Towards dimension theory for spectral semi-orthogonal decompositions

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1. Recall: semi-orthogonal decompositions from GW invariants

\( X \): projective algebraic variety over \( \mathbb{C} \), with given ample line bundle.

\( \rightsquigarrow \) Gromov-Witten invariants in genus zero, giving potential \( \mathcal{F}_0 \): formal series on \( H^\bullet(X) \) with coefficients in \( \mathbb{Q}[[T]] \).

**Conjecture:** \( \mathcal{F}_0 \) is convergent both for a point in \( H^\bullet(X) \) close to zero, and for \( T \) close 0.

Assume for today’s talk

Assuming additionally \( \Gamma \)-conjecture \( \rightsquigarrow \) nc Hodge structure parametrized by a domain \( M \subset H^\bullet(X, \mathbb{C}) \): connection on the trivial bundle \( H^\bullet(X) \) over \( u \)-plane \( \mathbb{C}_u \)

\[
\left( u \partial_u + \frac{1}{u} K + G \right) \psi(u) = 0 \quad (\Gamma \text{-conjecture gives a lattice which is hypothetically is compatible with Stokes filtrations along rays at } u \to 0)
\]

Operator \( K \) is the operator of quantum product with \( c_1(T_X) \), depends on the point in Frobenius manifold \( M \).

Operator \( G \) is constant, given by

\[
G|_{H^i(X)} = \frac{i-\dim_{\mathbb{C}} X}{2} \cdot \text{id}_{H^i(X)}
\]
Definition: **quantum spectrum** is the spectrum of $K$, a finite subset $\text{Spec}_X \subset \mathbb{C}$ (depends on a point in $\mathcal{M}$).

Consider a purely even affine submanifold $\mathcal{M}^{alg} \subset \mathcal{M}$, given by deformations of quantum product by linear combinations of algebraic classes $H^{alg}_\mathbb{Q}(X) \subset H^{even}(X, \mathbb{Q})$.

**Conjecture:** for any point in $\mathcal{M}^{alg}$ and a choice of disjoint paths from $-\infty$ to points of the corresponding spectrum (Gabrielov paths):

we obtain a semi-orthogonal decomposition $D^b(\text{Coh}(X)) = \langle C_1, \ldots, C_r \rangle$ where $r$ is the number of elements of the spectrum.

All categories $C_1, \ldots, C_r$ are saturated (i.e. smooth and proper), equal to local Fukaya-Seidel categories for the mirror LG dual $(Y, W : Y \to \mathbb{C})$, if it exists.
Examples: 1. $X = \mathbb{P}^n$, the spectrum is $\mu_{n+1} = \{z \in \mathbb{C} | z^{n+1} = 1\}$ (for some point in $\mathcal{M}$)

$\leadsto D^b(Coh(X)) = \langle \mathcal{O}, \ldots, \mathcal{O}(n) \rangle$.

2. Conjectural blow-up formula: If $\tilde{X} = Bl_Y(X)$ where $Y \subset X$ is a smooth closed subvariety of codimension $m \geq 2$, then the spectrum $\text{Spec} \tilde{X}$ is close to

with $(m - 1)$ copies of $\text{Spec}_Y$ around one copy of $\text{Spec}_X$.

3. If $X$ is Calabi-Yau or of general type, the spectrum is always $\{0\}$, get $\langle D^b(Coh(X)) \rangle$. 
2. Towards dimension theory of elementary pieces

We see that sometimes elementary pieces \( C_i = C_{z_i}, \quad z_i \in \text{Spec}_X \) (could be combined as some points of the spectrum collide), are themselves equivalent to derived categories of sheaves on some varieties, of certain \textit{dimensions} \( \leq \dim X \). In general, for a saturated category \( \mathcal{C} \) one can define its \textbf{Serre dimension} as

\[
\dim_{\text{Serre}} \mathcal{C} := \lim_{|k| \to +\infty} \left\{ \frac{i}{k} |\text{Ext}^i(\text{Id}_\mathcal{C}, S^k_\mathcal{C}) \neq 0 \} \subset \mathbb{R}
\]

Here \( S^k_\mathcal{C} : \mathcal{C} \to \mathcal{C} \) is the \textit{Serre functor} as defined by A.Bondal and M.Kapranov:

\[
\text{Hom}_\mathcal{C}(E, F)^* = \text{Hom}_\mathcal{C}(F, S^k_\mathcal{C}E), \quad \forall E, F \in \text{Ob}(\mathcal{C})
\]

In general, Serre dimension could be an empty set, or an interval. For categories \( D^b(\text{Coh}(X)) \) it is exactly the dimension \( \dim X \in \mathbb{Z}_{\geq 0} \).

For Fukaya-Seidel category of \( Y = \mathbb{C}_x, W = x^d, d \geq 2 \) the Serre dimension is \textit{fractional},

\[
= 1 - \frac{2}{d}.
\]

\textbf{Conjecture:} \textit{Serre dimension of an elementary piece is always a number in} \( \mathbb{Q} \cap [0, \dim X] \).
2.1. Hypothetical formula for Serre dimension of an elementary piece

The answer is given purely in terms of differential equation \( \left( u \partial_u + \frac{1}{u} K + G \right) \psi(u) = 0 \):

**Conjecture:**
\[
\dim_{\text{Serre}} C_{z_i} = -2 \min \left\{ s \in \mathbb{Q}_{\leq 0} \mid \exists \text{ solution } \sim \psi_{s,k} \cdot u^s \log(u)^k e^{\frac{z_i}{u}} + \ldots \right\}
\]

**Evidence:** I’ve checked 100s of examples of complete intersections in projective spaces. If \( X \) is a Fano complete intersection of hypersurfaces of degrees \( d_1, \ldots, d_r \) in \( \mathbb{P}^{N-1} \), then the corresponding semi-orthogonal decomposition is

\[
D^b(Coh(X)) = \langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(N - 1 - d_{\text{tot}}) \rangle, \text{ Kuznetsov component}
\]

where \( d_{\text{sum}} := \sum_i d_i \), the spectrum is \( \{0\} \cup \mu_{N-d_{\text{tot}}} \) and the predicted Serre dimension for Kuznetsov component \( C_{z=0} \) is

\[
\dim_{\text{Serre}} C_{z=0} = (N - r - 1) - 2 \frac{N - \sum_i d_i}{\max_i d_i} = \dim X - 2 \frac{N - d_{\text{sum}}}{d_{\text{max}}}
\]
Moreover, there is always a very striking equality
\[
\max\{i \in 2\mathbb{Z} + \dim X \mid i \leq \dim_{\text{Serre}} C_{z=0}\} = \\
= \max\{k \in \mathbb{Z} \mid HH_k(C_{z=0}) \neq 0\} = \\
= \max\{p - q \mid H^{p,q}(X) \neq 0\}
\]

**Analogy:** for any smooth projective variety $X$ we have:
\[
\dim X \geq \max\{p - q \mid H^{p,q}(X) \neq 0\}
\]

The decomposition $\langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(N - 1 - d_{\text{tot}}), \text{Kuznetsov component } C_{z=0} \rangle$ corresponds to a very special point in the Frobenius manifold, with the trivial correction to the ample class. In general, we should deform it by other algebraic cycles in $X$. The spectrum can a priori split into a finer one, and elementary pieces will be decomposed into smaller pieces. This motivates the following
2.2. Semi-continuity conjecture

if we deform a bit parameters (a point in the algebraic part $M^{alg}$ of the Frobenius manifold), so that the deformed eigenvalues of $K$ split, then the corresponding Serre dimension can only decrease.

Together with the blow-up conjecture, this will give a very strong criterium of non-rationality:

$\implies$ if for at least one elementary pieces of s.o.d. of $D^b(Coh(X))$, corresponding to a generic point of $M^{alg}$, its Serre dimension is $> \dim X - 2$, then $X$ is not rational.

Corollary (one of many): any odd-dimensional complete intersection of degrees $(d_1, \ldots, d_r)$ in $\mathbb{P}^{N-1}$ is not rational if $d_{sum} + d_{max} > N$. 
3. Landau-Ginzburg model

Let $Y$ be a noncompact complex manifold of dimension $n \geq 0$, and $W : Y \to \mathbb{C}$ be a holomorphic map. Denote by $\text{Crit}(W) \subset Y$ the critical locus considered as a closed analytic subspace (possibly non-reduced).

**Assumptions:**

1. (the most crucial) $\text{Crit}(W)$ is compact
2. (also important) there exists a Kähler metric on $Y$ (the choice is not a part of the structure, only existence is required)
3. (technical, for later convenience) $\text{Crit}(W)$ is nonempty and connected, and moreover $f(\text{Crit}(W)) = \{0\} \subset \mathbb{C}$.

What follows will not change if we replace $Y$ by any open subset $Y' \subset Y$ containing $\text{Crit}(W)$ (alternatively, one can consider $Y$ as a germ at $\text{Crit}(W)$).

Consider $\mathbb{Z}$-graded complex

$$(\Gamma(Y, \Omega^\bullet_{C^\infty}(Y))[[u]], \text{ differential } d_{\text{tot}} := \bar{\partial} + ud + dW \wedge \cdot)$$

It calculates hypercohomology $R\Gamma(Y, \Omega^\bullet_Y [[u]], ud + dW \wedge \cdot)$. If $\alpha \in \Gamma(Y, \Omega^\bullet_{C^\infty}(Y))\{u\}$ (i.e. not only a formal series in $u$, but a germ of analytic [forms on $Y$]-valued function at $u = 0$ and near
the compact set \( \text{Crit}(W) \) then \( d_{\text{tot}} \alpha = 0 \) means that
\[
d\left(e^\frac{W}{u} u^{\text{Gr}}(\alpha)\right) = 0, \quad \text{Gr}_{\Omega_{C,\infty}^{p,q}} := \frac{q - p}{2}
\]

**Conjecture (Hodge-de Rham degeneration for LG models):** Cohomology of \( d_{\text{tot}} \) is a free finite rank (equivalently, flat) \( \mathbb{C}[[u]] \)-module.

It is true in algebro-geometric situation by irregular Hodge theory, but in general I don't think it follows from Mochizuki, or Barannikov-me (maybe from Saito or Sabbah).

Let us assume that \( Y \) is endowed with an everywhere non-vanishing holomorphic volume form \( \text{vol} \in \Gamma(Y, \Omega^n) \), \( n = \dim Y \).

Then in the case when \( \text{Crit}(W) \) is connected and non-empty, there is a canonical 1-dimensional subspace at the fiber at 0:
\[
[\text{vol}] \in H^n(Y, \Omega^\bullet_Y, dW \wedge \cdot) \rightarrow \\
\rightarrow H^n(\text{Crit}(W), (\Omega^\bullet_Y)_{\text{Crit}(W)}) \rightarrow \\
\rightarrow H^0(\text{Crit}^{\text{red}}(W), \Omega^n_Y) = \mathbb{C} \ni 1
\]

**Conjecture:** the leading growth of solutions appears in the exactly one-dimensional subspace of cohomology of \( d_{\text{tot}} \), and its reduction at \( u = 0 \) is \( \mathbb{C} \cdot [\text{vol}] \).
4. A general reason for semi-continuity?

Let \( G \in Mat(N \times N, \mathbb{C}) \) be a semisimple operator with spectrum in \( \mathbb{Z} \subset \mathbb{C} \) (i.e. a \( \mathbb{Z} \)-grading on \( \mathbb{C}^N \)), and we have two formal series
\[
A = A_0 + A_1 t + \ldots, \quad K = K_0 + K_1 t + \ldots \in Mat(N \times N, \mathbb{C}[[t]])
\]
satisfying equations
\[
\begin{align*}
[A, K] &= 0 \\
t \partial_t K + [A, G] &= 0
\end{align*}
\]
which means that we get a flat connection:
\[
\left[ t \partial_t + \frac{A}{u}, \partial_u + \frac{K}{u^2} + \frac{G}{u} \right] = 0
\]

Then one has special connection \( \partial_u + \frac{K_0}{u^2} + \frac{G}{u} \), and generic connection over \( \mathbb{C}((t)) \) given by
\[\partial_u + \frac{K}{u^2} + \frac{G}{u} \).

Remark: second equation \( t \partial_t K + [A, G] = 0 \) implies that series \( K \) is uniquely reconstructed from \( K_0 \) and series \( A \). Therefore, we get a system of equations on a collection of matrices
\[
A_0, A_1, A_2, \ldots; K_0, G
\]
given by
Conjecture: The largest growth of a regular solution of the special connection $u \partial_u + \frac{K_0}{u} + G$ is more singular than those of the general one $u \partial_u + \frac{K}{u} + G$ (over the base field $\mathbb{C}((t))$).

This universal conjecture about equations will imply necessary upper bounds for the applications to non-rationality.