Exact CY categories and odd-dimensional Lagrangian spheres

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Spherical object

$\mathbb{K}$ will be a field with $\mathrm{Char}(\mathbb{K}) = 0$.
Let $\mathcal{C}$ be a $\mathbb{K}$-linear, hom-finite triangulated category, and X a n-dimensional spherical object of $\mathcal{C}$, which means that

- $\hom_{\mathcal{C}}(X, X) \cong H^*(S^n; \mathbb{K})$;
- $\hom_{\mathcal{C}}(X, Y) \cong \hom_{\mathcal{C}}(Y, X[n])^\vee$ functorially in Y.

Question

Let X be an odd-dimensional spherical object, does $[X]$ define a non-trivial class in $K_0(\mathcal{C})$?

Note that if X is even-dimensional, then its Euler form $\chi(X, X) = 2$, which shows that $[X] \neq 0$ in $K_0(\mathcal{C})$.

Lagrangian spheres

A related question in symplectic topology is the following:

Question

Let M be a $2n$-dimensional Weinstein manifold, where n is odd, and $L \subset M$ a Lagrangian sphere. Then is $[L] \neq 0$ in $H_n(M;\mathbb{Z})$?

Again, when n is even, $[L] \cdot [L] = (-1)^{n/2}2$, which shows that $[L]$ is primitive.

Conjecture (Eliashberg)

Let M be any Weinstein manifold, and $L \subset M$ a closed exact Lagrangian submanifold with vanishing Maslov class. Then its homology class $[L] \in H_n(M;\mathbb{Z})$ is primitive.

Fukaya categories

Given a Weinstein manifold M with $c_1(M) = 0$, there are two $\mathbb{Z}$-graded A_∞-categories:

- $\mathcal{F}(M)$: Fukaya category of closed exact Lagrangian submanifolds;
- $\mathcal{W}(M)$: wrapped Fukaya category.

Theorem (Lazarev)

There is a surjective homomorphism

$$\mathcal{L} : H^n(M; \mathbb{Z}) \twoheadrightarrow K_0(\mathcal{W}(M)).$$

When $\mathcal{F}(M)$ and $\mathcal{W}(M)$ are related by A_∞-Koszul duality, one should expect an injective map

$$\mathcal{L}^\vee : K_0(\mathcal{F}(M)) \hookrightarrow H_n(M; \mathbb{Z}).$$

Koszul duality

Let $\mathcal{A}$ be an augmented, unital A_∞-algebra over $\mathbb{K}$. Define its Koszul dual

$$\mathcal{A}^! := R\,\hom_{\mathcal{A}}(\mathbb{K}, \mathbb{K}) = (\mathrm{B}\mathcal{A})^{\#},$$

where

$$\mathrm{B}\mathcal{A} =: \mathbb{K} \oplus \overline{\mathcal{A}}[1] \oplus \overline{\mathcal{A}}[1]^{\otimes 2} \oplus \cdots$$

is the bar construction, and $\overline{\mathcal{A}}$ is the augmentation ideal.

Definition

Two augmented A_∞-algebras $\mathcal{A}$ and $\mathcal{A}^!$ are Koszul dual if there is a quasi-isomorphism $\mathcal{A} \cong (\mathcal{A}^!)^!$.

Example

Let $\mathcal{A} = C_{-*}(\Omega_q Q; \mathbb{K})$, where Q is a smooth compact manifold, and $\Omega_q Q$ its based loop space. Then $\mathcal{A}^! \cong C^*(Q; \mathbb{K})$.

Quiver algebras

Important examples of A_∞-Koszul duality are given by quiver algebras. For a finite quiver $Q = (Q_0, Q_1)$ with potential w, there are two associated A_∞-algebras over the semisimple ring $\mathbb{k} = \bigoplus_{v \in Q_0} \mathbb{K} e_v$:

- $\mathcal{B}(Q, w)$: the proper 3-CY algebra defined by Kontsevich-Soibelman;
- $\mathcal{G}(Q, w)$: the Ginzburg dg algebra, which is smooth 3-CY.

It is well-known that the *completed* Ginzburg ag algebra $\widehat{\mathcal{G}}(Q, w)$ and $\mathcal{B}(Q, w)$ are Koszul dual.

There are many examples of 6-dimensional Weinstein manifolds whose $\mathcal{F}(M)$ and $\mathcal{W}(M)$ can be identified respectively with $\mathcal{B}(Q, w)$ and $\widehat{\mathcal{G}}(Q, w)$. These manifolds are generally referred to as quiver 3-folds.

Quiver 3-folds

Here are some typical examples of quiver 3-folds:

- plumbings of T^*S^3's according to any tree Γ (Ekholm-Lekili);
- plumbings of two T^*S^3's along an unknotted circle (Smith-Wemyss);
- Milnor fibers of once stabilizations of the hypersurface cusp singularities

  $$x^p + y^q + z^r + a \cdot xyz = 0, \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$$

  in $\mathbb{C}^3$ (Li);
- the quasi-projective CY 3-folds Y_ϕ associated to quadratic differentials ϕ on Riemann surfaces (Smith); [Need to twist by a non-trivial background class in $H^2(Y_\phi; \mathbb{Z}_2)$.]

Dilating $\mathbb{C}^*$-action

Definition (Seidel)

A $\mathbb{C}^*$-action on $\mathcal{F}(M)$ is dilating if for every equivariant Lagrangian brane $L \subset M$, the $\mathbb{C}^*$-action on $HF^n(L, L)$ has weight 1.

For Fukaya categories which carry dilating $\mathbb{C}^*$-actions, one can consider the equivariant Mukai pairing

$$C_0 \cdot_q C_1 = \sum_k q^k \chi\left(H^*\left(\hom_{\mathcal{F}(M)^{perf}_{\mathbb{C}^*}}(C_0\langle k\rangle, C_1)\right)\right) \in \mathbb{Z}[q, q^{-1}]$$

on the equivariant Grothendieck group $K_0^{\mathbb{C}^*}(\mathcal{F}(M))$, and then use the forgetful map

$$K_0^{\mathbb{C}^*}(\mathcal{F}(M)) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z} \to K_0(\mathcal{F}(M))$$

to show that any spherical object of $\mathcal{F}(M)^{perf}$ defines a non-trivial class in $K_0(\mathcal{F}(M))$.

Noncommutative vector field

A dilating $\mathbb{C}^*$-action on $\mathcal{F}(M)$ induces a noncommutative vector field $\mathrm{def} \in HH^1(\mathcal{F}(M))$ which satisfies

$$\Delta_{CY}(\mathrm{def} \cdot h) = h \tag{1}$$

for some $h \in HH^0(\mathcal{F}(M))^\times$, where Δ_{CY} is the BV operator. For a quiver 3-fold M, we have

$$HH^*(\mathcal{F}(M)) \cong HH^*(\mathcal{W}(M)) \cong HH_{*-n}(\mathcal{W}(M)),$$

so (1) means that the (weak) smooth CY structure on $\mathcal{W}(M)$ is the image of a class in $HH_{-n+1}(\mathcal{W}(M))$ under the map

$$HH_{-n+1}(\mathcal{W}(M)) \xrightarrow{I} HC_{-n+1}(\mathcal{W}(M)) \xrightarrow{B} HH_{-n}(\mathcal{W}(M)).$$

This is not true for most of the quiver 3-folds.

Exact CY structure

A homologically smooth A_∞-category is n-CY if there is a Hochschild cocycle $\eta \in HH_{-n}(\mathcal{A})$ which induces a bimodule isomorphism

$$R\operatorname{hom}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}^e)[n] \cong \mathcal{A}.$$

The CY structure is said to be smooth if it can be lifted to a negative cyclic cocycle $\tilde{\eta} \in HC^-_{-n}(\mathcal{A})$, and it is exact if it can be further lifted to $HC_{-n+1}(\mathcal{A})$.
There is a commutative diagram:

$$\begin{array}{ccc} HC_{-n+1}(\mathcal{A}) & \longrightarrow & HC^-_{-n}(\mathcal{A}) \\ & \searrow^{B} & \downarrow \\ & & HH_{-n}(\mathcal{A}) \end{array}$$

Every (completed) Ginzburg dg algebra is exact CY.

Cyclic open-closed map

Ganatra shows that the open-closed string map

$$OC^\dagger : CH_*^{nu}(\mathcal{W}(M)) \to SC^{*+n}(M)$$

admits a cyclic refinement

$$\widetilde{OC}^\dagger : CH_*^{nu}(\mathcal{W}(M)) \otimes \mathbb{K}((u))/u\mathbb{K}[[u]] \to SC^{*+n}(M) \otimes \mathbb{K}((u))/u\mathbb{K}[[u]]$$

which respects the S^1-complex structures on both sides. The complex on the R.H.S. computes the S^1-equivariant symplectic cohomology $SH^*_{S^1}(M)$.

Proposition

The wrapped Fukaya category $\mathcal{W}(M)$ of a Weinstein manifold M is exact CY if and only if there exists a class $\tilde{b} \in SH^1_{S^1}(M)$ such that its image under the marking map $\boldsymbol{B} : SH^1_{S^1}(M) \to SH^0(M)$ is an invertible element $h \in SH^0(M)^\times$.

Cyclic dilation

The class $\tilde{b} \in SH^1_{S^1}(M)$ will be called a cyclic dilation.

Related notions were introduced previously by Seidel-Solomon. A class $b \in SH^1(M)$ is called a quasi-dilation if $\Delta(b \cdot h) = h$ for some $h \in SH^0(M)^\times$. It has the following noteworthy properties:

- The image of b under the closed-open map

$$[CO] : SH^*(M) \to HH^*(\mathcal{F}(M))$$

  is a noncommutative vector field $\mathrm{def} \in HH^1(\mathcal{F}(M))$ satisfying (1).
- The image of b under the erasing map

$$\mathbf{I} : SH^*(M) \to SH^*_{S^1}(M)$$

  is a cyclic dilation.

Symplectic capacities

The Gutt-Hutchings capacities of a Liouville domain $\overline{M}$ with $c_1(M) = 0$ is defined to be

$$c_k^{GH}(M) := \inf\left\{ a | \delta_{eq}(x) = u^{-k+1} e \text{ for some } x \in F^{\leq a} SC_{S^1}^{-2k+1}(M) \right\}.$$

Proposition

M admits a cyclic dilation with $h = 1$ if and only if $c_1^{GH}(M) < \infty$.

Definition

We say that M has property (H) if there is a real number $\lambda > 0$ such that

- $\lambda < \min \mathcal{P}_M$, where $\mathcal{P}_M$ is the period spectrum;
- $2\lambda \notin \mathcal{P}_M$;
- there is a cyclic dilation $\tilde{b} \in HF_{S^1}^1(2\lambda)$.

When $h = 1$, it essentially means $c_1^{GH}(M)$ is sufficiently small.

The theorem

Theorem

Let M be a $2n$-dimensional Weinstein manifold, where $n \geq 3$ is odd, and $c_1(M) = 0$. Assume that M has property (H), then for any Lagrangian sphere $L \subset M$, its homology class $[L] \in H_n(M; \mathbb{K})$ is non-trivial.

According to Zhou, the affine hypersurfaces in $\mathbb{C}^{n+1}$ defined by the equations

$$z_1^k + \cdots + z_{n+1}^k = 1, \ k \leq n$$

have property (H). It is also expected to be the case for the Milnor fibers associated to the Brieskorn singularities

$$z_1^{k_1} + \cdots + z_{n+1}^{k_{n+1}} = 0, \ \sum_{i=1}^{n+1} \frac{1}{k_i} > 1.$$

Cieliebak-Latschev map

Let $L \subset M$ be an exact Lagrangian submanifold, which is *Spin*. Cohen-Ganatra showed that the Viterbo transfer map

$$CL_0 : SC^*(M) \to C_{n-*}(\mathcal{L}L; \mathbb{K})$$

also admits an S^1-equivariant enhancement

$$\widetilde{CL} : SC^*(M) \otimes \mathbb{K}((u))/u\mathbb{K}[[u]] \to C^{\dagger}_{n-*}(\mathcal{L}L; \mathbb{K}((u))/u\mathbb{K}[[u]]),$$

whose kth component CL_k is defined by closed discs with an interior puncture ζ_{in}, and k auxiliary marked points $p_1, \cdots, p_k$ satisfying

$$0 < |p_k| < \cdots < |p_1| < 1.$$

$\widetilde{CL} = \sum_{k=0}^{\infty} CL_k u^k$ is called the Cieliebak-Latschev map.

Lagrangian embedding

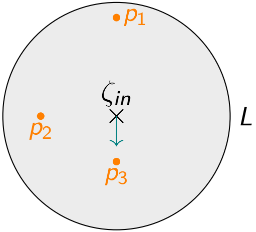


Figure: Domain of the operation CL_3

As an application of the Cieliebak-Latschev map, we have:

Proposition (Davision)

Let M be a Liouville manifold which admits a cyclic dilation, then M does not contain any hyperbolic exact Lagrangian submanifold. Moreover, if $h = 1$, then M does not contain any exact Lagrangian $K(\pi, 1)$.

Local triviality

More generally, for any Liouville subdomain $\overline{N} \subset M$, we have the S^1-equivariant enhancement of Viterbo functoriality

$$\tilde{v}^! = \sum_{k=0}^{\infty} v_k^! u^k : SC^*(M) \otimes \mathbb{K}((u))/u\mathbb{K}[[u]] \to SC^*(N) \otimes \mathbb{K}((u))/u\mathbb{K}[[u]],$$

which can be realized as an equivariant continuation map.

Proposition

Let M be a $2n$-dimensional Liouville manifold which admits a cyclic dilation $\tilde{\beta} = \sum_{k=0}^{\infty} \beta_k \otimes u^{-k}$, where n is odd. For any Lagrangian sphere $L \subset M$, the image of $\tilde{\beta}$ under the S^1-equivariant Viterbo functoriality $\tilde{v}^!$ is of the form

$$\sum_{k=0}^{\infty} v_k^!(\beta_k) \in SC^1(T^*L),$$

and is a non-zero scalar multiple of the dilation.

Parametrized closed-open maps

If we replace the interior puncture in the definition of CL_k with an output, we get a cochain

$$\phi_L^{1,0;k} \in CF^{n-2k}(\lambda).$$

More generally, we can add boundary punctures to the domain defining $\phi_L^{1,0;k}$ and obtain a sequence of maps

$$\begin{aligned}\phi^{1,d;k}_{L_1,\cdots,L_d} :& CF^{*+2k}(\lambda) \otimes CF^*(L_{d-1}, L_d) \otimes \cdots \otimes CF^*(L_1, L_2) \\ &\to CF^{*-d}(L_1, L_d).\end{aligned}$$

The auxiliary marked points $p_1, \cdots, p_k$ are now required to satisfy

$$0 < |p_k| < \cdots < |p_1| < \frac{1}{2} \tag{2}$$

near ζ_{in}.

Obstruction

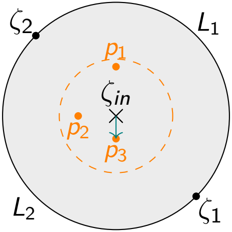


Figure: Domain of the operation $\phi^{1,2;3}_{L_1,L_2}$

Let $\sum_{k=0}^{\infty} \beta_k \otimes u^{-k}$ be the cochain level representative of a cyclic dilation, and $L \subset M$ an odd-dimensional Lagrangian sphere.

Proposition

$\sum_{k=0}^{\infty} \phi_L^{1,1;k}(\beta_k)$ is cohomologous to $\phi^{1,1}_{L\subset T^*L}\left(\sum_{k=0}^{\infty} v_k^!(\beta_k)\right)$, therefore defines a cocycle in $CF^1(L,L)$.

Endomorphism

Since $HF^1(L,L)=0$, we can choose a cochain $\gamma_L \in CF^0(L,L)$ such that $\mu^1(\gamma_L)=\sum_{k=0}^{\infty}\phi_L^{1,1;k}(\beta_k)$. For the pairing $\widetilde{L}=(L,\gamma_L)$, define an endomorphism

$$\phi_{\widetilde{L},\widetilde{L}} : CF^*(L,L) \to CF^*(L,L)$$

by

$$\phi_{\widetilde{L},\widetilde{L}}(\cdot) := \sum_{k=0}^{\infty} \phi_{L,L}^{1,2;k}(\beta_k,\cdot) - \mu^2(\gamma_L,\cdot) + \mu^2(\cdot,\gamma_L).$$

Proposition

$\phi_{\widetilde{L},\widetilde{L}}$ is a chain map, its cohomology level map $\Phi_{\widetilde{L},\widetilde{L}}$ acts trivially on $HF^0(L,L)$, and multiplies by $\alpha_L \in \mathbb{K}^\times$ on $HF^n(L,L)$. In particular, $\mathrm{Str}(\Phi_{\widetilde{L},\widetilde{L}}) = -\alpha_L \neq 0$.

Pairing

Equip the complex $\widetilde{C}^* = CF^*(-\lambda) \oplus CF^*(\lambda)$ with the pairing $\iota : \widetilde{C}^* \otimes \widetilde{C}^{2n-*} \to \mathbb{K}$ defined by

$$\iota((\xi_0, x_0), (\xi_1, x_1)) = \langle x_0, \xi_1 \rangle - (-1)^{|\xi_0|} \langle x_1, \xi_0 \rangle + \sum_{k=0}^{\infty} \langle \beta_k, \xi_0 *_k \xi_1 \rangle.$$

This is a chain map.

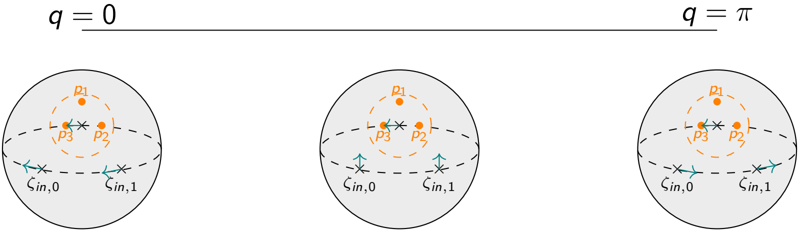


Figure: Defining the operation $*_3$

The proof

For any odd-dimensional Lagrangian sphere $L \subset M$, we can associate a cocycle

$$(\xi_L, x_L) := \left(\phi_L^{1,0}, (-1)^{n+1} \sum_{k=0}^{\infty} \phi_L^{2,0;k}(\beta_k) + \check{\phi}_L^{1,1}(\gamma_L)\right) \in \widetilde{C}^n,$$

whose cohomology class is $[[\widetilde{L}]] \in \widetilde{H}^n$.

Proposition

$(-1)^{n(n+1)/2}\mathrm{Str}(\phi_{\widetilde{L},\widetilde{L}}) = \iota((\xi_L, x_L), (\xi_L, x_L))$.

$\mathrm{Str}(\Phi_{\widetilde{L},\widetilde{L}}) = -\alpha_L \neq 0$ implies that $I\left([[\widetilde{L}]], [[\widetilde{L}]]\right) \neq 0$. Since $HF^n(\lambda) \subset \widetilde{H}^n$ is a half-dimensional subspace which is isotropic for $\widetilde{I}$, by projecting to $HF^n(-\lambda)$, we see that the class $[\phi_L^{1,0}]$ is non-zero. But by our assumption that λ is small, $HF^*(-\lambda)$ can be identified with $H^n_{cpt}(M; \mathbb{K})$, and $[\phi_L^{1,0}]$ is the Poincaré dual of $[L] \in H_n(M; \mathbb{K})$.

Moduli space of annuli

The proposition is proved by studying the moduli space of domains $(S; \ell_{in}, p_1, \cdots, p_k)$, where S is an annulus with an interior puncture ζ_{in}, and k auxiliary marked points $p_1, \cdots, p_k$ near ζ_{in} which satisfy (2). The asymptotic marker ℓ_{in} at ζ_{in} is required to point towards p_k if $k > 0$, and varies in a specific way if $k = 0$.

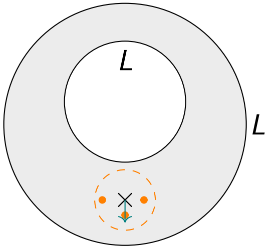


Figure: Domain when $k = 3$

A 3-parameter family

Let S be an annulus with an interior puncture ζ_{in} and a boundary puncture ζ_1 which is an output. The asymptotic marker ℓ_{in} points towards ζ_1, and ζ_1 is allowed to move along an open arc γ on the boundary. The compactified moduli space of these domains form a 3-parameter family, which is topologically the Cartesian product of a closed interval and a pentagon. Depending on whether ζ_1 is on the inner boundary or outer boundary, we get two different operations

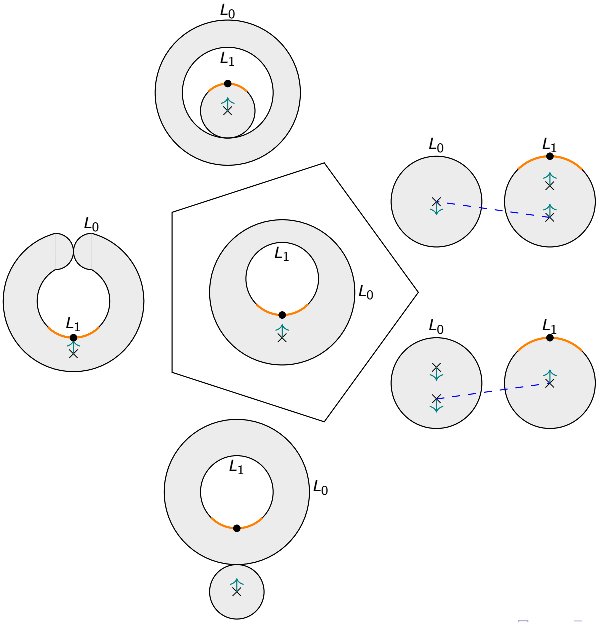
$$\psi^{1,1}_{L_0,L_1} : CF^{*+3}(\lambda) \to CF^{*+n}(L_1, L_1);$$

$$\check{\psi}^{1,1}_{L_0,L_1} : CF^{*+3}(\lambda) \to CF^{*+n}(L_0, L_0).$$

For particular choices of Floer data, these operations are related by

$$\psi^{1,1}_{L_0,L_1} - (-1)^n \check{\psi}^{1,1}_{L_1,L_0} = \text{null homotopy}.$$

The pentagon



Koszul duality, revisited

Theorem (Van den Bergh)

Let $\mathcal{A}$ be a homologically smooth, complete dg algebra over $\Bbbk$ so that $H^*(\mathcal{A})$ is concentrated in degrees $* \leq 0$. Then the following statements are equivalent:

- *$\mathcal{A}^!$ is a proper A_∞-algebra which, up to quasi-isomorphism, carries a minimal cyclic A_∞-structure of degree n;*
- *$\mathcal{A}$ is exact CY.*

Geometrically, this theorem shall be applied to the Fukaya A_∞-algebra of Lagrangian cocores in a Weinstein manifold.

Corollary

Let M_Γ be the plumbing of T^*Q_v according to a tree Γ, where $\dim(Q_v) \geq 3$ and Q_v is simply-connected for all $v \in \Gamma$. Then M_Γ admits a cyclic dilation.

3-fold triple point

As a more sophisticated application of Van den Bergh's theorem, we have:

Theorem

The Milnor fiber $M_{3,3,3,3}$ *of a 3-fold triple point*

$$x^3 + y^3 + z^3 + w^3 = 0$$

admits a cyclic dilation.

Remark

$M_{3,3,3,3}$ *does not admit a quasi-dilation. The same method should be applicable to more complicated examples, say* $M_{4,3,3,3}$.

As a byproduct, we have:

Proposition

The Fukaya A_∞*-algebra of a basis of vanishing cycles in* $M_{3,3,3,3}$ *is not formal.*

Vanishing cycles

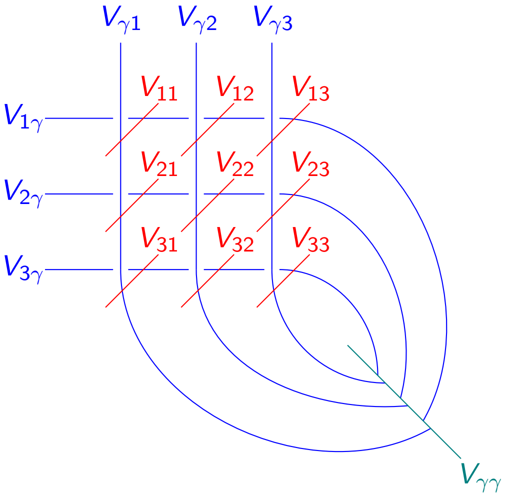


Figure: Configuration of vanishing cycles in $M_{3,3,3,3}$