From Family Floer cohomology to Homological mirror symmetry

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Outline

SYZ mirrors

Let X be a symplectic manifold. My goal is to explain a program for a local-to-global approach to proving HMS for X, which build upon ideas of SYZ, Fukaya, Kontsevich etc.
In the simplest class of examples, mirror symmetry should hold *without corrections.* Assume that $\pi\colon X \to Q$ is a Lagrangian torus fibration (with no singular fibres) which admits a Lagrangian section. Following SYZ, one can consider the *dual torus fibration,* Y which is naturally a complex analytic manifold.

Definition (SYZ)

X and Y are mirror.

One obtains a family by simply rescaling the symplectic form on X.

Novikov field

One problem with proving HMS in this context is that the Floer theory of a general Lagrangian $L \subset X$ is not well-defined over $\mathbb{C}$. We need to work over the Novikov field:

$$\Lambda = \{\sum_{i=0}^{\infty} a_i T^{\lambda_i} | a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lim_{i \to +\infty} \lambda_i = +\infty\}.$$

This field has a (non-archimedean) valuation

$$\begin{aligned} \mathrm{val}\colon \Lambda^* = \Lambda \setminus \{0\} &\mapsto \mathbb{R} \\ a_0 T^{\lambda_0} + \sum_{\lambda_0 < \lambda_i} a_i T^{\lambda_i} &\mapsto \lambda_0. \end{aligned}$$

We can consider $(\Lambda^*)^n$ as a *rigid analytic space* with ring of functions Laurent series which converge at every point of $(\Lambda^*)^n$ (i.e. a completion of Laurent series $\Lambda[z_1^{\pm}, \ldots, z_n^{\pm}]$).

Rigid analytic dual

Given $P \subset \mathbb{R}^n$, we obtain a ring of functions Γ_P consisting of Laurent series which converge at every point of $Y_P = \mathrm{val}^{-1}(P) \subset (\Lambda^*)^n$. If P is a $\mathbb{Z}$-affine polygon, then Y_P is an *affinoid domain*.
An inclusion $P \subset P'$ induces a ring map $\Gamma_{P'} \to \Gamma_P$. This means that we can glue spaces Y_P together along such inclusions to obtain a *rigid analytic variety.*

Example

If $X \to Q$ is a Lagrangian torus fibration, then Q has an integral affine structure. So we can pick a cover by subsets which can be identified with integral affine polygons in $\mathbb{R}^n$. Gluing the corresponding affinoid domains yields a rigid analytic space Y. This the rigid analytic T-dual of X.

If X admits a Lagrangian section, then Kontsevich-Soibelman proved that the Fukaya subcategory of Lagrangian sections is equivalent to the category of line bundles on Y (c.f. Fukaya).

Local Family Floer cohomology I

Assume that $\pi_2(Q) = 0$. This implies that $\pi_2(X, X_q) = 0$ for all fibres, hence that they do not bound any holomorphic discs.
Let $L \subset X$ be a Lagrangian, which is unobstructed.

Lemma (Fukaya)

Each $q \in Q$ admits a neighbourhood P so that the following holds: there is a complex $\mathcal{C}_P(L)$ of Γ_P-modules such that $CF^*(L, X_p)$ is isomorphic to $\mathcal{C}_P(L) \otimes_{\Gamma_P} \Lambda$.

Proof.

Choose a Hamiltonian perturbation of L which transverse to X_q, and a family of *diffeomorphism* of X which preserve L, and map X_q to nearby fibres. If the neighbourhood is small enough, these diffeomorphism preserve *tameness* of almost complex structures. □

Geometrically, this means that we have a complex of coherent sheaves on Y_P.

Local-global construction I

Theorem (A)

If $X \to Q$ admits a Lagrangian section, there is an A_∞ functor $\mathcal{C}\colon \mathcal{F}(X) \to \mathrm{Perf}(Y)$ which extends this local construction. This is a faithful embedding.

The proof is "heavy," but the main ideas are the following:

1. Say that $q \in P_i \cap P_j$, and we have chosen Hamiltonian perturbation L_i and L_j to define (complexes) of sheaves over Y_i and Y_j. Then a choice of Hamiltonian isotopy induces a chain equivalence *if P_i and P_j are small enough.*
2. Use a model for $\mathrm{Perf}(Y)$ in which objects are Γ_{P_i} modules glued together by quasi-isomorphism (and higher homotopical data ...).
3. Prove faithfulness by a version of the Cardy relation.

There is a version of this result in which X is not assumed to have a Lagrangian section.

Construction of mirror away from singularities

Theorem (Fukaya, 2009)

For an arbitrary Lagrangian torus of Maslov index 0*, there is a neighbourhood* $P \subset H^1(L, \mathbb{R})$*, and a bi-vector field* α_P *on* Y_P *which vanishes exactly at the unobstructed nearby Lagrangians (with local systems).*

So, we should take such pairs (Y_P, α_P) as our local charts. Fukaya further proved that Floer theory determines coordinates changes which preserve the bivector field.
Given a Lagrangian torus fibration, we can use this procedure to produce a space $\mathring{Y}$, with a non-commutative deformation, by considering only smooth fibres (c.f. Tu).

1. Extend across the singularities (i.e. construct Y).
2. Construct the mirror functor.
3. Show that the functor is an equivalence

Enlarging the Fukaya category

Recall that objects of the Fukaya category are Lagrangians together with *unitary finite-rank local systems*. For tori, it is sufficient to consider rank-1 systems, which are representations of $\pi_1(L)$ into

$$U_\Lambda = \{a_0 + \sum_{i=1}^{\infty} a_i T^{\lambda_i} | a_0 \neq 0, 0 < \lambda_i\}.$$

Non-unitary local systems, do not define objects because the counts of holomorphic discs may not converge with such coefficients. One way to state this is to replace the group ring $R_L = \Lambda[\pi_1(L)]$ by its *formal completion* $\hat{R}_L$ (completion with respect to the T-adic filtration. With this in mind, we can construct an enlargement $\hat{\mathcal{F}}(X)$ of $\mathcal{F}(X)$ with object Lagrangians L with *arbitrary* representations of $\hat{R}_L$ (throw out all obstructed objects).

The universal local system

If we consider $\hat{R}_L$ as a module over itself, then, if there are no corrections, we obtain, under mirror symmetry the structure sheaf of $Y_0 \subset (\Lambda^*)^n$. This is almost tautological, because

$$R_L = \Lambda[z_1^{\pm}, \ldots, z_n^{\pm}].$$

The completion with respect to T-adic topology corresponds to requiring convergence of Laurent series at all points with 0-valuation. To actually check this, we should compute morphisms on the two sides, and make sure they agree. The simplest answer arises when we use topological vector spaces throughout, in which case we have

$$\mathrm{Ext}^*_{\Gamma}(\Gamma_0, \Gamma_0) = \mathrm{Ext}^*_{\Gamma_0}(\Gamma_0, \Gamma_0) = \Gamma_0$$

On the symplectic side, we get $HF^*(\hat{R}_L, \hat{R}_L) = H^*(L, \mathrm{Hom}(\hat{R}_L, \hat{R}_L))$. Since we're using topological vector spaces, this is the completion of

$$H^*(L, \mathrm{Hom}(R_L, R_L)) = R_L.$$

Local Family Floer cohomology II

Unfortunately, the domains Y_p don't form a good affinoid cover. We should therefore be looking for the analogue of Y_P, for $P \subset H^1(L, \mathbb{R})$. Pick a norm $|\_|$ on H_1, and define the *analytic completion*:

$$\hat{R}_L^{\epsilon} = \left\{ \sum_{g \in \pi_1(L)} a_g g \middle| \epsilon |g| + \mathrm{val}(a_g) \to +\infty \right\}.$$

Fukaya's adic-convergence result can be interpreted to say that, for each Lagrangian L the self-Floer theory of $\hat{R}_L^{\epsilon}$ is well-defined. More generally, given another Lagrangian K, the Floer theory of the pair $(K, \hat{R}_L^{\epsilon})$ is well defined.

Replacing L by a fibre of $X \to Q$, we have an identification $\hat{R}_L^{\epsilon} = \Gamma_P$ for an appropriate choice of norm, so the functor

$$\mathcal{F}(X) \to \mathrm{Perf}_{Y_P}$$

becomes the restriction of a Yoneda functor for the object $\hat{R}_L^{\epsilon}$ in a category $\hat{\mathcal{F}}^{\epsilon}(X)$.

Local-global construction II

Given a a polyhedral cover $\mathcal{U}$ of Q, we consider Lagrangian fibres equipped with analytic completion of the group rings. The corresponding Fukaya category is quasi-isomorphic to the category $\mathcal{O}_{\mathcal{U}}$ of sheaves on Y given by structure sheaves of the affinoid domains. To prove full-faithfulness of the family Floer functor, we need two more steps:

1. Construct a category $\hat{\mathcal{F}}^{\mathcal{U}}(X)$ with subcategories $\mathcal{F}(X)$ and $\mathcal{O}_{\mathcal{U}}$.
2. Show that $\mathcal{O}_{\mathcal{U}}$ split-generates $\hat{\mathcal{F}}^{\mathcal{U}}(X)$.

Dropping the assumption that $X \to Q$ has no singular fibres, the first step still makes sense. Proving split-generation requires new tools in homological algebra (c.f. Kontsevich's first talk)

Missing objects

We have so far a strategy for understanding the functor

$$\mathcal{F}(X) \to \mathrm{Perf}(\mathring{Y})$$

where $\mathring{Y}$ is obtained by (rigid analytic) T-duality from the smooth part of X. The problem is to “compactify” $\mathring{Y}$ by adding objects corresponding to the singular fibres of $X \to Q$.

Example

Let $X = \mathbb{C}^2 \setminus \{xy = 1\} \to \mathbb{R}^2$ *be the local model for basic singularity of torus fibrations in dim 2. Then* $\mathring{Y} = \Lambda^2 \setminus \{uv = T\} \bigcup \{0\}$.

Theorem (Fukaya 2009–??)

The immersed Lagrangian given by the equations $|xy - 1| = 0$ *and* $|x| = |y|$ *represents the point* $\{0\}$. *Moreover, Family Floer cohomology gives a chart for* Y *near the origin.*

$X = \mathbb{C}^3 \setminus \{z_1 z_2 z_3 = 1\}$

Theorem (Gross, Matessi-Castano-Bernard,A-Auroux-Katzarkov)

X admits an SYZ fibration such that
$\mathring{Y}_\Lambda = \{uv = y_1 + y_2 + 1, (u, v) \neq (0, 0)\} \subset \Lambda^2 \times (\Lambda^*)^2$.

The expectation is that the mirror is obtained by "filling in" $u = v = 0$.

Theorem (A-Auroux 2014–??)

There is a fully faithful embedding $D^b(\mathrm{Coh}(Y)) \subset \mathcal{W}(X)$.

The missing objects correspond to solutions of $0 = y_1 + y_2 + 1$.
Immersed Lagrangians give all solutions with $\mathrm{val}(y_i) \neq 0$ for some i.
So we are missing *unitary* solutions to this equation.

Representing the missing object, I

Theorem (A-Auroux 2015–??)

There is an immersed Lagrangian torus $L \subset X$ with the following properties:

1. *The set of unobstructed unitary local systems is given by* $0 = y_1 + y_2 + 1$.
2. *The corresponding module over the mirror of* $\mathcal{O}_Y$ *is* $\mathcal{O}_y \oplus \mathcal{O}_y[1]$.

The intuition is to take two copies of the singular Lagrangian fibre, and appropriately smooth the result.
The same strategy works for $X = \mathbb{C}^n \setminus \{z_1 \dots z_n = 1\}$.

$X = \{uv = y_1 + y_2 + 1\}$

Theorem (Gross, Matessi-Castano-Bernard,A-Auroux-Katzarkov)

X admits an SYZ fibration such that
$\mathring{Y}_\Lambda = \Lambda^3 \setminus \{z_1 z_2 z_3 = T\} \cup_{i,j} \{z_i = z_j = 0\}$.

The expectation is that the mirror is obtained by "filling in" the coordinate axes. Point which are away from the origin correspond to immersed Lagrangians. So the origin is the unique missing point.

Theorem (A 2015–??)

The origin is represented by an immersed Lagrangian sphere $L \subset X$.

The idea is to take Seidel's immersed curve in the pair of pants, fill it by a disc in $(\mathbb{C}^*)^2$, and then lift to a circle bundle in the 3-fold. There are 3 self-intersection points.
The same construction work for $X = \{uv = y_1 + \cdots + y_n\}$, using Sheridan's Lagrangian.

Hypersurfaces and complete intersections

Theorem (A-Auroux-Katzarkov 2012)

SYZ mirror symmetry for hypersurfaces in projective space only requires Morse-Bott versions of the models $\mathbb{C}^n \setminus \{z_1 \dots z_n = 1\}$ *and* $\{uv = y_1 + \cdots + 1\}$.

This means that we can now make sense of Lagrangian Floer theory for all fibres. The problem is to combine them with the tools of Family Floer cohomology ...