# On integrable Systems and 3D Mirror Symmetry

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# Introduction

The 3D mirror Symmetry conjecture predicts that 3d, $\mathcal{N} = 4$ theories come in pairs such that the A-twist of each is equivalent to the B-twist of the other. The moduli of these theories have two branches, Higgs and Coulomb, that are supposed to be interchanged.

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An important source of examples is furnished by theories of class $S$. These are obtained by compactifying a 6d theory, depending on a group $G$, on a Riemann surface with punctures. The dual theory is a quiver gauge theory $S^\vee$.
The works of Hanany et al (in physics) and of Braverman, Finkelberg and Nakajima (math), give an explicit construction of the Coulomb branch of $S^\vee$, which is the Higgs branch of $S$. The other branch (Coulomb of $S$, Higgs of $S^\vee$) is the total space of the Hitchin fibration associated to the theory.

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Joint work with Aswin Balasubramanian and Jacques Distler.

# Higgs bundles and integrable systems

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Given a base curve $C$, a reductive group $G$, and a line bundle $L$ on $C$, the total space $Higgs_{C,G,L}$ is the moduli space of $L$-valued $G$-Higgs bundles on the curve $C$. For $G = GL(N)$, this means pairs $(V, \phi)$ where $V$ is a rank $N$ vector bundle and $\phi : V \to V \otimes L$ is the Higgs field. For general $G$, replace $V$ by a principal $G$-bundle $P$ over $C$, and $\phi \in \Gamma(C, ad(P) \otimes L)$. (You can impose a stability condition and get a moduli space, or not, and get a stack.)

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The line bundle L is often taken to be $K_C$ or $K_C(D)$ for some effective divisor $D$: the punctures.

# Higgs bundle and integrable systems

The base $\mathcal{B}$ is $\Gamma(C, L \otimes \mathfrak{t}/W)$. A point $b \in B$ determines a $W$-Galois cover, called the cameral cover $\widetilde{C} \to C$, namely the inverse image in $\Gamma(C, L \otimes \mathfrak{t})$ of the corresponding section. The choice of a representation $\rho$ of $G$ maps the cameral cover to the spectral cover $\widetilde{C}_\rho$ sitting in the total space of $L$ and parametrizing the spectra of $\rho(\phi)$ over points of $C$. For faithful $\rho$, the cameral and spectral data are equivalent data.

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The quotient map $\mathfrak{g} \to \mathfrak{t}/W$ applied to the Higgs field $\phi$ determines a family parametrized by $C$ of $L$-valued points of $\mathfrak{t}/W$. This is the Hitchin map $h : Higgs \to \mathcal{B}$: it sends $\phi$ to its cameral cover, or spectral cover, or the family of its spectra.

# Hitchin and Markman systems

For $L = K_C$, Hitchin proved that *Higgs* is (holomorphically) symplectic and the map to $\mathcal{B}$ is Lagrangian. In the meromorphic case, i.e. for $L = K_C(D)$, where $D$ is an effective divisor on $C$, Markman and Bottacin proved that *Higgs* is Poisson. Symplectic leaves are the inverse images of adjoint orbits in $\mathfrak{g}$ under the residue map $\mathcal{B} \to \Gamma(D, O_D \otimes \mathfrak{t}/W)$.

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We will consider only the tame case ($D$ is reduced) and focus mostly on nilpotent orbits.

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Four dimensional superconformal field theories have been a subject of study for many years. Recently, a class of such theories that admit a geometric construction from six dimensions have received greater attention. This includes several familiar theories with UV Lagrangians and more mysterious theories for which there is no known UV Lagrangian.

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In this realization from six dimensions, the Hitchin system plays an important role. Specifically, the Coulomb branch associated to the four dimensional theory can be described as the base $\mathcal{B}$ of Hitchin's integrable system associated to a simply laced Lie algebra $\mathfrak{g}$ and the UV curve $C_{g,n}$. The choice of the lie algebra $g$ parameterizes the available 6d $(0, 2)$ theories and the choice of $C_{g,n}$ determines the 2d surface on which we compactify the 6d theory (together with a partial twist). At the locations of the $n$ punctures, we insert four dimensional defects of the 6d $(0, 2)$ theory. The insertions of these defects affect the behaviour of the Hitchin system at these punctures.

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In order to obtain superconformal field theories (SCFT) using tame defects, we additionaly require that $Res(\phi) = a$ be a nilpotent element in the Lie algebra $\mathfrak{g}$. What really matters is the $\mathfrak{g}$-conjugacy class to which the element $a$ belongs. So we label the Hitchin boundary condition by the nilpotent orbit $\mathcal{O}_a$ to which the element $a$ belongs. We will sometimes call this nilpotent orbit the Hitchin orbit $\mathcal{O}_H$ associated to the defect.

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# The physics background

The absence of such discrete data for defects in type $A$ is related to the fact that component groups of centralizers of nilpotent orbits are always trivial in type $A$. Let us define

$$A(\mathcal{O}_a) = C_{\mathfrak{g}}(a)/C^0_{\mathfrak{g}}(a)$$

to be the group of components of the centralizer of nilpotent orbit $\mathcal{O}_a$. Here, $C_{\mathfrak{g}}(a)$ is the centralizer of $a$ and $C^0_{\mathfrak{g}}(a)$ is its connected component.

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# Weakly coupled gauge groups

An important feature of this geometric realization from six dimensions is that the space of marginal parameters associated to the SCFT is identified with the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ of complex structures on $C_{g,n}$. Moving/restricting to a stratum of (complex) codimension one in $\overline{\mathcal{M}}_{g,n}$ corresponds to the appearance of a weakly coupled gauge group with an associated gauge coupling that is related to plumbing fixture parameter $q$ by

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In basic examples, this gauge group is a compact group $G$ associated to the complex Lie algebra $\mathfrak{g}$.

# Weakly coupled gauge groups

But, interestingly, there are cases where the weakly coupled gauge group is a proper subgroup $H$ of $G$. This often corresponds to the surprising physical phenomenon of a certain SCFT with $G$ group being S-dual to a SCFT which can be described as an $H$ gauge theory coupled to non-Lagrangian matter. The first such example was due to Argyres-Seiberg and this was later given a Class S interpretation in Gaiotto's work. This has then been generalized further in the Tinkertoys program of Distler et al.

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# Nilpo orbits and spectral curves: local story, smooth curve

We work with the Hitchin system for $G = SL(n)$ on a smooth curve $C$ with marked points in a reduced divisor $D = \sum_k p_k$.

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| Nahm $\mathcal{O}_N$ | Nahm WDD | Hitchin $\mathcal{O}_H$ | $\phi_2$ | $\phi_3$ | $\phi_4$ |
|---|---|---|---|---|---|
| $[1^4]$ | 0-0-0 | $[4]$ | 1 | 1 | 1 |
| $[2,1^2]$ | 1-0-1 | $[3,1]$ | 1 | 1 | 2 |
| $[2^2]$ | 0-2-0 | $[2^2]$ | 1 | 2 | 2 |
| $[3,1]$ | 2-1-0 | $[2,1^2]$ | 1 | 2 | 3 |

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We will continue to use the Hitchin labels to identify the defects at the punctures.

# Nilpo orbits and spectral curves: local story, smooth curve

The spectral curve has equation $w^n = \sum_{i=2,\dots,n} a_i w^{n-i}$. Here $a_i$ is a pluridifferntial on $C$ i.e. a section of $(K_C)^i$ with allowed pole of order $\leq \pi^i$. Alternatively, it is a section on $C$ of $(K_C(D))^i$ with a zero of order $\geq \chi^i$, where $\pi^i + \chi^i = i$.

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The coefficient $a_i$ is a section of a line bundle of degree:

$$d^i := i(2g-2+deg(D)) - \sum_{k\in D} \chi_k^i = (2g--)2i + \sum_{k\in D} \pi_k^i.$$

The space of all such sections is a vector space of dimension:
$b^i \geq max(d^i - g + 1, 0)$.
We say the system is **OK** if equality holds, i.e. if these line bundles have at most one non vanishing cohomology. This is a simplification of the physicists trichotomy of **good, bad and ugly** theories.
(OK <=> not bad <=> good or ugly.)

# Hitchin system on a singular curve

Replace the smooth base curve $C$ by a Gorenstein curve: it can be singular, as long as there is still a good canonical line bundle $K_C$. Any curve that is a divisor in a smooth surface will do: the canonical line bundle is given by the adjunction formula. This includes any curve whose only singularities are nodes: sections of the canonical bundle are 1-forms on the normalization with first order poles allowed at the (inverse images of) the nodes, with opposite residues at the two inverse images of each node.

# Hitchin system on a Gorenstein curve

As in the smooth case, the Hitchin system for $C$ and a reductive group $G$ is the space *Higgs* of (isomorphism classes of) $K_C$-valued $G$-Higgs bundles on $C$. A $G$-Higgs bundle is a pair $(V, \phi)$ where $V$ is a principal $G$-bundle on $C$ and $\phi \in H^0(C, ad(V) \otimes K_C)$. For now we will focus on the case $G = GL(n)$, so $V$ is a vector bundle and $\phi : V \to V \otimes K_C$; or $G = SL(n)$, where $det(V)$ is required to be $O_C$ and the trace of $\phi$ is required to vanish . As in the smooth case, one can consider a GIT version where the Higgs bundles are subject to a stability condition; or one can allow all Higgs bundles and work with the resulting stack.

# Hitchin system on a Gorenstein curve

Also as in the smooth case, the spectral curve of $(V, \phi)$ is the curve in the total space of $K_C$ defined by the vanishing of the characteristic polynomial of the endomorphism $\phi$. The Hitchin base $B$ is defined to be the space of all spectral curves. This can be identified with the vector space:
$B := \oplus_i H^0(C, (K_C)^i)$
where $i$ runs over the degrees of the $G$-invariants. For $G = SL(n)$, these degrees are $i = 2, \ldots, n$. The Hitchin map $h : Higgs \to B$ sends $(V, \phi)$ to (the coefficients of) the characteristic polynomial of $\phi$. There is a natural analog of the above allowing poles on a divisor $D$ consisting of distinct smooth points of $C$.

# The standard node

When the Hitchin orbits at the punctures are sufficiently big, we get a node that is unrestricted. We call such a node the "standard" node. When $\mathfrak{g} = A_1$ or if $g \geq 1$, then every node is a standard node. On a four punctured sphere, a (separating) standard node occurs when the Hitchin orbits are sufficiently big. We will see an example where some of the Hitchin orbits are small, resulting in a node that is restricted, or non-standard.

# Degeneration of Hitchin systems

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# Degeneration of Hitchin systems

Sufficient condition for flatness: If the basic line bundle $K_{C_0}(D)$ is very ample on the limiting base curve, then the (Poisson) Hitchin system for $(C_0, D, G)$ is a flat limit of the Hitchin systems for $(C_t, D, G)$. In particular, the same holds for each symplectic leaf. For example, if the family of curves is the pencil of conics through 4 points in $P^2$ as above, the condition is simply that $K_{C_0}(D)$ has positive degree on each of the two components of $C_0$. This simply requires at least two marked points on each of the two components.

# Degeneration of B: L, C, R.

As we saw, the coefficient $a_i$ on a smooth $C_t$ is a section of a line bundle $L_{i,t}$ of degree: $d^i := -2i + \sum_{k \in D} \pi^i_k$.
Let $L_{i,0}$ be the limiting line bundle on the reducible curve $C_0$, let $L_{i,l}$ and $L_{i,r}$ be its restrictions to the left or right component of $C_0$, and let $d^i_l, d^i_r$ be their degrees. These are given by analogous formulas, except that the sum on the left is now over $k \in D_l \cup \{p\}$, with $\pi^i_p := i - 1$, and similarly for the sum on the right. (The pole orders $\pi^i_p = i - 1$ allow for arbitrary (e.g. regular) nilpotents at the node $p$.)
Let $B_l$ be the subspace of $B_0$ parametrizing spectral curves whose coefficients vanish on $C_r$, and vice versa, $B_r$ is the subspace of $B_0$ parametrizing spectral curves whose coefficients vanish on $C_l$. Let $B_c$ be the quotient $B_0/(B_l + B_r)$.

# Degeneration of B: L, C, R.

The dimensions of the spaces of sections are:
$b_l^{i'} := max(d_l, 0)$
$b_r^{i'} := max(d_r, 0)$
$b_c^{i} := 1$ if $d_l \geq 0$ and $d_r \geq 0$,
$b_c^{i} := 0$ otherwise
The question is whether these dimensions equal their unprimed counterparts $b_l^i, b_r^i$ arising as limits from the Hitchin systems on the smooth curves.

# Degeneration of B: L, C, R.

We may as well consider the following:
General setup : we have a family of smooth rational curves $C_t$ degenerating to a 2-component reducible rational curve $C_0$, and a line bundle $L_t = O_{C_t}(d)$ specializing to $L_0 = (O(d_l), O(d_r))$ with $d_l + d_r = d$. What is the limit of the vector space of sections $H^0(P^1, O(d))$?
If $d_r \geq 0, d_l < 0$, the limit gives a proper subspace of $H^0(P^1, O(d_r))$. The codimension is $-d_l$, which equals $h^1(P^1, O(d_l)) + 1 = h^1(P^1, O(d_l - 1))$. One sees immediately that this limiting subspace is the kernel of the coboundary map: $H^0(P^1, O(d_r)) \to H^1(P^1, O(d_l - 1))$, coming from the restriction SES:
$0 \to O_{P^1_l}(d_l - 1) \to L_0 \to O_{P^1_r}(d_r) \to 0.$

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Returning to our degenerating Hitchin systems, we see that the limit can be a **proper** subspace of $H^0(P^1, O(d^i_r))$ only when the line bundle $L_{i,l}$ has higher cohomology, i.e. when its degree $d^i_l$ satisfies $d^i_l \leq -2$. (The condition is necessary and sufficient when $L_0$ itself has no higher cohomology.)

# Example

Consider as usual a reducible curve $C_0 = C_l \cup C_r$ in $M_{1,4}$.
$\pi = 1, 1, 1$ and $1, 1, 1$ on left, $1, 2, 3$ and $1, 2, 3$ on right.
$\chi = 1, 2, 3$ and $1, 2, 3$ on left, $1, 1, 1$ and $1, 1, 1$ on right.
(The three entries for each quantity correspond to $i = 2, 3, 4$.)
On the smooth curves:
degrees $= 0, 0, 0$
dims $= 1, 1, 1$
On the limiting nodal curve:
$d_l = 0, -1, -2$ (note the occurrence of $-2$!)
$d_r = 0, 1, 2$
$b_l = 0, 0, 0$
$b'_r = 0, 1, 2$ (This is the dim on right side at $t = 0$, ignoring the limit.)
$b_c = 1, 0, 0$

# Example, continued

For $i = 4$, the SES for restricting from the nodal curve to its right component is:
$0 \to O_{P^1_l}(-3) \to L_{4,0} \to O_{P^1_r}(2) \to 0.$
On cohomology this gives:
$0 \to H^0(L_{4,0}) \to H^0(P^1_r, O(2)) \to H^1(P^1_l, O(-3)) \to 0,$
so: $b^4_r = h^0(L_{4,0}) = 1$. This is the first example when $d^i_l \leq -2$ and consequently $b^i_r < b^{i'}_r$.
For a regular nilpotent, we saw in section 1 that the vanishing orders of the $a_i$ are $1, 1, 1$. This what we get from the $b^{i'}_r$. The discrepancy between these and $b^i_r$ for $i = 4$ means that on the right component, the orders of vanishing are $1, 1, 2$. This corresponds to the subregular orbit of SL(4), with Hitchin partition $(3, 1)$.

# Bundle of bases

From the above example one learns that in order for the Hitchin bases to fit in a flat family, the naive line bundles need to be modified so as to eliminate their $h^1$. This turns out to be the only obstruction.

Let $\overline{\mathcal{C}}$ be the universal curve over $\overline{\mathcal{M}_{g,n}}$. For $k = 2, \ldots, n$, let $L_k$ be the line bundle whose fiber over $(C, D)$ is:

$$L_k := (K_C(D))^{\otimes k}(-\sum_{p_i \in D} \chi_k^i p_i),$$

where $\chi_k^i$ are the orders of vanishing imposed by the specified nilpotent orbits at the $p_i$.

# Bundle of bases

For a partition $S \cup \check{S} = D$, let $M_S$ be the Cartier divisor in $\overline{\mathcal{M}_{g,n}}$ of reducible curves $C_S \cup C_{\check{S}}$ such that $S \subset C_S$ and $\check{S} \subset C_{\check{S}}$ and $g(C_S) = 0$. Let:

$$L'_k := L_k(-\sum_S n_k^S M_S),$$

where

$$n_k^S := max(0, k-1-\sum_{p_i \in S}(k-\chi_k^i)) = h^1(C_S, L_k).$$

# Bundle of bases

THEOREM: The family of $SL(N)$-Hitchin bases $\mathcal{B} := \oplus_{k=2}^{N} \pi_* L'_k$ is a vector bundle over $\overline{\mathcal{M}_{g,n}}$.

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Idea: careful pruning of branches.

# Fiberwise structures

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The proofs require relatively minor modifications of the corresponding proofs for smooth $C$ in Markman.

# Properness

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# Classification of non-standard nodes

THEOREM: In an OK $SL(n)$ Hitchin system, the $\chi_k := 1 + n_k$ are the vanishing orders for a (unique) nilpotent orbit $O$ in $sl(N)$: the Hitchin orbit at the node. The graded dimension of the center $B_C$ is either $b_k^C = (1, \ldots, 1, 0, \ldots, 0)$ or $b_k^C = (1, 0, 1, 0, 1, \ldots, 1, 0, \ldots, 0)$, corresponding to a weakly coupled group $H = SL(k)$ or $H = Sp(k)$.

# OK theories and semistable Higgs bundles

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Simpson: the irreducible character variety is non empty iff $D := \sum_{a=1}^{n+1} dim(C_a) - 2(N^2-1) \geq 0$ and $\sum_{a=1}^{n} r_a \geq N$, where $C_a \subset SL(N)$ is the holonomy conjugacy class, $m_a$ is the largest multiplicity of its eigenvalues, and $r_a := N - m_a$.

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In this case $D$ is the dimension of the character variety, or, by NAHT, the dimension of $Higgs_C$.
THEOREM: In the regular case, Simpson's conditions are equivalent to the OK condition for the line bundles $L_k$.

# OK theories and semistable Higgs bundles

In the regular case, demanding that the strongly parabolic Higgs bundle be OK is necessary and sufficient for the corresponding irreducible character variety to be non-empty. By the NAHT, this is the same as demanding the non-emptiness of the corresponding moduli space of semistable Higgs bundles. The novel feature here is that semi-stability (in the Higgs sense) admits a translation to a condition on the line bundles $L_k$ appearing in the description of the Hitchin base. If we relax the assumption that one of the residues of the Higgs field is regular, then Simpson's two conditions are known to be necessary, but not sufficient for the nonemptiness of the character variety. A natural guess is that the OK condition on the line bundles $L_k$ , which is stronger than Simpson's conditions in this case, might be sufficient. Is the OK condition on strongly parabolic Higgs bundles sufficient to ensure the non-emptiness of the corresponding character variety? Is it necessary and sufficient?

# Further singularities

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# Resolution of singularities

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Blow up once. The new curve is still a spectral cover, for a smaller group. The new nilpotent orbit is obtained by removing the first column of the Nahm partition, or the first row of the Hitchin partition.

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# Comments and questions

(1) A key puzzle for physicists is that Coulomb branches of S-class theories decompose as hyperKahler quotients of products of Coulomb branches of limiting components. The group you need to divide by is often the full gauge group, but sometimes a proper subgroup. We give an algorithm for determining which is the case and what the weakly coupled gauge subgroup is. There is also a realization due to BFN which - counter to physics intuition - is a HK quotient by the full group. We are trying to explain this alternative.

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(2) Another intriguing question is the structure of the base of the fibration on a symplectic leaf. In type A our algorithm shows that this is always a vector space. In other types, extensive experimentation suggests the same, but all that is known in general is that it has a flat structure.

## Comments and questions

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(4) Construct the proper Hitchin map over $\overline{\mathcal{M}_{g,n}}$.

THANK YOU FOR YOUR ATTENTION!