+e_2} &\quad \downarrow &\quad \leftarrow &\quad 1 + Z^{e_2}
\end{align*}$$

$$\begin{pmatrix}
(1,0) \\
(1,1) \\
(0,1)
\end{pmatrix}$$
Another mutation

Scattering diagram with monodromy

The monodromy: \((1, 0) \mapsto (1, 1), (0, 1) \mapsto (0, 1)\).
Another mutation

Mutation of polytope
Another mutation

Mutation of polytope
Mutation cycle

[C-Vianna] same as symplectic mutation compactification: singular Lagrangian fibration (almost toric fibration)
[C-Vianna] same as symplectic mutation
compactification: singular Lagrangian fibration (almost toric fibration)
Thank you!