From categories to curve counts II: variations of semi-infinite Hodge structures

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November 6, 2015

Joint with Timothy Perutz and Nick Sheridan
Homological Mirror Symmetry Conference, University of Pennyslvania
preprints: `arXiv:1510.03839` and `in preparation`

Outline

- Main Result
- Enumerative Mirror symmetry
- Hodge-theoretic mirror symmetry
- HMS implies Hodge MS

Overview

This talk is about relating different types of mirror symmetry (homological and enumerative). The idea that there should be such a relationship goes back to Kontsevich's 1994 ICM address where homological mirror symmetry was first proposed.

We take an approach to this that is strongly based on ideas of Barannikov and Kontsevich. This perspective (which passes through 'non-commutative Hodge structures') has been clarified and developed by many others [Katzarkov-Kontsevich-Pantev].

Geometric setup

A side

$X := (X^n, \mathbf{D})$ *integral* Calabi-Yau symplectic manifold, $\mathbf{D} \subset X$ simple normal crossings divisor representing $[\omega]$.
$\mathbb{K}_A := \mathbb{C}((Q))$ coefficients, i.e., invariants defined over $\mathcal{M}_A = \textit{Spec}\, \mathbb{K}_A$.

B side

$Y^n \to \mathcal{M}_B$ connected n-dimensional smooth scheme over $\mathcal{M}_B := \textit{Spec}\, \mathbb{K}_B$, $\mathbb{K}_B := \mathbb{C}((q))$ which is *Calabi-Yau*, i.e., with trivial canonical sheaf. Assume *maximally unipotent monodromy*.

Running example

A side: quintic 3-fold

$$X^5 := \{\sum_i z_i^5 = 0\} \subset \mathbb{CP}^4,$$

$$D \subset X^5,\ D := \{\prod z_i = 0\}, [\omega] := P.D.([D])$$

$\mathbb{K}_A := \mathbb{C}((Q))$ coefficients, i.e., invariants defined over $\mathcal{M}_A = \mathit{Spec}\ \mathbb{K}_A$.

B side: mirror quintic family

Y^5: a family over $\mathcal{M}_B = \mathit{Spec}\ \mathbb{K}_B$, $\mathbb{K}_B := \mathbb{C}((q))$:

$$Y^5 := \text{ (a crepant resolution of) } \tilde{Y}^5/G,$$

$$\tilde{Y}^5 := \left\{ -z_1 \dots z_5 + q \sum_{j=1}^{5} z_j^5 = 0 \right\} \subset \mathbb{P}^4_{\mathbb{K}_B},\ G \cong (\mathbb{Z}/5)^3$$

Enumerative Mirror Symmetry

Yukawa couplings

Part of the enumerative *A*-model

$n_d := \#$ degree d rat'l curves on X^5. ($n_1 = 2875$, $n_2 = 609250$, $n_3 = 317206375, \ldots$)
$\rightsquigarrow$ a power series in $\mathbb{K}_A := \mathbb{K}((Q))$: interpret counts as $Yuk_A \in Sym^3(\Omega^1\mathcal{M}_A) \cong \mathbb{K}_A$:

$$\begin{aligned} Yuk_A(Q\partial_Q, Q\partial_Q, Q\partial_Q) &:= \langle H, H, H\rangle_{0,3} \\ &= 5 + \sum_{d=1}^{\infty} n_d d^3 \frac{Q^d}{1-Q^d}. \end{aligned}$$

Part of the enumerative *B*-model

Periods of differential forms. Define
$Yuk_B(q) \in Sym^3(\Omega^1\mathbb{K}_B) \cong \mathbb{K}_B$ by
$Yuk_B(q\partial_q, q\partial_q, q\partial_q) = \int_Y \Omega \wedge \nabla^3_{q\partial q}\Omega$ for a particular Ω.

Enumerative mirror symmetry (continued)

Predictions

(Enumerative) mirror symmetry predicts:

There exists $\psi : \mathcal{M}_A \xrightarrow{\sim} \mathcal{M}_B$ the *mirror map* intertwining Yuk_A and Yuk_B.
(moreoever it predicts an explicit formula for $\psi^* : \mathbb{K}_B \to \mathbb{K}_A$, $Q \mapsto Q(q) := q + a_2 q^2 + \cdots)$

Yuk_B can be computed in terms of solutions to Picard Fuchs.

$\rightsquigarrow$ predictions for the numbers n_d [CdOGP, 1991]. Proved in 1996 by Givental, Lian-Liu-Yau.

Homological Mirror Symmetry

$(X, \mathbf{D})$ and Y a pair as before. (i.e., (X^5, Y^5). $\mathbb{K}_A = \mathbb{C}((Q))$, $\mathbb{K}_B = \mathbb{C}((q))$.

Definition

X and Y are *homologically mirror* if there is a map $\tilde{\psi}^* : \mathbb{K}_A \xrightarrow{\sim} \mathbb{K}_B$ and a quasi-equivalence

$$tw^{\pi}\mathcal{F}uk(X, \mathbf{D}) \cong \tilde{\psi}^* D^b_{dg}Coh(Y).$$

Yesterday: [Perutz-Sheridan] X and Y are homologically mirror under 'Core HMS' hypotheses.

Example

$X := (X^5, \mathbf{D})$ quintic threefold, Y^5 its mirror.

Theorem* (Sheridan)

X and Y are homologically mirror; that is $tw^{\pi}\mathcal{F}(X, \mathbf{D}) \cong \tilde{\psi}^* D^b_{dg}Coh(Y)$ for some (undetermined) $\tilde{\psi} : \mathcal{M}_A \to \mathcal{M}_B$ of the form $\tilde{\psi}^*(q) := \pm Q + O(Q^2)$.

Questions:

- Does $\tilde{\psi}$ agree with the standard mirror map? **Theorem** [G-P-S]: *Yes*.
- Does this statement imply, without any further computations, enumerative mirror symmetry for X and Y? **Theorem** [G-P-S]: *Yes*.
- HMS should ideally be a statement about equivalences of *Calabi-Yau categories*...? **Theorem** [G-P-S]: *It can be canonically enhanced to one*.

Remarks/caveats

- The proof goes via showing HMS implies a form of *Hodge-theoretic mirror symmetry*, following the ideas of Barannikov and Kontsevich.
- The result as stated is contingent on a (partially proven in the literature) B side conjecture. Even without it, we can conclude that $D^b_{dg}Coh(Y)$ (and $Fuk(X, \mathbf{D})$) knows about the relevant genus-0 'enumerative' GW counts in X.
- An interesting aspect of this, going back to Kontsevich, is that known HMS proofs to date (including Sheridan's) involve only finitely many computations. Known proofs of enumerative mirror symmetry involve computations to all orders.

Hodge theoretic mirror symmetry

Definition (c.f., Barannikov)

A *(polarized) pre-* VSHS *(Variation of Semi-infinite Hodge Structure), resp.* VSHS *over a base* $\mathcal{M}$*, is the data of*

- *A* sheaf $\mathcal{E}$ *(resp. a* finite dimensional vector bundle*) over* $\mathcal{M}$*, equipped with:*
- *A* filtration *of* $\mathcal{E}$*:* $\cdots \subset F^p\mathcal{E} \subset F^{p-1}\mathcal{E} \subset \cdots$*,*
- *A* flat connection $\nabla : T\mathcal{M} \otimes \mathcal{E} \to \mathcal{E}$ *satisfying* Griffiths transversality*:* $\nabla F^p \subset F^{p-1}$*,*
- *A* covariantly constant *(resp.* non-degenerate*) pairing* $\langle \cdot, \cdot \rangle$ *on* $\mathcal{E}$ *(that behaves well with respect to the filtration).* (polarization)

Example (On the B model)

$Y \to \mathcal{M}_B$ a complex CY family as before, *classical variation of Hodge structure*:

$$V^B(Y) = \begin{cases} \mathcal{E} \to \mathcal{M}_B,\ \mathcal{E}_m := H^\bullet(\mathcal{Y}_m) \\ F^p\mathcal{E} \text{ Hodge filtration} \\ \nabla \text{ Gauss-Manin connection} \\ (\alpha, \beta)_m = i^{|\alpha|} \int_{\mathcal{Y}_m} \alpha \cup \beta \end{cases}$$

Example (On the A model)

X an (integral) symplectic CY as before, $\mathcal{M}_A := Spec\, \mathbb{C}((Q))$:

$$V^A(X) = \begin{cases} \mathcal{E} \to \mathcal{M}_A,\ \mathcal{E} := H^\bullet(X; \mathbb{C}((Q))) \\ F^p\mathcal{E} := H^{\bullet \leq p}(X) \\ \nabla_{Q\partial_Q}\alpha := Q\partial_Q(\alpha) - [\omega] \star_Q \alpha \\ (\alpha, \beta) := (-1)^{n(n+1)/2} \int_X \alpha \cup \beta \end{cases}$$

Hodge-theoretic mirror symmetry

An *isomorphism* of VSHS $\mathcal{E}_1 \to \mathcal{M}_1$, $\mathcal{E}_2 \to \mathcal{M}_2$ is a commutative diagram

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow[\sim]{f} & \mathcal{E}_2 \\ \downarrow & & \downarrow \\ \mathcal{M}_1 & \xrightarrow[\sim]{\psi} & \mathcal{M}_2 \end{array}$$

where f intertwines the VSHS data.

Definition

X and *Y* as before are *Hodge-theoretically mirror* if there is an isomorphism of VSHS:

$$V^A(X) \cong V^B(Y).$$

Canonical coordinates

- **Classical (crucial) fact**: A VSHS of correct type (i.e., one associated to a maximally unipotent family) over $\mathcal{M}$ determines *canonical coordinates* on the base $\mathcal{M}$, unique up to $\mathbb{C}^*$-scaling.
- Given any isomorphism of VSHS of this type

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow[\sim]{f} & \mathcal{H}_2 \\ \downarrow & & \downarrow \\ \mathcal{M}_1 & \xrightarrow[\sim]{\psi} & \mathcal{M}_2 \end{array}$$

  $\psi : \mathcal{M}_1 \to \mathcal{M}_2$ is *uniquely* determined (up to $\mathbb{C}^*$), as it must match up canonical coordinates.
- Thus, for Hodge-theoretically mirror pairs, the corresponding *mirror map* $\psi : \mathcal{M}_A \to \mathcal{M}_B$ is uniquely determined, again up to $\mathbb{C}^*$ ambiguity.

Canonical coordinates and normalizations (cont'd)

- A *volume form*, following the classical notion, is a non-vanishing section of $F^n\mathcal{E}$ (i.e., in $H^0(Y,\Omega^n)$). If one exists, there is a $\mathbb{K}^*$ ambiguity.
  Any VSHS of 'correct type' determines a unique (up to $\mathbb{C}^*$) *Hodge-theoretically normalized volume form*.
- In particular, the VSHS also determines the Yukawa couplings (and other enumerative invariants…).

Main theorem

(X, Y) as before.

Theorem* (G.-Perutz-Sheridan)

Suppose $(X, \mathbf{D})$ *and* Y *are homologically mirror, i.e., for some* $\tilde{\psi} : \mathcal{M}_A \xrightarrow{\sim} \mathcal{M}_B$, $tw^{\pi} Fuk(X, \mathbf{D}) \cong \tilde{\psi}^* D^b_{dg} Coh(Y)$.
Then, X *and* Y *are Hodge theoretically mirror, over the same map* $\tilde{\psi}$.

Corollary

1. $\tilde{\psi}$ *must preserve canonical coordinates. As* $\tilde{\psi}'(0) = \pm 1$ *this determines* $\tilde{\psi}$ *up to* ± 1.
2. *It follows that the* A *and* B *Yukawa couplings agree.*
3. *HMS here refines to an equivalence of saturated (or smooth and proper)* Calabi-Yau *categories, for instrinsically chosen CY structures on each side.*

Categorical VSHS

Proposition (well known)

To a proper $\mathbb{Z}$*-graded* A_∞ *category* $\mathcal{C}$ *over* $\mathbb{K} := Spec\ \mathbb{C}((t))$*, one can associate the following* Morita invariant *pre-* VSHS*:*

1. $\mathcal{E} := \mathrm{HP}_\bullet(\mathcal{C})$ periodic cyclic homology *over* $\mathcal{M} := Spec\ \mathbb{K}$*, together with its* non-commutative Hodge filtration $F^p\mathcal{E} := \mathrm{im}(u^p\mathrm{HC}^-_\bullet(\mathcal{C}) \to \mathrm{HP}_\bullet(\mathcal{C}))$.
2. $\nabla : T\mathcal{M} \otimes \mathrm{HP}_\bullet(\mathcal{C}) \to \mathrm{HP}_\bullet(\mathcal{C})$ Getzler's Gauss-Manin connection *[Getzler] (where* $\mathcal{M} := Spec\ \mathbb{K}$*).*
3. *If* $\mathcal{C}$ *is weakly Calabi-Yau, Shklyarov's higher residue pairing:* $(-,-)_{res} : \mathrm{HP}_\bullet(\mathcal{C}) \otimes \mathrm{HP}_\bullet(\mathcal{C}) \to \mathbb{K}$. *[Shklyarov]*

If $\mathcal{C}$ is furthermore *homologically smooth*, and the *non-commutative Hodge-de Rham conjecture* holds, then this is a VSHS. See [Sheridan:2015] for formulae with uniform conventions, along with an explicit proof of Morita invariance.

Remark

In the symplectic CY ($\mathbb{Z}$-graded situation), there is nothing really *semi-infinite* really going on. (but the semi-infinite versions naturally arises non-commutatively, and e.g., for Fanos/non-$\mathbb{Z}$-graded settings).

Proof of main Theorem

There is a diagram of isomorphisms of VSHS:

$$\begin{array}{ccc}
\mathrm{HP}_\bullet(tw^\pi Fuk(X,\mathbf{D})) & \xrightarrow[\cong]{HMS_*} & \mathrm{HP}_\bullet(D^b_{dg}Coh(Y)) \\
\textbf{Thm } [\mathrm{GPS}] \downarrow \cong & & \cong \downarrow Conj \\
V^A(X) & & V^B(Y)
\end{array}$$

Comparing VSHS: B side

There are known (filtration-preserving) isomorphisms

$$\mathfrak{I} : \mathrm{HP}(D^b_{dg}Coh(Y)) \xrightarrow{\sim} \mathrm{HP}(Y) \xrightarrow{I_{HKR}} H^{odd/even}(Y; \mathbb{K}_B).$$

[Keller, Weibel] Denote by $\tilde{\mathfrak{I}} := (td^{1/2}(Y) \wedge -) \circ \mathfrak{I}$.

Conjecture

- $\tilde{\mathfrak{I}}$ *intertwines connections (the Getzler-Gauss-Manin connection with the usual Gauss-Manin connection). [Cattaneo-Felder-Willwacher]*
- $\tilde{\mathfrak{I}}$ *intertwines pairings. [Markarian, Ramadoss on associated-graded]*

Comparing VSHS: A side

Theorem* (G., adapted to compact CY case by G.-Perutz-Sheridan. See also FOOO)

There is a filtered map $\widetilde{\mathcal{OC}} : \mathrm{HP}(Fuk(X, \mathbf{D})) \to H^\bullet(X, \mathbb{K}_A)$*, the* 'cyclic open-closed map'*, which is an isomorphism whenever the associated graded map* $\mathcal{OC} : \mathrm{HH}(Fuk(X, \mathbf{D})) \to H^\bullet(X, \mathbb{K}_A)$ *is.*

Corollary (of the last talk)

Under HMS hypotheses, $\widetilde{\mathcal{OC}}$ *is an isomorphism.*

([Perutz-Sheridan]) imply that (a) there is a *non-degenerate subcategory* of $Fuk(X, \mathbf{D})$, and [G.-Perutz-Sheridan, AFOOO] imply that (b) that hence $\mathcal{OC}$ is an isomorphism.

Remark

Whenever $\mathcal{OC}$ (and $\widetilde{\mathcal{OC}}$) are isomorphisms, the non-commutative Hodge-de Rham degeneration conjecture holds for the Fukaya category.

Comparing of VSHS: A side

Our main technical pseudoholomorphic curve theory input is:

Theorem* (G.-Perutz-Sheridan)

The cyclic open-closed map

$$\widetilde{\mathcal{OC}} : \mathrm{HP}_\bullet(Fuk(X, \mathbf{D})) \to H^{\bullet+n}(X, \mathbb{K}_A)$$

is a morphism of pre- VSHS*:* $\mathrm{HP}_\bullet(Fuk(X, \mathbf{D})) \to V^A(X)$*.*

Corollary (last slide)

$\widetilde{\mathcal{OC}}$ *is an isomorphism of* VSHS*. So the Fukaya category 'knows'* $V^A(X)$*.*

Remarks:

- Last talk, we saw that $\mathcal{OC} : \mathrm{HH}_\bullet(Fuk(X, \mathbf{D})) \to QH^{\bullet+n}(X)$ respects pairings. That is the *associated graded* of this Thm.
- One can reformulate VSHS using the 'Rees correspondence': the associated categorical VSHS is *negative cyclic homology* $\mathrm{HC}^-(\mathcal{C})$*, a* $\mathbb{K}_A[u]$ *module.*

Compatibility with connections and pairings

Some remarks about why the result should hold:

- Compatibility with connections is the most technically involved part.
  On the associated graded of the Hodge filtration, $\nabla^{Getzler}_{Q\partial Q}$ is contraction with the *categorical Kodaira spencer class* $[- \cap \mathrm{KS}_{cat}(Q\partial Q)] : \mathrm{HH}_\bullet(\mathcal{F}) \to \mathrm{HH}_{\bullet+2}(\mathcal{F})$.
  In part 1, it was shown that $\mathrm{KS}_{cat}(Q\partial Q) = \mathcal{CO}([\omega])$, and $\mathcal{OC} : \mathrm{HH}_\bullet(\mathcal{F}) \to QH^{\bullet+n}(X)$ is a module map; hence $[\alpha \cap \mathrm{KS}_{cat}(Q\partial Q)]$ is sent to $[\omega] \star_Q \mathcal{OC}(\alpha)$ by *compatibility of* $\mathcal{OC}$ *with module structures.*
- Compatibility with pairings, like the associated graded compatibility with Mukai pairing, follows from a *Cardy relation*, but keeping track of cyclic homotopies…

The cyclic open-closed map

Suppose $\mathcal{C}$ is *strictly unital*, so there are strict unit elements $e_X^+ \in \mathcal{C}$.

Then, $\mathrm{HC}^-_\bullet(\mathcal{C}) := H^\bullet(\mathrm{CH}_*^{red}(\mathcal{C})[[u]], b_{eq} = b + uB)$, where $|u| = 2$,

- *CH^{red}* is the *reduced Hochschild chain complex* (generators: chains $x_k \otimes \cdots \otimes x_1$ where only x_k can be a strict unit).
- The *Connes' B operator* acts, mod signs, as $x_k \otimes \cdots \otimes x_1 \mapsto \sum_i \pm e^+ \otimes x_i \otimes \cdots \otimes x_1 \otimes x_k \otimes \cdots \otimes x_{i+1}$.

Assuming $Fuk(X, \mathbf{D})$ is *strictly unital* (with say $e_L^+ = [L]$), the *cyclic open-closed map* is a u-linear chain map

$$\widetilde{\mathcal{OC}} : (\mathrm{CH}_*^{red}(\mathcal{C})[[u]], b_{eq}) \to (H^*(X)[u], 0).$$

(the map from HP comes by inverting u). Expand

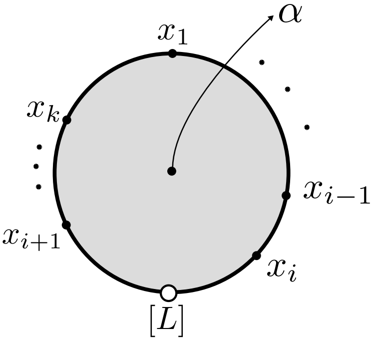
$$\widetilde{\mathcal{OC}} = \sum_{i=0}^{\infty} \mathcal{OC}_i u^i,$$

with $\mathcal{OC}_0 = \mathcal{OC}$.(Remark: $Fuk(X, \mathbf{D})$ is *not strictly unital!*)

The cyclic open-closed map (cont'd)

First, we might hope that $\widetilde{\mathcal{OC}} = \mathcal{OC}_0 u^0$.
Meaning: $\mathcal{OC} \circ (b + uB) = 0$. We know $\mathcal{OC} \circ b = 0$, so does $\mathcal{OC} \circ B = 0$? **Not always.**

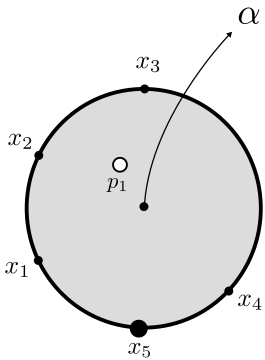
$\mathcal{OC} \circ B(x_k \otimes \cdots \otimes x_1)$ is a sum of terms of the form $\mathcal{OC}(e^+ \otimes x_i \otimes \cdots \otimes x_1 \otimes x_k \otimes \cdots \otimes x_{i+1})$, which looks like:



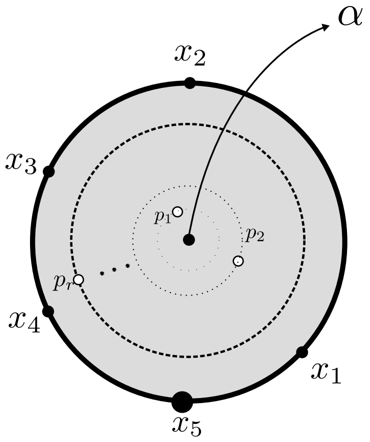
We could only guarantee such a term is zero by a form of *cyclic permutation invariance* of $\mathcal{OC}$, which in turn requires a type of *equivariant transversality*. The category *Fuk*$(X, \mathbf{D})$ won't have such transversality (though see [Fukaya-Oh-Ohta-Ono]). Also, there are algebraic obstructions making such symmetry *impossible* over $\mathbb{Z}((Q))$.

The cyclic open-closed map (cont'd)

Instead: there should exist $\mathcal{OC}_1$ with $\mathcal{OC} \circ B = \mathcal{OC}_1 \circ b$, so $\mathcal{OC}_1$ is a chain-homotopy between $\mathcal{OC} \circ B$ and 0. Roughly, $\mathcal{OC}_1$ counts:



$\mathcal{OC}_0 + u\mathcal{OC}_1$ may not be a chain map either. So we add an infinite hierarchy of correcting terms: $\widetilde{\mathcal{OC}} := \sum_{i=1}^{\infty} \mathcal{OC}_i u^i$.
$\mathcal{OC}_r$ counts, roughly:



Comparing Calabi-Yau structures

[*Kontsevich-Soibelman*]: A *strong proper Calabi-Yau structure* of degree n on a proper A_∞ category is a map

$$\widetilde{\phi} : HC^+_\bullet(\mathcal{C}) \to \mathbb{K}$$

of degree $-n$, which is *non-degenerate*: the associated pairing

$$\hom^\bullet(X,Y) \otimes \hom^{n-\bullet}(Y,X) \stackrel{\mu^2}{\to} \hom^n(X,X) \to CC^+_n(\mathcal{C}) \to \mathbb{K}$$

should be (cohomologically) perfect (Poincaré duality).
[*Kontsevich-Soibelman*]: In char. 0, if $\widetilde{\phi}_{\mathcal{C}}$ exists, $\mathcal{C}$ is quasi-isomorphic to a *cyclic A_∞ category*.

Theorem* (G., G.-Perutz-Sheridan in relative Fukaya category setting, see also Fukaya)

The Fukaya category has a canonical strong proper Calabi-Yau structure, $\tilde{\phi}_{\mathcal{F}}$.

Comparing Calabi-Yau structures (cont'd)

- Since $\mathcal{F}$ is smooth and proper, there is a corresponding *smooth strong Calabi-Yau structure*, in the sense of [Kontsevich-Vlassopolous]; or a *non-degenerate* element $\tilde{\Omega} \in \mathrm{HC}^-_{-n}(\mathcal{F})$.
- $\tilde{\phi}_{\mathcal{F}}$ and $\tilde{\Omega}$ can both be defined in terms of $\widetilde{\mathcal{OC}}$, and our main result implies that: $\tilde{\Omega}$ corresponds to a *normalized volume form*. Hence:

Theorem* (G.-Perutz-Sheridan)

The equivalence $tw^{\pi} Fuk(X, \mathbf{D}) \cong D^b_{dg}Coh(Y)$ preserves strong (smooth and proper) CY structures, *where*

- *we use the canonical CY structure on $Fuk(X, \mathbf{D})$,*
- *and the (canonical) CY structure coming from a* normalized volume form *on $D^b_{dg}Coh(Y)$.*

- $\Rightarrow$ an equivalence of associated TFTs [Costello, Kontsevich-Soibelman, . . .]