Towards dimension theory for spectral semi-orthogonal decompositions

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1. Recall: semi-orthogonal decompositions from GW invariants

X: projective algebraic variety over $\mathbb{C}$, with given ample line bundle.

$\rightsquigarrow$ Gromov-Witten invariants in genus zero, giving potential $\mathcal{F}_0$: formal series on $H^\bullet(X)$ with coefficients in $\mathbb{Q}[[T]]$.

Conjecture: $\mathcal{F}_0$ *is convergent both for a point in* $H^\bullet(X)$ *close to zero, and for* T *close 0.*
Assume for today's talk

Assuming additionally Γ-conjecture $\rightsquigarrow$ $\mathbf{nc}$ Hodge structure parametrized by a domain $\mathcal{M} \subset H^\bullet(X, \mathbb{C})$: connection on the trivial bundle $H^\bullet(X)$ over u-plane $\mathbb{C}_u$
$\left(u\partial_u + \frac{1}{u}\mathbf{K} + \mathbf{G}\right)\psi(u) = 0$ (Γ-conjecture gives a lattice which is hypothetically is compatible with Stokes filtrations along rays at $u \to 0$)

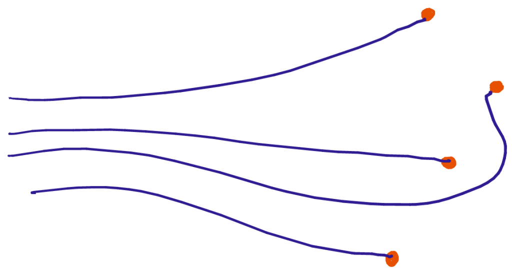
Operator $\mathbf{K}$ is the operator of quantum product with $c_1(T_X)$, depends on the point in Frobenius manifold $\mathcal{M}$.

Operator $\mathbf{G}$ is constant, given by $\mathbf{G}_{|H^i(X)} = \frac{i - \dim_{\mathbb{C}} X}{2} \cdot \mathrm{id}_{H^i(X)}$

Definition: ***quantum spectrum*** *is the spectrum of* $\mathbf{K}$*, a finite subset* $\mathsf{Spec}_X \subset \mathbb{C}$ (depends on a point in $\mathcal{M}$).

Consider a purely even affine submanifold $\mathcal{M}^{alg} \subset \mathcal{M}$, given by deformations of quantum product by linear combinations of *algebraic classes* $H^{alg}_{\mathbb{Q}}(X) \subset H^{even}(X, \mathbb{Q})$.

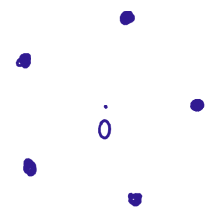
Conjecture: *for any point in* $\mathcal{M}^{alg}$ *and a choice of disjoint paths from* $-\infty$ *to points of the corresponding spectrum (Gabrielov paths):*



we obtain a semi-orthogonal deocomposition $D^b(Coh(X)) = \langle \mathcal{C}_1, \ldots, \mathcal{C}_r \rangle$ *where* r *is the number of elements of the spectrum.*

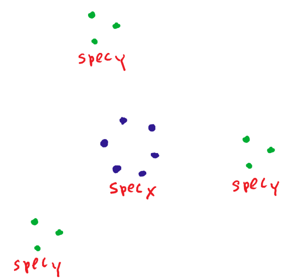
All categories $\mathcal{C}_1, \ldots, \mathcal{C}_r$ are *saturated* (i.e. smooth and porper), equal to *local Fukaya-Seidel categories* for the mirror LG dual $(Y, W : Y \to \mathbb{C})$, if it exists.

Examples: $\mathbf{1}$. $X = \mathbb{P}^n$, the spectrum is $\mu_{n+1} = \{z \in \mathbb{C} | \, z^{n+1} = 1\}$ (for some point in $\mathcal{M}$)



$\rightsquigarrow D^b(Coh(X)) = \langle \mathcal{O}, \ldots, \mathcal{O}(n) \rangle$.

$\mathbf{2}$. *Conjectural blow-up formula*: If $\tilde{X} = Bl_Y(X)$ where $Y \subset X$ is a smooth closed subvariety of codimension $m \geq 2$, then the spectrum $\mathsf{Spec}_{\tilde{X}}$ is close to



with $(m-1)$ copies of Spec_Y around one copy of Spec_X.

$\mathbf{3}$.. If X is Calabi-Yau or of general type, the spectrum is always $\{0\}$, get $\langle D^b(Coh(X)) \rangle$.

2. Towards dimension theory of elementary pieces

We see that sometimes elementary pieces $\mathcal{C}_i = \mathcal{C}_{z_i}, \quad z_i \in \mathsf{Spec}_X$ (could be combined as some points of the spectrum collide), are themselves equivalent to derived categories of sheaves on some varieties, of certain *dimensions* $\leq \dim X$. In general, for a saturated category $\mathcal{C}$ one can define its **Serre dimension** as

$$\dim_{\mathrm{Serre}} \mathcal{C} := \lim_{|k| \to +\infty} \left\{ \frac{i}{k} | Ext^i(\mathrm{Id}_{\mathcal{C}}, S_{\mathcal{C}}^k) \neq 0 \right\} \subset \mathbb{R}$$

Here $S_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is the *Serre functor* as defined by A.Bondal and M.Kapranov:

$$\mathrm{Hom}_{\mathcal{C}}(E, F)^* = \mathrm{Hom}_{\mathcal{C}}(F, S_{\mathcal{C}} E), \qquad \forall\, E, F \in Ob(\mathcal{C})$$

In general, Serre dimension could be an empty set, or an interval. For categories $D^b(Coh(X))$ it is exactly the dimension $\dim X \in \mathbb{Z}_{\geq 0}$.

For Fukaya-Seidel category of $Y = \mathbb{C}_x, W = x^d, d \geq 2$ the Serre dimension is *fractional*, $= 1 - \frac{2}{d}$.

Conjecture: *Serre dimension of an elementary piece is always a number in* $\mathbb{Q} \cap [0, \dim X]$.

2.1. Hypothetical formula for Serre dimension of an elementary piece

The answer is given purely in terms of differential equation $\left(u\partial_u + \frac{1}{u}\mathbf{K} + \mathbf{G}\right)\psi(u) = 0$:

Conjecture:

$$\dim_{\mathrm{Serre}} C_{z_i} = -2\min\{s \in \mathbb{Q}_{\leq 0} |\ \exists \text{ solution } \sim \psi_{s,k} \cdot u^s \log(u)^k e^{\frac{z_i}{u}} + \dots\}$$

Evidence: I've checked 100s of examples of complete intersections in projective spaces. If X is a Fano complete intersection of hypersurfaces of degrees $d_1, \dots, d_r$ in $\mathbb{P}^{N-1}$, then the corresponding semi-orthogonal decomposition is

$$D^b(Coh(X)) = \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(N-1-d_{tot}),\ \text{Kuznetsov component}\rangle$$

where $d_{sum} := \sum_i d_i$, the spectrum is $\{0\} \cup \mu_{N-d_{tot}}$ and the predicted Serre dimension for Kuznetsov component $\mathcal{C}_{z=0}$ is

$$\dim_{\mathrm{Serre}} C_{z=0} = (N-r-1) - 2\frac{N - \sum_i d_i}{\max_i d_i} = \dim X - 2\frac{N - d_{sum}}{d_{max}}$$

Moreover, there is always a very striking equality

$$\max\{i \in 2\mathbb{Z} + \dim X | i \leq \dim_{\text{Serre}} C_{z=0}\} =$$
$$= \max\{k \in \mathbb{Z} | HH_k(\mathcal{C}_{z=0}) \neq 0\} =$$
$$= \max\{p - q | H^{p,q}(X) \neq 0\}$$

Analogy: for any smooth projective variety X we have:

$$\dim X \geq \max\{p - q | H^{p,q}(X) \neq 0\}$$

The decomposition $\langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(N - 1 - d_{tot}), \text{ Kuznetsov component } \mathcal{C}_{z=0} \rangle$ correponds to a very special point in the Frobenius manifold, with the trivial correction to the ample class. In general, we should deform it by other algebraic cycles in X. The spectrum can a priori split into a finer one, and elementary pieces will be decomposed into smaller pieces. This motivates the following

2.2. Semi-continuity conjecture

if we deform a bit parameters (a point in the algebraic part $\mathcal{M}^{alg}$ of the Frobenius manifold), so that the deformed eigenvalues of $\mathbf{K}$ split, then the corresponding Serre dimension can only decrease.

Together with the blow-up conjecture, this will give a very strong criterium of *non-rationality*:

$\Longrightarrow$ *if for at least one elemenatry pieces of s.o.d. of $D^b(Coh(X))$, corresponding to a generic point of $\mathcal{M}^{alg}$, its Serre dimension is $> \dim X - 2$, then X is* ***not rational***.

Corollary (one of many): *any odd-dimensional complete intersection of degrees $(d_1, \ldots, d_r)$ in $\mathbb{P}^{N-1}$ is not rational if $d_{sum} + d_{max} > N$.*

3. Landau-Ginzburg model

Let Y be a *noncompact* complex manifold of dimension $n \geq 0$, and $W : Y \to \mathbb{C}$ be a holomorphic map. Denote by $\mathsf{Crit}(W) \subset Y$ the critical locus considered as a closed analytic subspace (possibly non-reduced).

Assumptions:

1. (the most crucial) $\mathsf{Crit}(W)$ is compact
2. (also important) there exists a Kähler metric on Y (the choice is *not* a part of the structure, only existence is required)
3. (technical, for later convenience) $\mathsf{Crit}(W)$ is nonempty and connected, and moreover $f(\mathsf{Crit}(W)) = \{0\} \subset \mathbb{C}$.

What follows will not change if we replace Y by any open subset $Y' \subset Y$ containing $\mathsf{Crit}(W)$ (alternatively, one can consider Y as a *germ* at $\mathsf{Crit}(W)$).

Consider $\mathbb{Z}$-graded complex

$$(\Gamma(Y, \Omega^{\bullet}_{C^\infty}(Y)[[u]], \text{ differential } d_{\text{tot}} := \bar{\partial} + u\partial + dW \wedge \cdot)$$

It calculates hypercohomology $R\Gamma(Y, \Omega^{\bullet}_{Y}[[u]], ud + dW \wedge \cdot)$. If $\alpha \in \Gamma(Y, \Omega^{\bullet}_{C^\infty}(Y))\{u\}$ (i.e. not only a formal series in u, but a germ of analytic [forms on Y]-valued function at $u = 0$ and near the compact set $\mathsf{Crit}(W)$) then $d_{\mathrm{tot}}\alpha = 0$ means that

$$d\left(e^{\frac{W}{u}} u^{\mathbf{Gr}}(\alpha)\right) = 0, \qquad \mathbf{Gr}_{|\Omega^{p,q}_{C^\infty}} := \frac{q-p}{2}$$

Conjecture (Hodge-de Rham degeneration for LG models): *Cohomology of d_{tot} is a free finite rank (equivalently, flat) $\mathbb{C}[[u]]$-module*.

It is true in algebro-geometric situation by irregular Hodge theory, but in general I don't think it follows from Mochizuki, or Barannikov-me (maybe from Saito or Sabbah).

Let us assume that Y is endowed with an everywhere non-vanishing holomorphic volume form $vol \in \Gamma(Y, \Omega^n), \quad n = \dim Y$.

Then in the case when $\mathsf{Crit}(W)$ is *connected and non-empty*, there is a *canonical* 1-dimensional subspace at the fiber at 0:

$$\begin{aligned} &[vol] \in \mathbb{H}^n(Y, \Omega^\bullet_Y, dW \wedge \cdot) \to \\ &\to \mathbb{H}^n(\mathsf{Crit}(W), (\Omega^\bullet_Y)_{|\mathsf{Crit}(W)}) \to \\ &\to H^0(\mathsf{Crit}^{red}(W), \Omega^n_Y) = \mathbb{C} \ni 1 \end{aligned}$$

Conjecture: *the leading growth of solutions appears in the exactly one-dimensional subspace of cohomology of d_{tot}, and its reduction at $u = 0$ is $\mathbb{C} \cdot [vol]$.*

4. A general reason for semi-continuity?

Let $G \in Mat(N \times N, \mathbb{C})$ be a semisimple operator with spectrum in $\mathbb{Z} \subset \mathbb{C}$ (i.e a $\mathbb{Z}$-grading on $\mathbb{C}^N$), and we have two formal series

$$A = A_0 + A_1 t + \ldots, K = K_0 + K_1 t + \cdots \in Mat(N \times N, \mathbb{C}[[t]])$$

satisfying equations

$$[A, K] = 0$$

$$t\partial_t K + [A, G] = 0$$

which means that we get a flat connection:

$$\left[t\partial_t + \frac{A}{u}, \partial_u + \frac{K}{u^2} + \frac{G}{u}\right] = 0$$

Then one has special connection $\partial_u + \frac{K_0}{u^2} + \frac{G}{u}$, and generic connection over $\mathbb{C}((t))$ given by $\partial_u + \frac{K}{u^2} + \frac{G}{u}$.

Remark: second equation $t\partial_t K + [A, G] = 0$ implies that series K is uniquely reconstructed from K_0 and series A. Therefore, we get a system of equations on a collection of matrices

$$A_0, A_1, A_2, \ldots ; K_0, G$$

given by

$$[G, A_0] = 0$$
$$[A_0, K_0] = 0$$
$$[A_1, K_0] + \frac{1}{1}[A_0, [G, A_1]] = 0$$
$$[A_2, K_0] + \frac{1}{1}[A_1, [G, A_1]] + \frac{1}{2}[A_0, [G, A_2]] = 0$$
$$[A_3, K_0] + \frac{1}{1}[A_2, [G, A_1]] + \frac{1}{2}[A_1, [G, A_2]] + \frac{1}{3}[A_0, [G, A_3]] = 0$$
$$\cdots$$

Conjecture: *The largest growth of a regular solution of the special connection* $u\partial_u + \frac{K_0}{u} + G$ *is more singular than those of the general one* $u\partial_u + \frac{K}{u} + G$ (over the base field $\mathbb{C}(\!(t)\!)$).

This universal conjecture about equations will imply necessary upper bounds for the applications to non-rationality.