Fukaya-Seidel category, Hilbert scheme and category O

Tuesday, December 1, 2020 11:05 AM

Khovanov homology

cube of resolution

Coherent sheaves in iterated $\mathbb{P}^1$ bundles
Cautis-Kamnitzer

mirror symmetry?

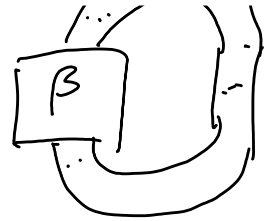
Quantum group representation, arc algebra, diagrammatic calculus
Khovanov, Brundan-Stroppel

Lag Fiber in open subset of $\mathrm{Hilb}^n(A_{2n-1})$
Seidel-Smith, Manolescu

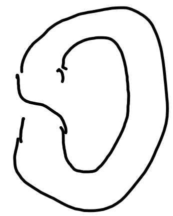
This talk: describe the Fukaya-Seidel category of $\mathrm{Hilb}^n(A_{2n-1})$ w.r.t. a Lefschetz fibration + some consequences.

§1 Kh^{symp}

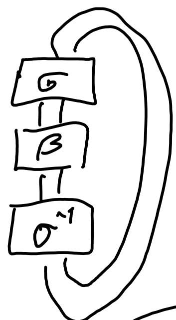
- K = a link

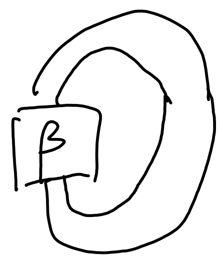
$\rightsquigarrow$ braid closure $K =$ 

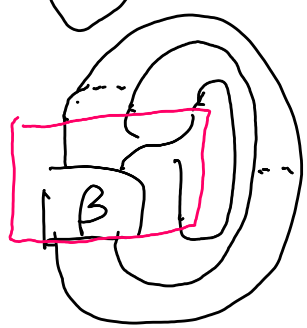
$\beta \in Br_n$

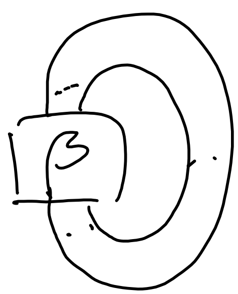
e.g. $\beta =$  $\Rightarrow K =$  $=$ unknot

Different braid closures are related by a seq of Markov moves

(M1) 

$=$ 

(M2) 

$=$ 

Seidel-Smith: associate to each n

- a symp mfd $\mathcal{Y}_n$
- a Lag submfd $L_{\cap} \subseteq \mathcal{Y}_n$

- action $$\overset{\cup}{Br_n} \longrightarrow Symp(\mathcal{Y}_n)$$ $$\beta \longmapsto \phi_\beta$$

Thm (Seidel-Smith)

$$Kh^{symp}(K) \overset{def}{=\!=} HF(L_{\frown}, \phi_\beta(L_{\frown}))$$ is well-defined (Markov invariant)

Symp mfd:

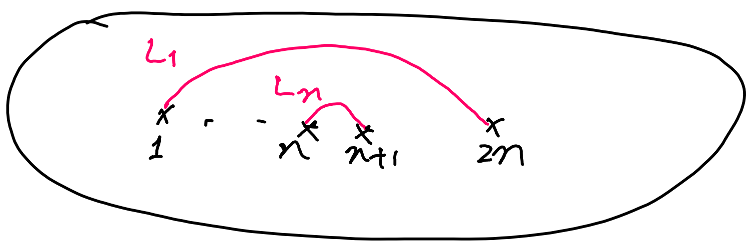
$$A_{2n-1} = \{x^2+y^2+(z-1)\cdots(z-2n)=0\} \overset{\pi}{\longrightarrow} \mathbb{C}_z$$

$$\mathcal{Y}_n = Hilb^n(A_{2n-1}) \setminus D$$

exact

$D = \overline{\left\{\begin{array}{l}\text{Unordered } n\text{-tuples of}\\ \text{distinct pts } (x_1, \dots x_n)\\ \text{w/ } \# \pi(\{x_1, \dots x_n\}) < n\end{array}\right\}}$

Lag submfd:



$L_{\frown} = \{(x_1, \cdots x_n)$ s.t. $x_i \in L_i\}$

| then forget ordering $\} \subseteq \mathcal{Y}_n$

In general, for any collection of n pairwise disjoint Lag $\underline{L} = \{L_1, \dots L_n\}$, we can use the same definition to define a Lag in $\mathcal{Y}_n$,

We call it $Sym(\underline{L})$

Braid group action:

Varying critical values of A_{2n-1}

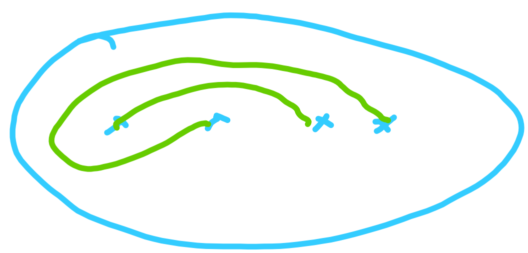
$\Rightarrow \pi_1(Conf_{2n}(\mathbb{C}), \{1, \dots 2n\}) = Br_{2n} \curvearrowright A_{2n-1}$

Claim: it also acts on $\mathcal{Y}_n$

in particular
$$\begin{tikzcd} Br_n \arrow[r, hook, "\text{left } n"', "\text{strands}"] \arrow[rr, bend right] & Br_{2n} \arrow[r] & Symp(\mathcal{Y}_n) \end{tikzcd}$$

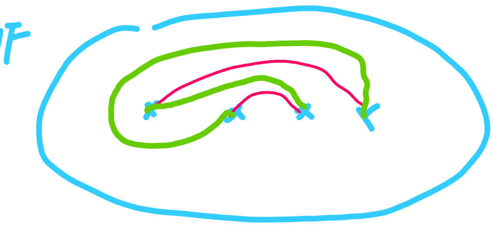
e.g. $\sigma = \underset{1}{\overset{\backslash\ /}{\times}} \in Br_2$

$\phi_\sigma(L \frown) =$



Summary

$$\overset{0}{Kh}{}^{symp}(K) = HF(L_{\frown}, \phi_\beta(L_{\frown}))$$

e.g. $K =$  , $Kh^{symp}(K) \overset{HF}{=}$ 

Thm (Abouzaid-Smith)

In char 0, $Kh^{symp}(K) = Kh(K)$.

§2 A Fukaya-Seidel category associated to Y_n

Symplectic mfd

$$Y_{n,m} \overset{def}{=} Hilb^n(A_{m-1}) \setminus D \qquad (\text{so } Y_{n,2n} = Y_n)$$

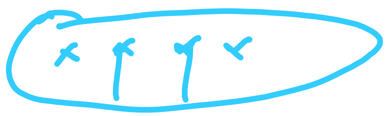
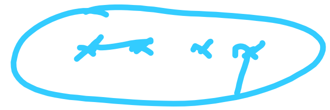
$$\downarrow \Sigma(A_{m-1} \to \mathbb{C}) =: \pi_Y$$

$$\mathbb{C}$$

$D \overset{def}{=} \left\{ \begin{array}{l} \text{unordered } n\text{-tuples of} \\ \text{distinct pts } (x_1, \cdots x_n) \\ \text{w/ } \#\pi(\{x_1, \cdots x_n\}) < n \end{array} \right\}$

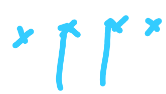
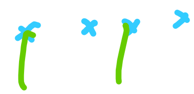
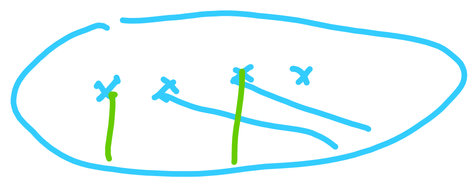
Lag submanifolds: we are interested in

$\mathrm{Sym}(\underline{L})$, where $\underline{L} = \{L_1, \ldots L_n\}$ each L_i is either compact of a Lefschetz thimble of A_{m-1}

e.g.  

represents Log in $\mathcal{Y}_{2,4}$

Denote the Fukaya-Seidel category w.r.t. π_y by $FS(\mathcal{Y}_{n,m})$

e.g. $\hom_{FS}\big($  ,  $\big) =$ 

Claim: The thimbles in $FS(\mathcal{Y}_{n,m})$ are precisely $\mathrm{sym}(\underline{L})$ s.t. $\underline{L} = \{L_1, \ldots L_n\}$ is a collection of pairwise disjoint thimbles

(for Sym^n(open Riemann surface), it is an observation of Auroux)

Cor: $FS(\mathcal{Y}_{n,m}) = \begin{cases} 0 & \text{if } n > m \\ \mathbb{K} & \text{if } n = m \end{cases}$

<u>Thm</u> (M. Smith) In char 0, for any positive integers n, m

$$FS(\mathcal{Y}_{n,m}) = \mathrm{perf}(K_{n,m})$$

$K_{n,m}$ is a graded associative algebra (called extended arc algebra by Brundan-Stroppel)

Moreover, $FS(\mathcal{Y}_{n,m}) = FS(\mathcal{Y}_{m-n,m})$ ← we'll come back to this in a geometric way later

and $FS(\mathcal{Y}_{n,m})^{op} = FS(\mathcal{Y}_{n,m})$

<u>Remarks</u>:

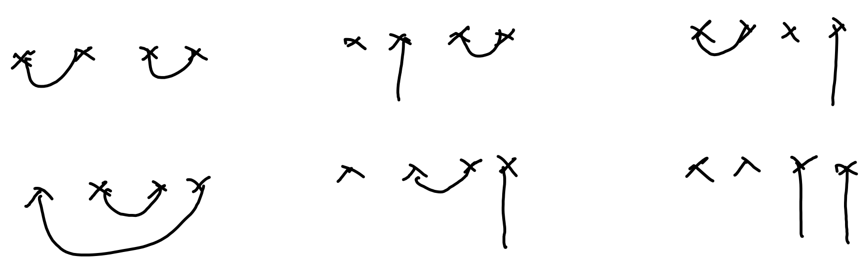
① $K_{n,m}$ is also the $\mathrm{End}\left(\text{Proj generator of parabolic BGG cat } \mathcal{O} \text{ of } \mathfrak{sl}_m \text{ w.r.t. } n\{\cdots\}m\right)$

② The proof is given by guessing the Lagrangians $\in FS$ corresponding to indecomposable projective modules, and then check that

$\mathrm{End}(\bigoplus_{\mathcal{C}} \mathrm{Lag}) = K_{n,m}$, and $\mathcal{C}$ generates FS.

A big hint is given in the work of Abouzaid-Smith

E.g. For $n=2, m=4$, $\mathcal{C}$ consists of



③ $\mathcal{Y}_{n,m}$ can be obtained from nilpotent slice of $\left(\begin{array}{c|c} J_n & 0 \\ \hline 0 & J_{m-n} \end{array}\right)$ J_k = Jordan block of size $k \times k$

$\iff 2n \le m$

In this case, elements of $\mathcal{Y}_{n,m}$ can be expressed as matrices of the form

$$\left\{ \left(\begin{array}{ccc|ccc} a_1 & 1 & 0 & b_1 & & \\ \vdots & & \ddots & \vdots & & 0 \\ a_n & 0 & & b_n & & \\ \hline c_1 & & & d_1 & 1 & 0 \\ \vdots & & 0 & \vdots & & \ddots \\ c_n & & & d_{m-n} & 0 & \end{array} \right) \middle| \begin{array}{c} \text{fixed generic} \\ \text{characteristic} \\ \text{poly} \end{array} \right\} \xrightarrow{\pi_y = a_1} \mathbb{C}$$

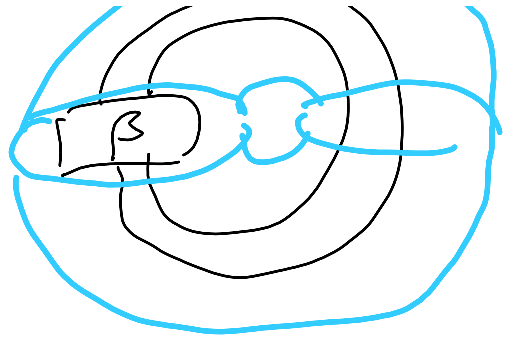
§3 Application to symplectic annular Khovanov homology AKh^{symp}

Braid: $Br_m \curvearrowright \mathcal{Y}_{n,m}$

$\beta \longmapsto M^n_\beta$ bimod over $FS(\mathcal{Y}_{n,m})$

Intuitively, could think of it as certain fixed pt Floer of ϕ_β

Def: $AKh^{symp}(\beta) = \bigoplus_{n=0}^{m} HH_*(FS(\mathcal{Y}_{n,m}), M^n_\beta)$

It is an invariant of  in $D^2 \times S^1$

Thm (M-Smith)
In char 0, it is isomorphic to AKh.

Historical remarks

- $Y_{1,m} = A_{m-1}$, it was proved by Auroux-Grigsby-Wehrli that $HH_*(FS(A_{m-1}), M^1_\beta)$ is a direct summand of AKh
- Motivated by AGW, Beliakova-Putyra-Wehrli show that AKh can be written as a direct sum of Hochschild homology
- We apply the idea from Abouzaid-Smith to show that
$$AKh^{symp} = AKh$$

Rest of the talk: $AKh^{symp} \rightrightarrows Kh^{symp}$

b)

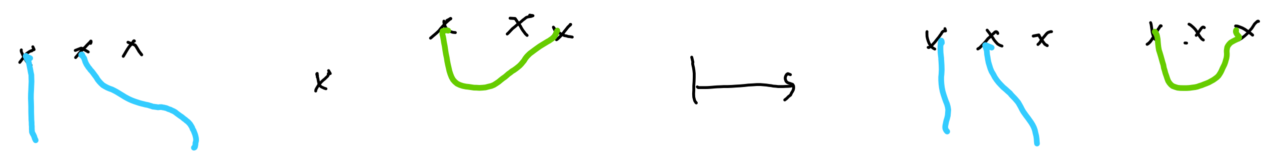
$$\bigoplus_{n=0}^{m} HH_*(FS(Y_{n,m}), M_\beta^n) \Rightarrow HF^*(L_\frown, \phi_\beta(L_\frown)) \text{ in } Y_{m,2m}$$

guess: $L_\frown \in FS(Y_{m,2m})$ should be related to Δ over $FS(Y_{n,m})$

Q: How are $FS(Y_{n,m})$ and $FS(Y_{m,2m})$ related?

A: A m-tuple is the union of a n-tuple and a m-n tuple

$$I_n : FS(Y_{n,m}) \times FS(Y_{m-n,m}) \longrightarrow FS(Y_{m,2m})$$



Lemma

① I_n is a cohomologically full and faithful embedding

② $FS(Y_{m,2m})$ admits a semiorthogonal decomposition

$$\langle \mathrm{Im}(I_m), \mathrm{Im}(I_{m-1}), \ldots, \mathrm{Im}(I_0) \rangle$$

e.g.

$\underline{L} \in \mathrm{Im}(\mathcal{I}_3)$ $\quad \underline{K} \in \mathrm{Im}(\mathcal{I}_2)$ $\quad \hom(\underline{K}, \underline{L}) = 0$

Recall: $FS(\mathcal{Y}_{n,m}) \simeq FS(\mathcal{Y}_{m-\,,m})^{op}$

Prop: The map

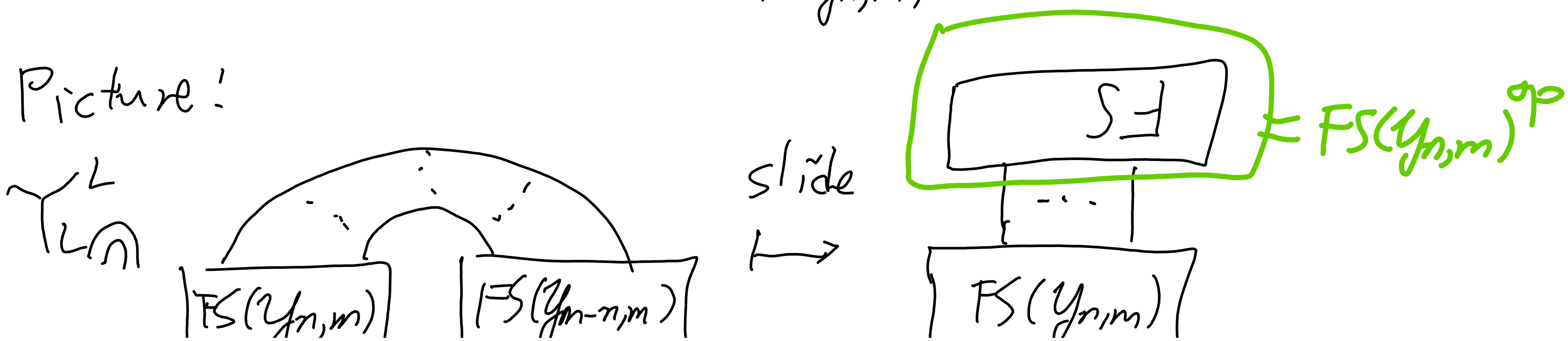
$$FS(\mathcal{Y}_{m,2m})\text{-mod} \xrightarrow{\mathcal{I}_n^*} FS(\mathcal{Y}_{n,m}) \times FS(\mathcal{Y}_{m-n,m})\text{-mod}$$

$$\xrightarrow{\simeq} FS(\mathcal{Y}_{n,m}) \times FS(\mathcal{Y}_{n,m})^{op}\text{-mod}$$

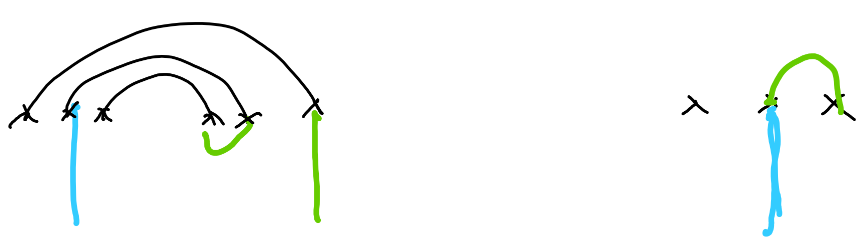
$$\xrightarrow{\simeq} FS(\mathcal{Y}_{n,m})\text{-Mod-}FS(\mathcal{Y}_{n,m})$$

sends $Y^{L}_{L\cap} \longmapsto \Delta_{FS(\mathcal{Y}_{n,m})}$

Picture!



Example

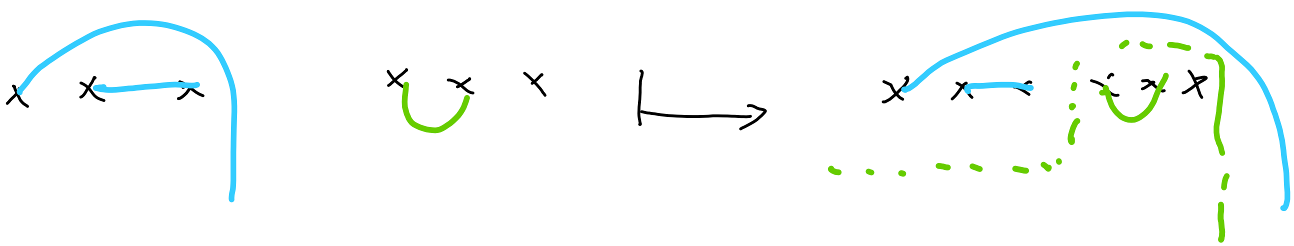


$$\hom(L_{\cap}, \underline{L} \cup \underline{K}) \simeq \hom(D(\underline{K}), \underline{L})$$

$\Rightarrow$ "the components of $Y_{L_\cap}$ w.r.t. $\langle Im(I_m), \dots Im(I_0) \rangle$" are diagonal bimodules

☆ Koszul dual semi-orthogonal decomposition

$$J_n : FS(Y_{n,m}) \times FS(Y_{m-n,m}) \longrightarrow FS(Y_{m,2m})$$



Lemma

① J_n is a cohomologically full and faithful embedding

② $FS(Y_{m,2m})$ admits a semi-orthogonal decomposition

$$\langle \mathrm{Im}(J_0), \mathrm{Im}(J_1), \ldots, \mathrm{Im}(J_m) \rangle$$

<u>Prop</u>: The map

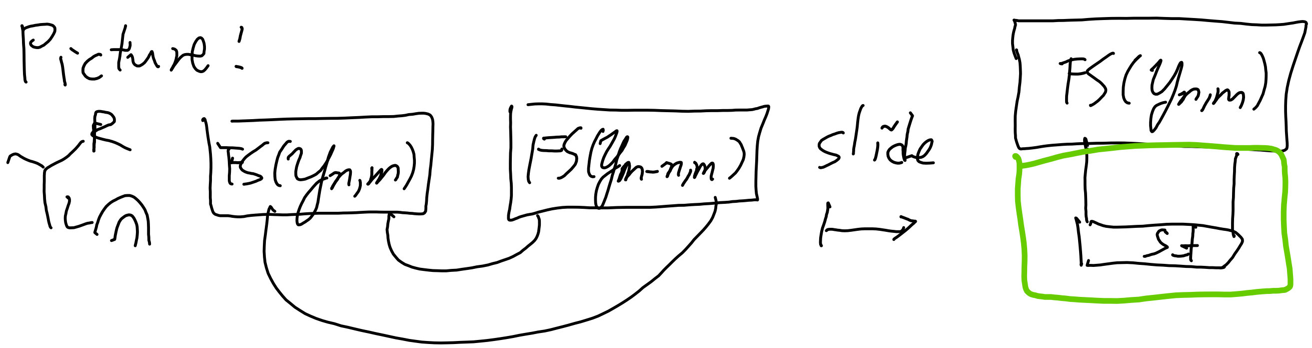
$$\text{mod-}FS(Y_{m,2m}) \xrightarrow{J_n^*} \text{mod-}FS(Y_{n,m}) \times FS(Y_{m-n,m})$$

$$\xrightarrow{\simeq} \text{mod-}FS(Y_{n,m}) \times FS(Y_{n,m})^{op}$$

$$\xrightarrow{\simeq} FS(Y_{n,m})\text{-Mod-}FS(Y_{n,m})$$

sends $Y^{R}_{L\cap} \longmapsto \Delta_{FS(Y_{n,m})}$

Picture:



$\simeq FS(Y_{n,m})^{op}$

Example

$$\hom(L \cup K, L_{\cap}) \simeq \hom(L, D(K))$$

Beilinson-type spectral sequence:

$$\bigoplus_{n=0}^{m} \left(J_n^* Y^R_{L_\cap} \underset{FS(Y_{n,m})\text{-mod-}FS(Y_{n,m})}{\otimes} I_n^* Y^L_{L_\cap} \right) \rightrightarrows \mathrm{Hom}(L_\cap, L_\cap)$$

$$\| $$

$$\bigoplus_{n=0}^{m} HH_*(FS(Y_{n,m}))$$

In general, we apply a twist on $Y^L_{L_\cap}$ to get the result.