From categories to curve counts I: Automatic split-generation and non-degeneracy

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Mirror symmetry context: A-model

- (X, ω) = connected $2n$-dimensional Calabi-Yau Kähler manifold.
- $D \subset X$ a simple normal-crossings divisor, supports an effective ample divisor.
- $\omega|_{X \setminus D} = d\alpha$, and the corresponding 'Liouville vector field' points outwards at infinity.

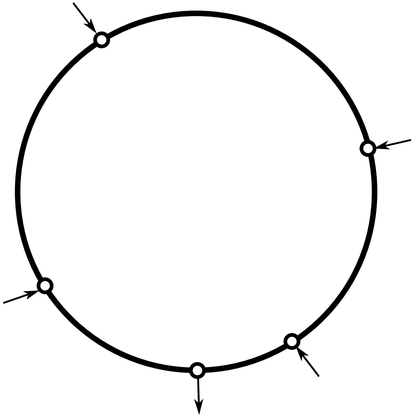
A-model category

- We consider $\mathcal{F}uk(X, D)$, the Fukaya category of X relative to D (Seidel): its objects are compact exact Lagrangian branes $L \subset X \setminus D$.
- It is a $\mathbb{Z}$-graded R-linear A_∞ category, where R is some Novikov field, e.g.

$$R := \left\{ \sum_{j=1}^{\infty} c_j q^{\lambda_j} : c_j \in \mathbb{C}, \lambda_j \in \mathbb{R}, \lim_{j \to \infty} \lambda_j = +\infty \right\}.$$

A-model category

- The A_∞ structure maps count pseudoholomorphic discs $u : \mathbb{D} \to X$:



weighted by $q^{\omega(u)-\alpha(\partial u)} \in R$.

- There should be a fully faithful embedding

$$\mathcal{F}uk(X, D) \hookrightarrow \mathcal{F}uk(X).$$

Mirror symmetry context: *B*-model

- $Y \to \mathcal{M}$ is a connected *n*-dimensional smooth projective scheme over $\mathcal{M} := \mathrm{Spec} R$, with trivial relative canonical sheaf.
- We have the classical Kodaira–Spencer map

$$KS_{class} : T\mathcal{M} \to H^1(Y, TY).$$

- We say that Y is *maximally unipotent* if

$$KS_{class}(\partial_q)^n \neq 0,$$

where the power is taken with respect to the natural product on *tangential cohomology*

$$HT^\bullet(Y) := H^\bullet(Y, \wedge^\bullet TY).$$

- If Y is obtained by base change from a family Y_q parametrized by $q \in \mathbb{D}^*$, and the family Y_q has maximally unipotent monodromy about $q = 0$, then Y is maximally unipotent.

B-model category

- $D^b_{dg}Coh(Y)$ is a DG enhancement of the bounded derived category of coherent sheaves on Y.
- We regard it as an R-linear A_∞ category.

Homological mirror symmetry

- We say that (X, D) and Y are *homologically mirror* if there is a quasi-equivalence of R-linear A_∞ categories

$$tw^\pi \mathcal{F}uk(X, D) \cong D^b_{dg}Coh(Y).$$

  'tw^π' denotes the split-closed triangulated envelope.
- More generally, we say X and Y are homologically mirror if there is a quasi-equivalence

$$tw^\pi \mathcal{F}uk(X) \cong D^b_{dg}Coh(Y).$$

Core homological mirror symmetry

We say that (X, D) and Y satisfy *core HMS* if

$$\begin{array}{ccc} \mathcal{F}uk(X,D) & & D^b_{dg}Coh(Y) \\ \cup & & \cup \\ \mathcal{A} & \cong & \mathcal{B} \end{array}$$

where:

- $\mathcal{A}$ and $\mathcal{B}$ full subcategories;
- $\mathcal{A}$ and $\mathcal{B}$ are quasi-equivalent;
- $\mathcal{B}$ split-generates $D^b_{dg}Coh(Y)$;
- Y is maximally unipotent.

- Note: core HMS implies that $D^b_{dg}Coh(Y)$ embeds into $tw^{\pi}\mathcal{F}uk(X, D)$.

Main theorem

Theorem (Perutz–S.)

If (X, D) *and* Y *satisfy core HMS, then* $\mathbb{A}$ *split-generates* $\mathcal{F}uk(X, D)$*: it follows that* (X, D) *and* Y *are homologically mirror.*

- We expect a similar result to hold for $\mathcal{F}uk(X)$.
- The main tool in the proof is Abouzaid's split-generation criterion.

Applications

Core-HMS-type results have been proved for:

- Abelian varieties (Kontsevich–Soibelman, Fukaya);
- Calabi–Yau hypersurfaces in projective space (Seidel, S.);
- A sketch proof for Calabi–Yau Gross–Siebert mirror pairs also exists (Abouzaid–Gross–Siebert).

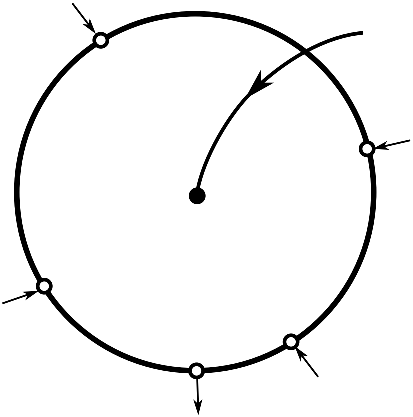
Modulo technical details, our result shows that these core HMS proofs extend to HMS proofs.

Closed-open string map

- The *closed-open string map* is a graded R-algebra homomorphism

$$\mathcal{CO}\colon QH^\bullet(X) \to HH^\bullet(\mathcal{F}uk(X, D)),$$

defined by counting the following pseudoholomorphic discs:

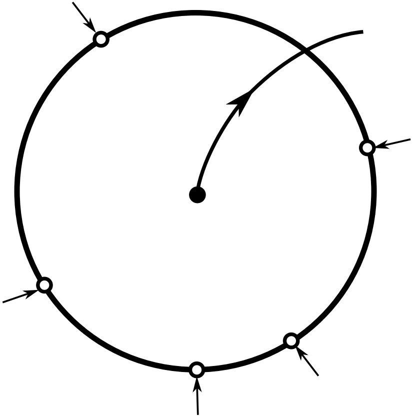


Open-closed string map

- The *open-closed string map*

$$\mathcal{OC}\colon HH_\bullet(\mathcal{F}uk(X, D)) \to QH^{\bullet+n}(X)$$

is defined by counting the following pseudoholomorphic discs:



Properties of $\mathcal{OC} : HH_\bullet(\mathcal{F}uk(X,D)) \to QH^{\bullet+n}(X)$

- $\mathcal{OC}$ is a homomorphism of $QH^\bullet(X)$-modules, where $HH_\bullet$ acquires its $QH^\bullet$-module structure via its natural $HH^\bullet$-module structure and $\mathcal{CO}$.
- The map

$$\begin{array}{rcl} HH_n(\mathcal{F}uk(X,D)) & \to & R \\ \alpha & \mapsto & \int \mathcal{OC}(\alpha) \end{array}$$

  defines a *weak proper CY[n]* structure on $\mathcal{F}uk(X,D)$.
- I.e., the pairing

$$\begin{array}{rcl} Hom^\bullet(K,L) \otimes Hom^{n-\bullet}(L,K) & \to & R \\ a \otimes b & \mapsto & \int \mathcal{OC}(a \cdot b) \end{array}$$

  is non-degenerate.

Duality of $\mathcal{CO}$ and $\mathcal{OC}$

- A weak proper CY[n] structure induces an isomorphism

$$HH_{\bullet+n}(\mathcal{F}uk(X,D)) \cong HH^{-\bullet}(\mathcal{F}uk(X,D))^\vee.$$

- We also have the identification

$$QH^{\bullet+2n}(X) \cong QH^{-\bullet}(X)^\vee$$

from Poincaré duality.

- With respect to these isomorphisms, the maps

$$HH_{\bullet+n}(\mathcal{F}uk(X,D)) \xrightarrow{\mathcal{OC}[n]} QH^{\bullet+2n}(X),$$
$$HH^{-\bullet}(\mathcal{F}uk(X,D))^\vee \xrightarrow{\mathcal{CO}^\vee} QH^{-\bullet}(X)^\vee$$

coincide.

Non-degeneracy

- We say that a full subcategory $\mathcal{A} \subset \mathcal{F}uk(X, D)$ is *non-degenerate* if

$$\mathcal{OC}|_{\mathcal{A}} \colon HH_{-n}(\mathcal{A}) \to QH^0(X)$$

hits the unit, i.e., is non-zero.

- Because $\mathcal{OC}$ is dual to $\mathcal{CO}$,

$$\mathcal{A} \text{ non-deg } \iff \mathcal{CO}|_{\mathcal{A}} \colon QH^{2n}(X) \to HH^{2n}(\mathcal{A}) \text{ is non-zero.}$$

Abouzaid's split-generation criterion

Theorem (Abouzaid, adapted to this setting by Perutz–S.)

If $\mathcal{A}$ is non-degenerate, it split-generates $\mathcal{F}uk(X, D)$.

- We prove our main theorem by showing

$$\text{Core HMS} \Rightarrow \mathcal{A} \text{ is non-degenerate.}$$

- This is also a key ingredient in the following theorem, which we will explain if time permits:

Theorem (Ganatra–Perutz–S.)

If (X, D) and Y are homologically mirror, and Y is maximally unipotent, then $\mathcal{CO}$ and $\mathcal{OC}$ are isomorphisms.

Proving core HMS $\Rightarrow \mathcal{A}$ non-degenerate

- Recall:

$$\mathcal{A} \text{ non-deg} \iff \boxed{\mathcal{CO}|_{\mathcal{A}} \colon QH^{2n}(X) \to HH^{2n}(\mathcal{A}) \text{ is non-zero.}}$$

- Kähler form ω is non-degenerate $\Rightarrow [\omega]^{\cup n} \neq 0 \Rightarrow [\omega]^{\star n} \neq 0$.
- Hence, $[\omega]^{\star n}$ generates $QH^{2n}(X)$: so

$$\mathcal{A} \text{ non-deg} \iff \boxed{\mathcal{CO}|_{\mathcal{A}}([\omega]^{\star n}) \neq 0.}$$

- $\mathcal{CO}|_{\mathcal{A}}$ is an algebra homomorphism, so

$$\mathcal{A} \text{ non-deg} \iff \boxed{\mathcal{CO}|_{\mathcal{A}}([\omega])^{n} \neq 0 \text{ in } HH^{2n}(\mathcal{A}).}$$

Kaledin class

- If $\mathcal{C}$ is an R-linear A_∞ category with structure maps $\mu^* \in CC^2(\mathcal{C})$, then

$$\frac{\partial \mu^*}{\partial q} \in CC^2(\mathcal{C})$$

  is a Hochschild cocycle. The corresponding class in $HH^2(\mathcal{C})$ is called the *Kaledin class*.
- More generally, we have the *categorical Kodaira–Spencer map*

$$\begin{aligned} KS_{cat} : Der(R) &\to HH^2(\mathcal{C}) \\ KS_{cat}(v) &:= [v(\mu^*)]. \end{aligned}$$

Kaledin class and $\mathcal{CO}$

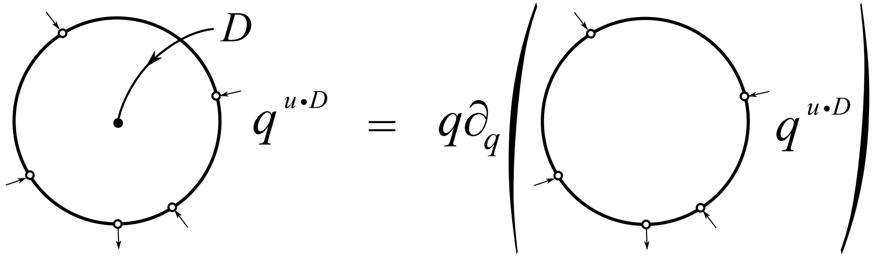
Proposition

If $\mathcal{A} \subset \mathcal{F}uk(X, D)$, the closed-open map

$$\mathcal{CO}|_{\mathcal{A}} : QH^{\bullet}(X) \to HH^{\bullet}(\mathcal{A})$$

satisfies

$$\mathcal{CO}|_{\mathcal{A}}([\omega]) = KS_{cat}(q\partial_q).$$



Using core HMS

- Hence,

$$\mathcal{A} \text{ non-deg} \iff \boxed{KS_{cat}(q\partial_q)^n \neq 0 \text{ in } HH^{2n}(\mathcal{A}).}$$

- Assume core HMS: so $\mathcal{A} \cong \mathcal{B} \subset D^b_{dg}Coh(Y)$, and $\mathcal{B}$ split-generates.
- The statement $KS_{cat}(q\partial_q)^n \neq 0$ is Morita-invariant: so

$$\mathcal{A} \text{ non-deg} \iff \boxed{KS_{cat}(q\partial_q)^n \neq 0 \text{ in } HH^{2n}(D^b_{dg}Coh(Y)).}$$

The HKR map: mirror to $\mathcal{CO}$

- There are isomorphisms

$$HT^\bullet(Y) \cong HH^\bullet(Y) \cong HH^\bullet(D^b_{dg}Coh(Y))$$

(Hochschild–Kostant–Rosenberg, Gerstenhaber–Schack, ...; Toën, Lowen–Van den Bergh).

- Pre-composing this isomorphism with contraction with $td_Y^{1/2}$ gives an isomorphism of graded R-algebras,

$$I^K_{cat} : HT^\bullet(Y) \xrightarrow{\cong} HH^\bullet(D^b_{dg}Coh(Y))$$

(Kontsevich, Huybrechts–Nieper-Wisskirchen, Calaque–Van den Bergh–Rossi).

Hochschild–Kostant–Rosenberg and Kodaira–Spencer

- The classical Kodaira–Spencer map is a map

$$KS_{class} : Der(R) \to H^1(Y, TY) \subset HT^2(Y).$$

- The categorical Kodaira–Spencer map is a map

$$KS_{cat} : Der(R) \to HH^2(D^b_{dg}Coh(Y)).$$

Lemma

These are related as you expect:

$$I^K_{cat} \circ KS_{class} = KS_{cat}.$$

- In particular, $KS_{cat}(q\partial_q) = I^K_{cat}(KS_{class}(q\partial_q))$.

End of proof

- Recall: in the context of core HMS,

$$\mathcal{A} \text{ non-deg} \iff \boxed{KS_{cat}(q\partial_q)^n \neq 0 \text{ in } HH^{2n}(D^b_{dg}Coh(Y)).}$$

- By the previous Lemma, this is equivalent to

$$KS_{class}(q\partial_q)^n \neq 0 \text{ in } HT^{2n}(Y).$$

- This is the definition of maximal unipotence of Y.

Summary of proof

- We have graded algebra homomorphisms

$$\begin{array}{ccccc} HH^\bullet(\mathcal{A}) & \cong & HH^\bullet(\mathcal{B}) & \cong & HH^\bullet(D^b_{dg}Coh(Y)) \\ \uparrow \mathcal{CO} & & & & I^K_{cat} \uparrow \cong \\ QH^\bullet(X) & & & & HT^\bullet(Y). \end{array}$$

- These send

$$\begin{array}{ccc} KS_{cat}(q\partial_q) & \longleftrightarrow & KS_{cat}(q\partial_q) \\ \uparrow & & \uparrow \\ [\omega] & & KS_{class}(q\partial_q). \end{array}$$

- Hence they send the nth powers to each other: since the nth power in the bottom right is non-vanishing by maximal unipotence, the nth power in the top left is non-vanishing, and $\mathcal{A}$ is non-degenerate.

Second main theorem

We now present the proof of:

Theorem (Ganatra–Perutz–S.)

If (X, D) and Y are homologically mirror, and Y is maximally unipotent, then $\mathcal{CO}$ and $\mathcal{OC}$ are isomorphisms.

- The preceding argument shows $\mathcal{F}uk(X, D)$ is non-degenerate.
- Because

$$\mathcal{OC} : HH_\bullet(\mathcal{F}uk(X, D)) \to QH^{\bullet+n}(X)$$

  is a homomorphism of $QH^\bullet(X)$-modules, and hits the unit, it immediately follows that it is surjective.
- It remains to prove injectivity of $\mathcal{OC}$.

The Mukai pairing (after Shklyarov)

- For any proper R-linear A_∞ category $\mathcal{C}$, the diagonal bimodule defines a functor

$$\begin{array}{rcl} \mathcal{C} \otimes \mathcal{C}^{op} & \to & Perf(R) \\ K \otimes L & \mapsto & hom^\bullet(K, L). \end{array}$$

- This induces a map on Hochschild homologies:

$$HH_\bullet(\mathcal{C}) \otimes HH_\bullet(\mathcal{C}^{op}) \to HH_\bullet(Perf(R)) \cong R.$$

- We have $HH_\bullet(\mathcal{C}^{op}) \cong HH_\bullet(\mathcal{C})$, so we obtain a pairing on $HH_\bullet(\mathcal{C})$, called the *Mukai pairing*, denoted $\langle -, - \rangle_{Muk}$.

Formula for the Mukai pairing

Lemma (S.)

If the morphism spaces of $\mathcal{C}$ are finite-dimensional on the cochain level, we have a formula for $\langle -, - \rangle_{Muk}$:

$$\langle a_0 \otimes a_1 \otimes \ldots \otimes a_s, b_0 \otimes b_1 \otimes \ldots \otimes b_t \rangle_{Muk} = \sum (-1)^{\dagger} Tr\left(\mu^*(\ldots, a_0, \ldots, \mu^*(\ldots, -, \ldots, b_0, \ldots), \ldots)\right).$$

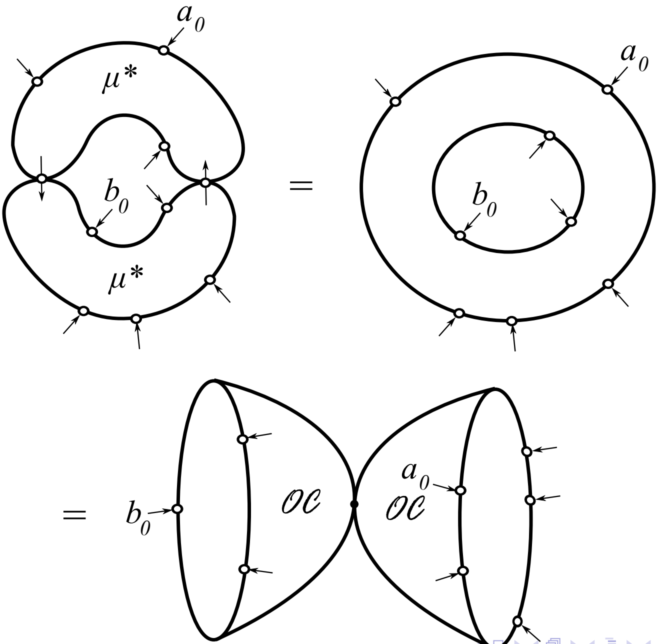
The open-closed map respects pairings

Theorem (Ganatra–Perutz–S.)

The open-closed map intertwines the Mukai pairing with the intersection pairing:

$$\langle \alpha, \beta \rangle_{Muk} = (-1)^{n(n+1)} \int \mathcal{OC}(\alpha) \cup \mathcal{OC}(\beta).$$

Proof that $\mathcal{OC}$ respects pairings: Cardy relation



Non-degeneracy of pairing

Theorem (Shklyarov)

If the category $\mathcal{C}$ is proper and smooth, then the Mukai pairing is non-degenerate.

- Because Y is smooth, $D^b_{dg}Coh(Y)$ is smooth.
- Hence the Mukai pairing on $HH_\bullet(D^b_{dg}Coh(Y))$ is non-degenerate.

End of proof

- If (X, D) and Y are homologically mirror, the Mukai pairing on $HH_\bullet(\mathcal{F}uk(X, D))$ is non-degenerate.
- For any $0 \neq \alpha \in HH_\bullet(\mathcal{F}uk(X, D))$, choose β so that $\langle \alpha, \beta \rangle_{Muk} \neq 0$: then

$$0 \neq \langle \alpha, \beta \rangle_{Muk} = (-1)^{n(n+1)} \int \mathcal{OC}(\alpha) \cup \mathcal{OC}(\beta)$$

  so $\mathcal{OC}(\alpha) \neq 0$.
- Therefore, $\mathcal{OC}$ is injective; we already saw that it is surjective, so it is an isomorphism.
- $\mathcal{CO}$ is dual to $\mathcal{OC}$, so it is also an isomorphism.