Polytopes, wall crossings, and cluster varieties

Man Wai, Mandy, Cheung

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Current Advances in Mirror Symmetry, 2020

Last episode

Previously in the ”Floer-theoretic and algebro-geometric aspects of SYZ mirror symmetry” conference

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Today: how to describe the compactifictaion

Gross-Hacking-Keel mirror

(X, D) log Calabi-Yau pair, where X is a smooth complex projective surface and $D \in |-K_X|$ reduced, normal crossing, singular, cycle of rational curves.

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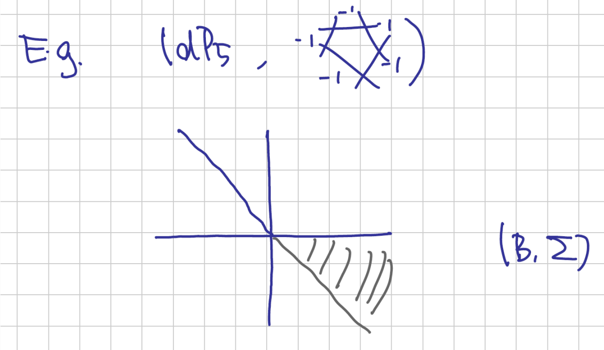

Canonical scattering diagram ($\dim 2$)

Wall $(\mathfrak{d}, f_{\mathfrak{d}})$, where

- $\mathfrak{d} \subset B$ is a ray enumerating from the origin with rational slope v.
- $f_{\mathfrak{d}} = 1 + \sum_{k \geq 1} a_k z^{kv}$, a_k come from some curve counting invariants.

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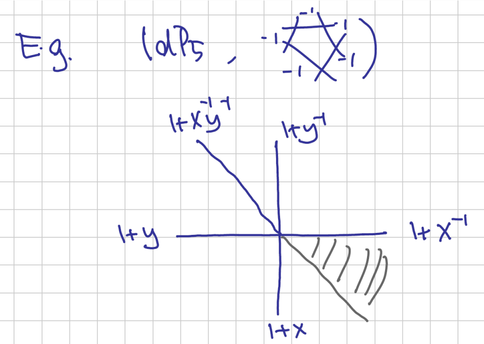
[Gross-Hacking-Keel] The canonical scattering diagrams are consistent.



GHK mirror construction

Associate each chambers with an algebraic torus $\mathbb{G}_m^2$ and glue the tori by the wall crossing

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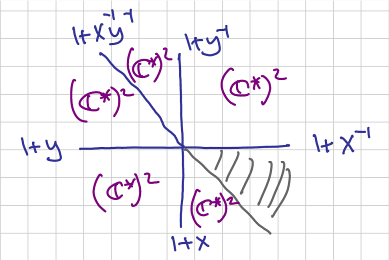

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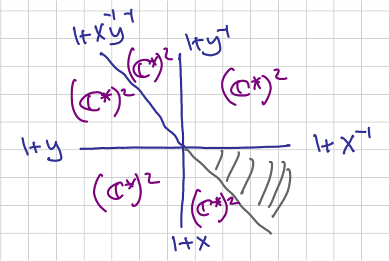

Define global $\theta_q \in \Gamma(X_{\mathfrak{D}}^{\circ}, \mathcal{O}_{X_{\mathfrak{D}}^{\circ}})$

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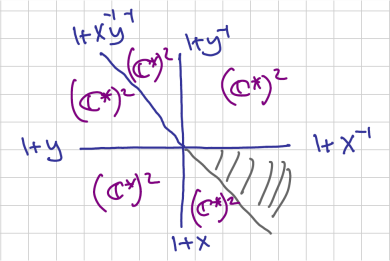

Define global $\theta_q \in \Gamma(X^\circ_{\mathfrak{D}}, \mathcal{O}_{X^\circ_{\mathfrak{D}}})$

Construct partial compactificationion $X_{\mathfrak{D}} = \operatorname{Spec} \Gamma(X^\circ_{\mathfrak{D}}, \mathcal{O}_{X^\circ_{\mathfrak{D}}})$

family Floer mirror

[C-Lin] Rational elliptic surfaces

family Floer SYZ	**Gross-Hacking-Keel-Siebert mirror**
large complex structure limit	toric degeneration
base of SYZ fibration with complex affine structure	dual intersection complex of the toric degeneration
loci of SYZ fibres bounding holomorphic discs	rays in scattering diagram
homology of boundary of a holomorphic disc	direction of the ray
exp of generating function of open Gromov-Witten invariants of Maslov index zero	slab functions attached to the ray
isomorphisms of Maurer-Cartan spaces induced by pseudo isotopies	wall crossing transformation
family Floer mirror	GS/GHK mirror

Construction

family Floer mirror	GHKS mirror
Tate algebra	$\mathbb{C}[L]$
rational domain analytic torus $\mathbb{G}_{\mathrm{an}}^n$	$\mathbb{G}_m^n$
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[C-Lin] The family Floer mirror of the hyperKahler rotation of complement of a II^*, III^*, IV^* fibres in a rational elliptic surfaces have the compactifications which are the analytification of dP5 and dP6, and cluster varieties of type G_2 respectively.

Compactification

Note that the resulting schemes are not compact, e.g. $\mathbb{C}^*$, so we want to compactify it

$$\mathbb{C}^* \subset \mathbb{C} \subset \mathbb{CP}^1.$$

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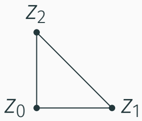
Projective toric varieties:

Example: $\mathbb{CP}^2$

Homogeneous coordinate ring of $\mathbb{CP}^2$ is the graded ring $\mathbb{C}[z_0, z_1, z_2]$. The grading can be described by a convex polytope.



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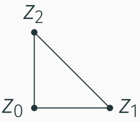

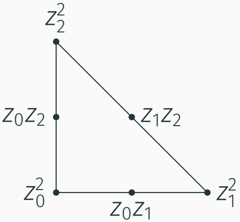

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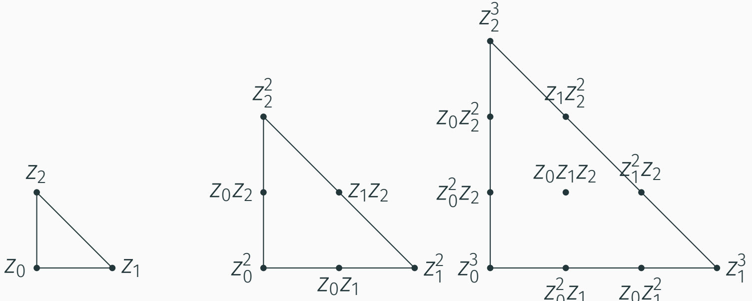

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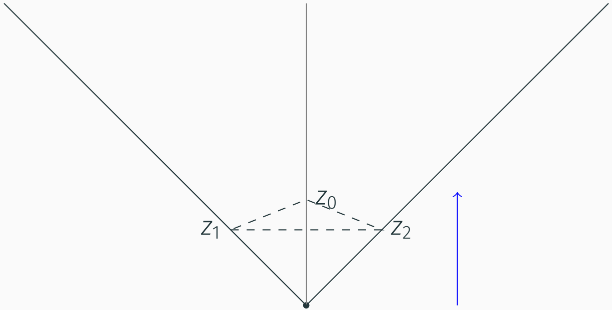

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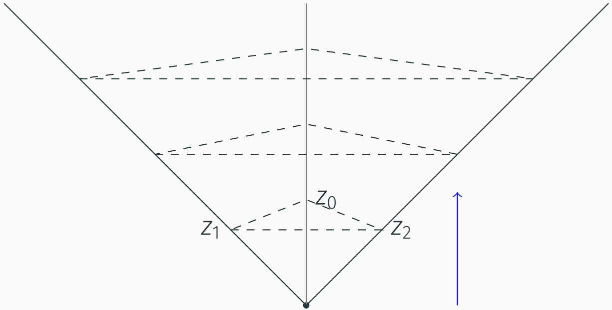

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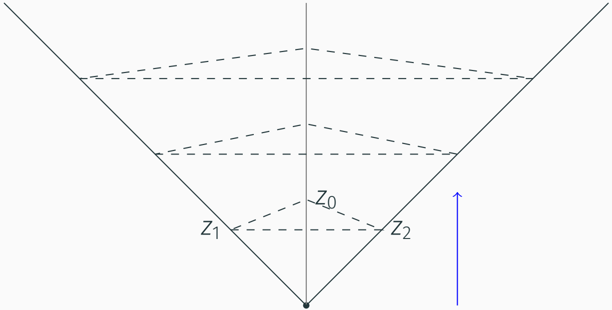

Polytope construction:

Consider a convex lattice polytope Δ in $\mathbb{R}^n$.

$\rightsquigarrow$ define a graded ring

$$S_\Delta = \langle z^m \rangle_{m \in k\Delta}.$$

$\rightsquigarrow$ projective toric geometry $\mathbb{P}_\Delta = \mathrm{Proj}(S_\Delta)$.

Cluster varieties

Cluster varieties can be defined by gluing tori by birational maps given by cluster transformation, e.g. $x \mapsto x(1+y)$.

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Fix a lattice $N \cong \mathbb{Z}^n$, $M = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. $N_{\mathbb{R}} = N \otimes \mathbb{R}$, $M_{\mathbb{R}} = M \otimes \mathbb{R}$.

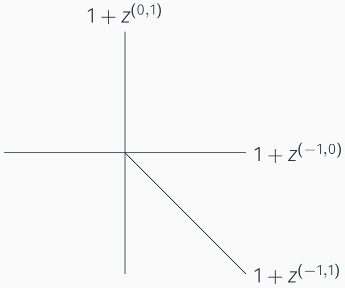
Cluster scattering diagram $\mathfrak{D}$ = collection of walls with finiteness and consistent condition

Wall : $(\mathfrak{d}, f_{\mathfrak{d}})$

- $\mathfrak{d} \subseteq M_{\mathbb{R}}$ support of walls - convex rational polyhedral cone of codim 1, contained in $n^{\perp} \in N$.
- $f_d = 1 + \sum c_k z^{kv}$, where $v \in n^{\perp}$.

Cluster scattering diagrams

Example: A_2



Path-ordered product

Path-ordered product (wall crossing transformation):

Consider a path γ passing a wall $\mathfrak{d}$, we define a map

$$\mathfrak{p}_\gamma : z^m \mapsto z^m f_{\mathfrak{d}}^{\pm \langle n_0, m \rangle},$$

where n_0 is the primitive normal of the wall $\mathfrak{d}$.

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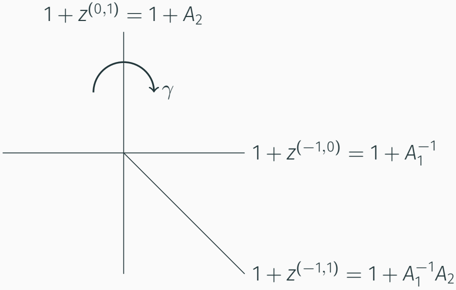
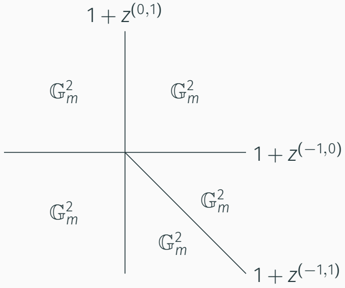


$$z^{(-1,0)} \mapsto z^{(-1,0)}(1 + z^{(0,1)}).$$

Scattering diagram as fan



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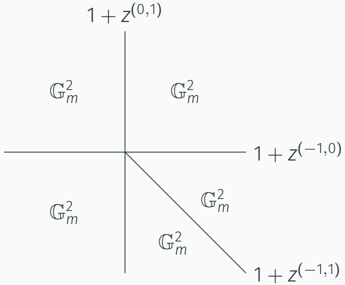

$f_{\mathfrak{d}} \rightsquigarrow$ wall crossing $\rightsquigarrow$ gluing the $\mathbb{G}_m^2$'s.

$\rightsquigarrow \mathcal{A}$-cluster variety of type A_2

Similar construction hold for general setting

Theta functions

Motivating example - $(\mathbb{C}^*)^2$: $H^0((\mathbb{C}^*)^2, \mathcal{O}) = \bigoplus_{m_1, m_2 \in \mathbb{Z}} \mathbb{C} z_1^{m_1} z_2^{m_2}$.

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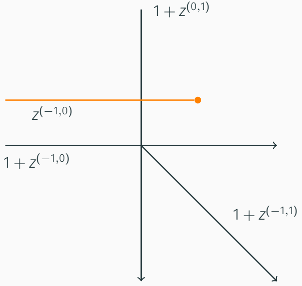

$$\vartheta_{Q,(-1,0)} = z^{(-1,0)} + z^{(-1,1)}.$$

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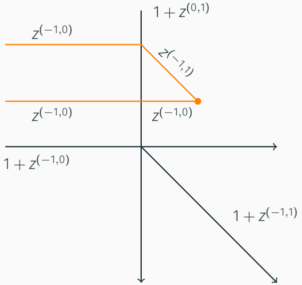

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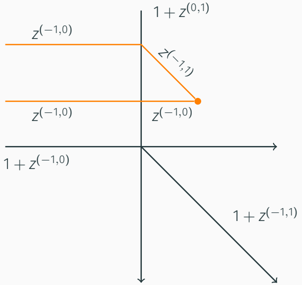

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Algebra structure

[Gross-Hacking-Keel-Konsevich]

$$\vartheta_p \cdot \vartheta_q = \sum_{r \in L} \alpha_{pq}^r \vartheta_r,$$

where $L = M^\circ$ or N, α_{pq}^r structure constant.

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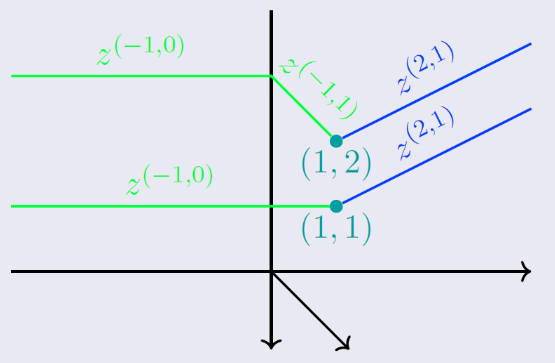
α_{pq}^r are expressed in terms of broken lines:

$$\alpha_{pq}^r := \sum c(\gamma^{(1)})\ c(\gamma^{(2)}),$$

where summing over pairs of broken lines $(\gamma^{(1)}, \gamma^{(2)})$ such that $I(\gamma^{(1)}) = p,\ I(\gamma^{(2)}) = q, \gamma^{(1)}(0) = \gamma^{(2)}(0) = r, F(\gamma^{(1)}) + F(\gamma^{(2)}) = r$

Example:

$$\vartheta_{(-1,0)} \cdot \vartheta_{(2,1)} = \vartheta_{(1,1)} + \vartheta_{(1,2)}.$$



Toric v.s. Cluster

Toric	Cluster
fan	scattering diagram
toric monomials	theta functions
convex polytope	positive polytope

Positive polytope

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Definition
A closed subset $S \subseteq L_{\mathbb{R}}$ is *positive* if
for every $a, b \in \mathbb{Z}_{\geq 0}$, $p \in aS(\mathbb{Z})$, $q \in bS(\mathbb{Z})$, and $r \in L$ with $\alpha_{pq}^r \neq 0$,

$\Rightarrow r \in (a+b)S$.

Notation: $L = M^\circ$ or N, $dS(\mathbb{Z})$ is the cone of S at the 'd'th-level.

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line	broken line
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Broken line convex

Definition (C-Magee-Nájera Chávez)
A closed subset S is called *broken line convex* if for any $x, y \in S(\mathbb{Q})$, every broken line segment connecting x and y is entirely contained in S.

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Idea: The structure constant α_{pq}^{r} in GHKK were expressed as two broken lines with initial slope p and q.

$\star$ [C-Magee-Nájera Chávez] construct the correspondence of those two broken lines with broken line segments with (scaling of) the endpoints p, q and r.

Compactification

Result:

$\rightsquigarrow$ get graded ring R

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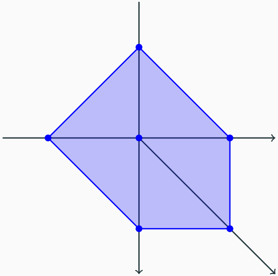
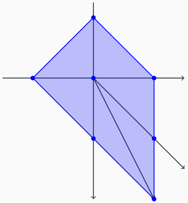
Example:

Type A_2:



[Gross-Hacking-Keel-Kontsevich]del Pezzo surface of degree 5

Type B_2:



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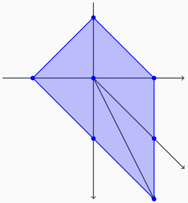

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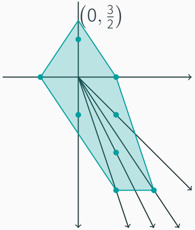

non-integral point coming from bending of broken line!

Any evidence?
Why we care?

Grassmannian

Grassmannian $\mathrm{Gr}(k, n)$ is the space that parameterizes all k-dimensional ear subspaces of the n-dimensional vector space $\mathbb{C}^n$.

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[Rietsch-Williams] NO body is a rational polytope.

$\mathrm{Gr}_3(\mathbb{C}^6)$

Non-integral example from NO body calculation: $\mathrm{Gr}_3(\mathbb{C}^6)$.

[Bossinger-C-Magee-Nájera Chávez] Get the non-integral point from broken line convexity!

Non-integral example from NO body calculation: $\mathrm{Gr}_3(\mathbb{C}^6)$.

[Bossinger-C-Magee-Nájera Chávez] Get the non-integral point from broken line convexity!

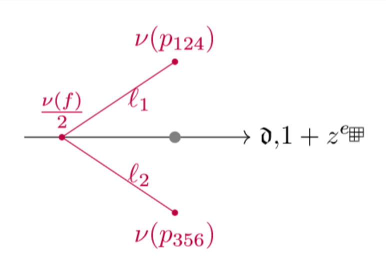

Figure 1: Part of the scattering diagram of $\mathrm{Gr}_3(\mathbb{C}^6)$.

$$\frac{\nu(f)}{2} = \left(\frac{1}{2}, 1, \frac{3}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}\right)$$

Newton Okounkov bodies

[Kaveh-Khovanskii, Lazarsfeld-Mustata, Rietsch-Williams]

$\mathbb{X} = \mathrm{Gr}_{n-k}(\mathbb{C}^n)$, with anticanonical divisor $D_{\mathrm{ac}} = D_1 + \cdots + D_n$.
$\mathbb{X}^\circ = \mathbb{X} \setminus D_{\mathrm{ac}}$.
Consider ample divisor $D = r_1 D_1 + \ldots r_n D_n$, and
the valuation $\mathrm{val} : \mathbb{C}(\mathbb{X}) \setminus \{0\} \to \mathbb{Z}^K$.

Then the NO body for the divisor D and val is

$$\Delta(D) = \overline{\mathrm{ConvexHull}\left(\bigcup_r \frac{1}{r}\mathrm{val}(H^0(\mathbb{X}, \mathcal{O}(rD)))\right)}$$

Intrinsic NO body

Define a map ν_θ: $\nu_\theta(\vartheta_p) = p$, where $p \in L$ (set of tropical points)

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Definition (Bossinger-C-Magee-Nájera Chávez)
[Intrinsic Newton-Okounkov body] Let $R(\mathcal{L})$ be the section ring of a line bundle $\mathcal{L}$. Consider

$$\Delta_{\mathrm{BL}}(\mathcal{L}) = \overline{\mathrm{conv}_{\mathrm{BL}} \left(\bigcup_{j \geq 1} \left(\bigcup_{f \in R_j(\mathcal{L})} \frac{1}{j} \mathrm{Newt}_\vartheta(f) \right) \right)}$$

Intrinsic NO body

Theorem (Bossinger-C-Magee-Nájera Chávez)

$$\Delta_{\mathrm{BL}}(\mathcal{L}) = \mathrm{conv}_{\mathrm{BL}}\left(\bigcup_{j\geq 1}\left(\bigcup_{f\in R_j(\mathcal{L})}\frac{1}{j}\mathrm{Newt}_{\vartheta}(f)\right)\right)$$

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$$\Delta_{\mathrm{BL}}(\mathcal{L}) = \mathrm{conv}_{\mathrm{BL}}\left(\bigcup_{j\geq 1}\left(\bigcup_{f\in R_j(\mathcal{L})}\frac{1}{j}\mathrm{Newt}_{\vartheta}(f)\right)\right)$$

Grassmannian: Let $\mathcal{L} \to \mathrm{Gr}_k(\mathbb{C}^n)$ be the pullback of $\mathcal{O}(1)$ under the Plücker embedding. (By definition of this embedding, the Plücker coordinates are a basis for $\Gamma(\mathrm{Gr}_k(\mathbb{C}^n), \mathcal{L})$.)

Theorem (Bossinger-C-Magee-Nájera Chávez)

$$\Delta_{\mathrm{BL}}(\mathcal{L}) = \mathrm{conv}_{\mathrm{BL}}\left(\left\{\nu_{\vartheta}(p_J) : J \in \binom{[n]}{k}\right\}\right)$$

Landau Ginzburg mirror

[C-Magee]

Toric	Cluster
$X \supset T$ toric Fano, $D = \sum_i D_{n_i}$ toric anticanonical divisor	
$D_{n_i} \rightsquigarrow z^{n_i}$ Laudau Ginzburg mirror $W = \sum_i z^{n_i} : T^\vee \to \mathbb{C}$	
Generic sections of $\mathcal{O}_X(D)$ mildly singular CY hypersurfaces	
level sets of W	
want W as sections of some $\mathcal{O}_Y(D')$, $M \subset T^\vee$	
$Y := \mathrm{TV}(\mathrm{Newt}(W))$	
Sections of $\mathcal{O}_X(D)$ and $\mathcal{O}_Y(D')$ are mirror	

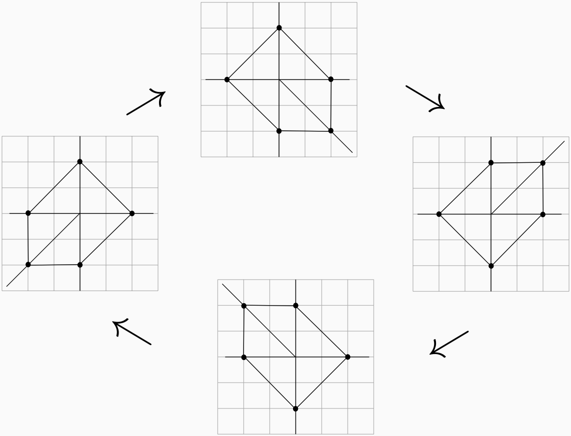
Landau Ginzburg mirror

[C-Magee]

Toric	Cluster
$X \supset T$ toric Fano, $D = \sum_i D_{n_i}$ toric anticanonical divisor	(X, D) Fano minimal model of cluster variety U, $D = \sum_i D_{v_i}$
$D_{n_i} \rightsquigarrow z^{n_i}$ Laudau Ginzburg mirror $W = \sum_i z^{n_i} : T^\vee \to \mathbb{C}$	$D_{v_i} \rightsquigarrow \vartheta_{v_i}$ Laudau Ginzburg mirror $W = \sum_i \vartheta_{v_i} : U^\vee \to \mathbb{C}$
Generic sections of $\mathcal{O}_X(D)$ mildly singular CY hypersurfaces	Generic sections of $\mathcal{O}_X(D)$ mildly singular CY hypersurfaces
level sets of W	level sets of W
want W as sections of some $\mathcal{O}_Y(D')$, $M \subset T^\vee$	want W as sections of some $\mathcal{O}_Y(D')$, $M \subset U^\vee$
$Y := \mathrm{TV}(\mathrm{Newt}(W))$	$\mathrm{Newt}_\vartheta(W) := \mathrm{conv}(v_i)$
Sections of $\mathcal{O}_X(D)$ and $\mathcal{O}_Y(D')$ are mirror	?

Mutation of polytopes

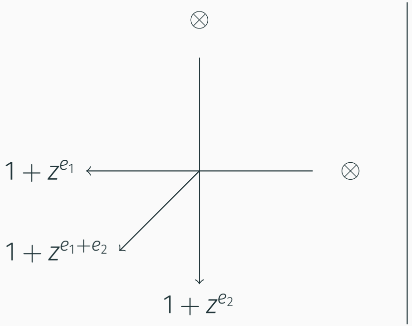
Cluster mutation of ($\mathcal{X}$-)scattering diagram / polytope



The underlying scheme is not changing

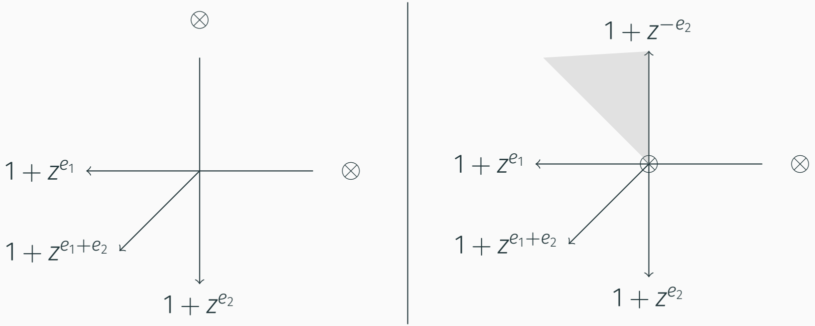
Another mutation

Scattering diagram with monodromy



Another mutation

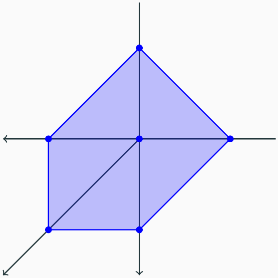
Scattering diagram with monodromy



the monodromy: $(1,0) \mapsto (1,1)$, $(0,1) \mapsto (0,1)$.

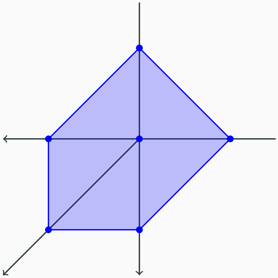
Another mutation

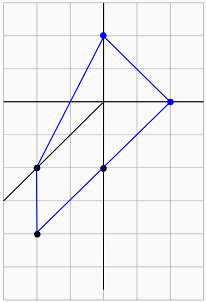
Mutation of polytope



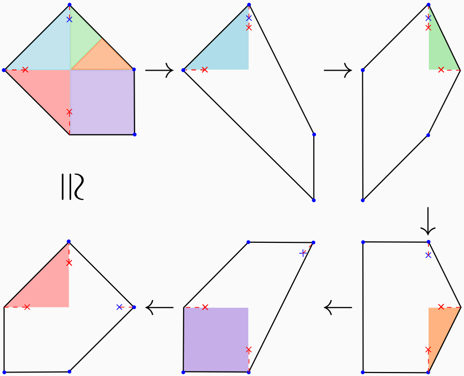
Another mutation

Mutation of polytope

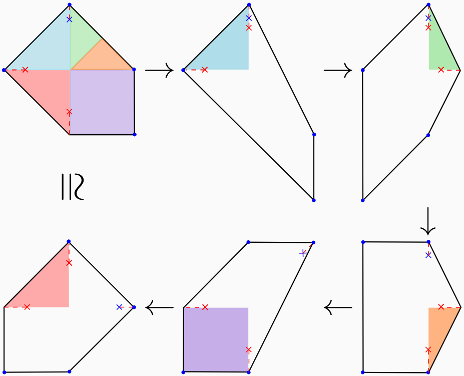




Mutation cycle



Mutation cycle



[C-Vianna] same as symplectic mutation compactification: singular Lagrangian fibration (almost toric fibration)

Thank you!