Periods, Residues and Mirror Symmetry

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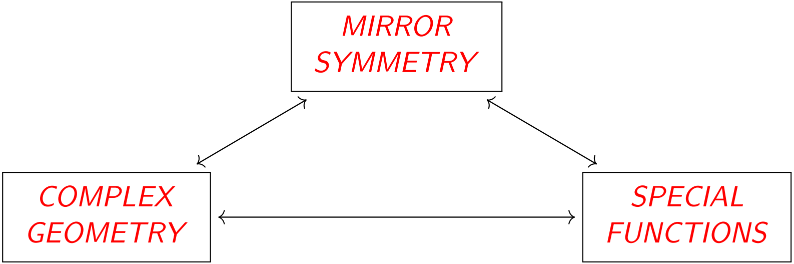
Study interplay between



2. Applications of mirror symmetry

To use periods of Calabi-Yau manifolds from 'B-models ' to prove statements about

- special functions (e.g. generalized hypergeometric functions), and their associated D-modules
- cohomological descriptions of the D-modules
- certain general type and Fano varieties

  Joint work with S. Bloch, L. Fu, A. Huang, D. Srinivas, S.-T. Yau and X. Zhu.

3. Lecture plan

- Riemann-Hilbert problem for period integrals
- Explicit D-module descriptions of periods and residues
- Chain integral solutions to GKZ systems
- LCSL problems
- Periods of general type and Fano toric hypersurfaces
- Open questions

4. Period sheaf

- Consider a family $\pi : \mathcal{Y} \to B$ of d-dimensional compact complex manifolds, with $Y_b := \pi^{-1}(b)$.

- Put $E := R^k \pi_* \mathbb{C}$, a local system over B with fibers $H^k(Y_b, \mathbb{C})$.

- Dual $E^\vee$ has fibers $H_k(Y_b, \mathbb{C})$, with Poincaré pairing:

$$\langle , \rangle : \mathcal{O}(E^\vee) \otimes \mathcal{O}(E) \to \mathcal{O}_B.$$

- Local parallel sections of the Gauss-Manin connection ∇ on $E^\vee$:

$$\ker \nabla \simeq H_k(Y_b, \mathbb{C}).$$

5. Period sheaf

- Fix a space of global section $\Lambda \subset \Gamma(B, E)$. For $\lambda \in \Lambda$, we want to integrate the de Rham class $\lambda(b) \in H^k(Y_b, \mathbb{C})$ on Y_b.
- The kth period sheaf $\mathbf{\Pi}^k(\Lambda)$ (with respect to Λ) of the family $\mathcal{Y}$ is the local system generated by the germs of period integrals

$$\langle \gamma, \lambda \rangle_b = \left(\int_\gamma \lambda \right)_b \in \mathcal{O}(B)_b, \qquad \gamma \in \ker \nabla.$$

Will consider the middle degree $k = d$, and write $\mathbf{\Pi}^d \equiv \mathbf{\Pi}$.

6. Residue sheaf

- Now suppose $\pi : \mathcal{Y} \to B$ is a family of *hypersurfaces* in a fixed projective manifold X^n, parameterized by sections $b \in B$ of a line bundle L on X. The groups $H^n(X - Y_b)$, $b \in B$, form a local system (E', ∇') on B.

- For $g \in \Gamma(X, K_X + L)$, we define a section $\lambda^g \in \Gamma(B, E')$ by

$$\lambda^g(b) = \frac{g\Omega}{b} \in \Omega^n(\log Y_b)$$

  Here Ω is a canonical holomorphic n-form on certain principal H-bundle on X. (See [L-Yau 12'], [Chen-L 15'].)

- Put

$$\Lambda' := \{\lambda^g | g \in \Gamma(X, K_X + L)\} \subset \Gamma(B, E').$$

7. Residue sheaf

- The residue sheaf $\Pi'(\Lambda')$ of the family $\mathcal{Y}$ is defined to be the sheaf on B generated by the germs of residue integrals of the form

$$\left(\int_{\gamma'} \lambda^g\right)_b \in \mathcal{O}(B)_b, \quad \gamma' \in \ker \nabla'.$$

- Put $\Lambda = \mathrm{Res}\, \Lambda' := \{\mathrm{Res}\, \lambda_g | g \in \Gamma(X, K_X + L)\}$, where Res is the Poincaré residue map, so that

$$\mathbf{\Pi}(\Lambda) \subset \mathbf{\Pi}'(\Lambda'), \quad \left(\int_{\gamma} \mathrm{Res}\, \lambda^g\right)_b = (\int_{\gamma'} \lambda^g)_b$$

where γ' =the 'tube' over the cycle $\gamma \in \ker \nabla'$ in $X - Y_b$.

8. Riemann-Hilbert problem

Problem:
Construct a complete differential system for each of the sheaves $\mathbf{\Pi} \equiv \mathbf{\Pi}(\Lambda)$ and $\mathbf{\Pi}' \equiv \mathbf{\Pi}'(\Lambda')$ above.

- Having a good description of such a differential system will give many interesting applications, including mirror symmetry.
- Until recently, only a few cases were known and they were mostly from physics . Important example: the family of mirror quintic threefolds [Candelas et al '91].
- Even the case of quintic threefolds (there was a conjectural answer) was not known until two years ago.

9. Hypersurfaces in $X = \mathbb{P}^n$

- Toy example: $X = \mathbb{P}^1$, $L = \mathcal{O}(2)$. A family of CY hypersurfaces defined by the polynomial

$$f(a,x) := a_0 x_0 x_1 + a_1 x_1^2 + a_2 x_2^2.$$

For $b \in B := \mathbb{C}^3 - \{a_1 a_2 - \frac{1}{4} a_0^2 = 0\}$, $Y_b \equiv \{f(b,x) = 0\}$ is a pair of points in $\mathbb{P}^1$.

- $sl_2 \curvearrowright Sym^2\mathbb{C}^2 \equiv \sum_i \mathbb{C} a_i$ acts by 1st order differential operators of the form

$$Z(\xi) := \sum_{ij} \xi_{ij} a_i \partial_{a_j}, \quad (\xi \in sl_2).$$

- Solution to the RH problem for both $\mathbf{\Pi}, \mathbf{\Pi}'$ in this case is the differential system τ:

$$\partial_{a_0}^2 - \partial_{a_1}\partial_{a_2}, \quad \sum_i a_i \partial_{a_i} + 1, \quad Z(\xi) \quad\quad (\xi \in sl_2).$$

10. Hypersurfaces in $X = \mathbb{P}^n$

- The solution sheaf of the system τ is

$$\mathbb{C}(a_1 a_2 - \frac{1}{4} a_0^2)^{-1/2}.$$

Therefore in this case

$$\mathbf{\Pi} = \mathbf{\Pi}' = \mathbb{C}(a_1 a_2 - \frac{1}{4} a_0^2)^{-1/2}.$$

- If we replace sl_2 in the system τ by its Cartan subalgebra $\mathfrak{t} \simeq \mathbb{C}$, the smaller system is a *GKZ hypergeometric system*.

11. Hypersurfaces in $X = \mathbb{P}^n$

- $X = \mathbb{P}^n$, $L = \mathcal{O}(n+1)$. The universal family of CY hypersurfaces in X is defined

$$Y_b : \quad f(b,x) := \sum_{\mu} b_\mu x^\mu = 0$$

$$x^\mu := x_0^{\mu_0} \cdots x_n^{\mu_n}, \quad \mu \in \mathbb{Z}_{\geq 0}^{n+1}, \ \sum_i \mu_i = n+1.$$

- In this case, there is a canonical section $\frac{\Omega}{f} \in \Gamma(B, E')$, $\Omega = \sum_i (-1)^{i-1} x_i dx_0 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots dx_n$. So we take $\Lambda' = \mathbb{C}\frac{\Omega}{f}$.

- Again $sl_{n+1} \curvearrowright Sym^{n+1}\mathbb{C}^{n+1} \equiv \sum_\mu \mathbb{C} a_\mu$ acts by 1st order differential operators of the form

$$Z(\xi) := \sum_{\mu,\nu} \xi_{\mu\nu} a_\mu \partial_{a_\nu}, \quad \xi \in sl_{n+1}.$$

12. Hypersurfaces in $X = \mathbb{P}^n$

Proposition: [Hosono-L-Yau '95] Both the period sheaf and residue sheaf of this family are solutions to the differential system τ:

$$\partial_{a_\mu}\partial_{a_\nu} - \partial_{a_{\mu'}}\partial_{a_{\nu'}} \quad (\mu + \nu = \mu' + \nu')$$
$$Z(\xi) \quad (\xi \in sl_{n+1})$$
$$\sum_\mu a_\mu \partial_{a_\mu} + 1.$$

- It's known that [Batyrev '93] residues of the family of affine toric hypersurfaces $Y_b \cap T$ are solutions to the GKZ system (with sl_{n+1} replaced by its Cartan.) This system, however, is never complete.
- It was conjectured in the 90's that τ above is complete for both $\mathbf{\Pi}, \mathbf{\Pi}'$.

13. RH for hypersurfaces in $X = \mathbb{P}^n$

- The completeness property was confirmed recently.
- **Theorem:** [Bloch-Huang-L-Srinivas-Yau '13] τ above is complete. In other words, it solves the RH problem for both period sheaf and residue sheaf of the family of CY hypersurfaces in $X = \mathbb{P}^n$.
- This is a consequence of a more general result for an arbitrary projective G-variety, to be discussed next.

14. Tautological systems – set up

Data:

G: complex algebraic group, with Lie algebra $\mathfrak{g}$
X: a projective G-manifold
L: a G-equivariant very ample line bundle on X

Notations:

$V := \Gamma(X, L)^\vee$
$\mathbb{C}[V]$: ring of polynomial functions on V
$Z : \mathfrak{g} \to \mathrm{End}\, V$, Lie algebra action
$\phi : X \hookrightarrow \mathbb{P}V$ the corresponding 'tautological' embedding

Ingredients:

I_ϕ: the ideal of $X \subset \mathbb{P}V$
$R_X := \mathbb{C}[V]/I_\phi$, the coordinate ring of X
$\langle , \rangle$: symplectic pairing on $TV^\vee \equiv V \times V^\vee$
D_V: the ring of polynomial differential operators on V
$D_{V^\vee}$: the ring of polynomial differential operators on $V^\vee$

15. Example to keep in mind

$G = \mathbb{P}GL_3$
$X = \mathbb{P}^2$
$L = O(3)$

$V = Sym^3\ \mathbb{C}^3 = \sum_{i=0}^{9} \mathbb{C}a_i, \quad V^\vee = \sum_{i=0}^{9} \mathbb{C}a_i^\vee$
$\mathbb{C}[V] = \mathbb{C}[a_0^\vee, .., a_9^\vee] \equiv \mathbb{C}[a^\vee]$
$Z(\xi) = \sum_i \xi_{ij} a_i \partial_{a_j}, \quad \xi \in \mathfrak{g}$
$\phi : X \hookrightarrow \mathbb{P}V$, $[z_0, z_1, z_2] \mapsto [z_0^3, z_0^2 z_1, .., z_2^3]$, Segre embedding

I_ϕ=the quadratic ideal generated by the Veronese binomials
$R_X \simeq \mathbb{C}[x_0^3, x_0^2 x_1, ..., x_2^3] \subset \mathbb{C}[x]$
$\langle a_i, a_j^\vee \rangle = \delta_{ij}$
$D_{V^\vee} = \mathbb{C}[a_0, ..., a_9, \partial_{a_0}, .., \partial_{a_9}]$, the Weyl algebra on $V^\vee$
$D_V = \mathbb{C}[a_0^\vee, ..., a_9^\vee, \partial_{a_0^\vee}, .., \partial_{a_9^\vee}] \supset \mathbb{C}[V]$.

16. Tautological systems

Consider the **Fourier transform**:

$$D_V \xrightarrow{\sim} D_{V^\vee}, \quad a_i^\vee \mapsto \partial_{a_i},\ \partial_{a_i^\vee} \mapsto -a_i.$$

For $p(a^\vee) \in \mathbb{C}[V] \equiv \mathbb{C}[a^\vee]$, $\widetilde{p(a^\vee)} = p(\partial_a) \in \mathbb{C}[\partial_a]$ is a differential operator with constant coefficients.

Definition: Fix $\beta \in \mathbb{C}$. Let $\tau \equiv \tau(X, L, G, \beta)$ be the left ideal in $D_{V^\vee}$ generated by the following differential operators:

$$\begin{array}{ll} p(\partial_a), \quad p(a^\vee) \in I_\phi & (\textit{polynomial operators}) \\ Z(\xi), \quad \xi \in \mathfrak{g} & (\textit{symmetry operators}) \\ \sum_i a_i \partial_{a_i} + \beta & (\textit{Euler operator}) \end{array}$$

We call such a differential system a **tautological system** .

17. Basics about τ

- τ makes sense if we replace I_ϕ by any G-invariant ideal $I \subset \mathbb{C}[V]$, and β by any Lie algebra character of $\hat{\mathfrak{g}} \equiv \mathfrak{g} \oplus \mathbb{C}$. The more general set-up is needed to deal with Fano and general type varieties.

- τ is **regular holonomic** if X has only finite number of G orbits. Thus, any formal power series solution is analytic; the sheaf of analytic solutions has finite rank at each point. [L-Song-Yau '11, L-Yau '12].

- τ has **solution rank** which is bounded above by the degree of $\phi : X \hookrightarrow \mathbb{P}V$ if the R_X is Cohen-Macaulay [L-Song-Yau '11, L-Yau '12].

18. Special cases of τ

- τ specializes to a **GKZ system** [GKZ '90] if X is a toric variety and $G = T$ is the dense torus. Basic results above were well-known in this case.

- τ specializes to a **Kapranov system** [Kapranov '97] if X a certain spherical variety of a semisimple group G.

- τ has the following explicit description if $X = G/P$: its polynomial operators are precisely given by

$$(C_{\mathfrak{g}} - \lambda) \sum_{ij} \mathbb{C} \partial_{a_i} \partial_{a_j}$$

  $C_{\mathfrak{g}}$ =quadratic Casimir of $\mathfrak{g}$; λ is certain distinguished eigenvalue of $C_{\mathfrak{g}} \in \text{End}\, V^{\vee}$. This follows from an old result of Kostant.

- **Example:** For $X = \mathbb{P}^n$, $L = -K_X$, the polynomial operators recover with the Veronese quadratic operators above.

19. Period sheaf, residue sheaf and τ

Assume X is Fano G-variety, and consider first the universal family of Calabi-Yau hypersurfaces $\mathcal{Y}$ in X.

Theorem: [L-Yau '12] The period sheaf and residue sheaf of $\mathcal{Y}$ are both subsheaves of the solution sheaf of

$$\tau = \tau(X, -K_X, G, 1).$$

- More generally, we can replace $-K_X$ by a very ample line bundle L such that $K_X + L$ is big with sections, and consider the family of **general type** hypersurfaces in X parameterized by $V^\vee = \Gamma(X, L)$. In this case, an analogous result holds, with τ being a D-module defined on the space

$$W^\vee \times V^\vee \equiv \Gamma(X, K_X + L)^\vee \times \Gamma(X, L)^\vee.$$

The ideal $I_\phi \subset \mathbb{C}[V]$ is replaced by certain G-invariant, but non-radical ideal in the ring $\mathbb{C}[W \times V]$.

20. Period sheaf, residue sheaf vs. τ

- Likewise, we can take L to be very ample such that $-K_X - L$ is ample with section, and consider the family of **Fano** hypersurfaces in X parameterized by $V^\vee = \Gamma(X, L)$. In this case, an analogous result holds, with τ being a D-module defined on

$$W^\vee \times V^\vee \equiv \Gamma(X, K_X + pL)^\vee \times \Gamma(X, L)^\vee$$

for some positive $p \in \mathbb{Z}$. The ideal $I_\phi \subset \mathbb{C}[V]$ is replaced by certain G-invariant ideal in $\mathbb{C}[W \times V]$.

- **Example:** $X = \mathbb{P}^4$, $L = \mathcal{O}(3)$, $p = 2$. In this case, the solution sheaf of the new τ contains the periods of cubic threefolds of the form

$$\int_\gamma \operatorname{Res} \frac{g\Omega}{f^2}.$$

21. Descriptions of τ

- To discuss two descriptions of τ: one geometric, the other algebraic. To focus on the CY case with $\beta = 1$ — there are analogous results for the general case.
- Assume the *finite-orbit condition* on the G-variety X.
- Recall $V^\vee \equiv \Gamma(X, L)$, $\hat{\mathfrak{g}} \equiv \mathfrak{g} \oplus \mathbb{C}$. Put

$$f \equiv \sum a_i a_i^\vee : V^\vee \times X \to L \qquad \text{universal section of } L$$

$$(b, x) \mapsto \sum a_i(b) a_i^\vee(x)$$

$$D_{X,\beta} := (D_X \otimes \mathbb{C}_\beta) \otimes_{\hat{\mathfrak{g}}} \mathbb{C}, \qquad \text{character D-module on } X$$

$$U := \{(b, x) \in V^\vee \times X | f(b, x) \neq 0\} \text{ universal complement}$$

$$U_b := \{(b, x) \in U | b(x) \neq 0\} \simeq X - Y_b$$

$$\pi^\vee : U \twoheadrightarrow V^\vee \qquad \text{projection (affine)}$$

$$\pi_+^\vee : D_h^b(U) \to D_h^b(V^\vee) \qquad \text{the derived pushforward}$$

$$M \mapsto R\pi_*^\vee(\Omega^\bullet_{U/V^\vee} \otimes M[\dim X]) \quad \text{(GM construction.)}$$

22. A cohomological description of τ

Theorem A: [Huang-L-Zhu '14] There is a canonical isomorphism of D-modules

$$\boxed{\tau \equiv \tau(X, G, -K_X, \beta) \simeq H^0\pi_+^\vee(\mathcal{O}_{V^\vee} \boxtimes D_{X,\beta})|_U.}$$

Hence at each $b \in V^\vee$

$$\mathrm{Hom}(\tau, \mathcal{O}_{V^\vee})_b \simeq H_c^n(U_b, \mathrm{Sol}(D_{X,\beta})|_{U_b}).$$

- We have $D_{X,\beta} = D_X/D_X\mathfrak{g} \twoheadrightarrow D_X/D_X T_X = \mathcal{O}_X$. So the isomorphism yields
$$\tau \twoheadrightarrow H^0\pi_+^\vee\mathcal{O}_U.$$
This is an isomorphism if G generates T_X.

- Taking Hom at each $b \in V^\vee$ yields a *canonical injection*
$$H_n(X - Y_b) \simeq \mathrm{Hom}(H^0\pi_+^\vee\mathcal{O}_U, \mathcal{O}_{V^\vee})_b \hookrightarrow \mathrm{Hom}(\tau, \mathcal{O}_{V^\vee})_b.$$

23. Application: completeness problems

- The canonical injection factors through the 'cycle-to-residue' map
$$H_n(X - Y_b) \to \mathbf{\Pi}'_b, \quad \gamma' \mapsto \left(\int_{\gamma'} \frac{\Omega}{f}\right)_b.$$
In particular, this map is injective $\forall b \in V^\vee$, hence gives a sharp lower bound on ranks
$$\boxed{\dim \mathrm{Hom}(\tau, \mathcal{O}_{V^\vee})_b \geq rk\ \mathbf{\Pi}'_b \geq \dim H_n(X - Y_b).}$$
- The canonical injection is isomorphic if G generates T_X. In this case, at generic $b \in V^\vee$, the solution rank
$$\boxed{\begin{aligned} \dim \mathrm{Hom}(\tau, \mathcal{O}_{V^\vee})_b \quad &= \dim H^n(X)_{prim} + \dim H^{n-1}(Y_b) \\ &\quad - \dim H^{n+1}(X). \end{aligned}}$$
- For $X = G/P$, we get $\mathbf{\Pi}' \simeq \mathrm{Hom}(\tau, \mathcal{O}_{V^\vee})$. In particular, $\mathbf{\Pi} \simeq \mathrm{Hom}(\tau, \mathcal{O}_{V^\vee})$ iff $H^n_{prim}(X) = 0$.
Example: $X = \mathbb{P}^n$, or $\dim X$ is odd. Cf. [BHLSY '13].

24. Application: chain integral solutions

- Consider now opposite extreme with $G \curvearrowright X$ having fixed points: take X toric variety and $G = T$, so τ is a GKZ system. Then $H_n(X - Y_b) \overset{\neq}{\hookrightarrow} \mathrm{Hom}(\tau, \mathcal{O}_{V^\vee})_b$.

  Theorem: [Huang-L-Yau-Zhu '15] Put $D =$union of T-divisors in X. Then the cycle-to-residue map above factors through the isomorphism

  $$H_n(X - Y_b, D - Y_b) \overset{\sim}{\to} \mathrm{Hom}(\tau, \mathcal{O}_{V^\vee})_b, \quad C \mapsto \left(\int_C \frac{\Omega}{f} \right)_b.$$

  In other words, every solution to the GKZ system τ is the integral over a **chain** with boundary on the toric divisor D.

- The 'extra' chain integral solutions to τ here generalize 'semi-periods' of the mirror quintics [Avram et al '95]. They are also similar to periods in type IIB open string theory [cf. Walcher et al '08], where the boundary of chains are holomorphic curves (D-branes) in a CY threefold.

25. Application: chain integral solutions

- It's known that solutions to GKZ systems can be represented as 'Euler integrals' [GKZ '90] – formal integrals of multivalued forms on algebraic tori T. They can also be represented as cohomology on complements of hypersurfaces in T with coefficients in a suitable local system. Our chain integral isomorphism can be viewed as a topological/geometric analogue of this.
- The chain integral isomorphism can also give a purely topological proof of the famous formula [GKZ '90]:

$$rk\ \mathrm{Hom}(\tau_{GKZ}, \mathcal{O}_{V^\vee}) = vol(P_D)$$

  where the right side is the volume of the Newton polytope of the space of sections of the line bundle $[D]$. It is done by first proving a vanishing theorem for the homology of the pair $(X - Y_b, D - Y_b)$ [Fu-Huang-L-Yau '15].
- The chain integral map holds for general G-variety with $D = \cup(G\text{-divisors})$ in X, but the map need not be surjective in general.

26. Application: rank one fibers

- **Problem in mirror symmetry:** To find LCSL points b_∞ in moduli space of Calabi-Yau.

- The MHS near a LCSL implies the period sheaf degenerates to rank 1 at such points. This suggests looking at singular points $b \in V^\vee$ of τ where

$$\dim H_n(X - Y_b) = 1$$

  We call such a singular Y_b a 'rank 1 fiber ' of the family $\mathcal{Y}$.

- **Example:** [Hosono-L-Yau '96] For X a toric variety, the union of T-divisors D in X is rank 1 fiber of the CY family $\mathcal{Y}$ in X. (E.g. $X = \mathbb{P}^n$, $D = \{x_0 x_1 \cdots x_n = 0\}$.)

- **Example:** [Bloch-Huang-L-Srinivas-Yau '13] For $X = G(k, n)$, $D = \{x_{1\cdots k} x_{2\cdots k+1} \cdots x_{n1\cdots k-1} = 0\}$ is a rank 1 fiber of the CY family $\mathcal{Y}$ in X.

27. Richardson stratification

- **Example:** [Huang-L-Zhu '14] Next, we construct rank 1 fibers in $X = G/P$. Recall that the flag variety G/B has a canonical stratification [Reitsch '05, Knutson-Lam-Speyer '10]:
$$G/B = \bigsqcup'_{(u,w)}[(B^- uB/B) \cap (BwB/B)].$$
The projection $G/B \twoheadrightarrow X = G/P$ yields a stratification on X.
- Put $Y_\star$:=closure of the codimension 1 stratum in X. Lam observed that the Kazhdan-Lusztig conjecture implies
$$\dim H_n(X - Y_\star) = 1.$$
- Moreover $[Y_\star] = -K_X$, hence $Y_\star$ is a rank 1 fiber of the universal CY family in X.
- A result of Kostant-Luna implies the point $\star \in V^\vee$ is GIT semistable (its G-orbit is closed), so $Y_\star$ survives as a point in the moduli space of CYs in X.

Conjecture: [Huang-L-Zhu '14]
$Y_\star$ is a LCSL of the CY family $\mathcal{Y}$ in X.

28. **Period mappings**

- We can also use the geometric description of τ (Theorem A) to translate our differential system τ for the periods of the family $\mathcal{Y}$ in X, to give an *explicit* PDE system for the each 'step' of the Hodge filtration, hence of the period mapping of the family.

- In fact, the pth step of the Hodge filtration gives rise to a *tautological system* of the form

$$\tau(X, I_p, G, \beta_p)$$

defined on the G-module $V^\vee \times V^\vee$ ($V^\vee \equiv \Gamma(X, L)$.) Here I_p is certain G-invariant ideal in the ring $\mathbb{C}[V \times V]$, and β_p is suitable $\hat{\mathfrak{g}}$ character. Details in [Chen-Huang-L '15].

29. Periods near LCSL

- Consider the a mirror pair of toric varieties X, X^* [Batyrev '93], and the CY hypersurface family $\mathcal{Y}$ in X. We consider the periods of this family in some neighborhood $\mathcal{U}_\infty$ of the standard LCSL $D \equiv$union of T-divisors in X.
- The local solution space of $\tau = \tau(X, -K_X, T, 1)$ at LCSL can be explicitly given by a cohomology-valued function $B_X : \mathcal{U}_\infty \to H^\bullet(X^*)$ [Hosono-L-Yau '95]

$$B_X(a) := \sum_{d \in M_+(X^*)} \prod_{k=1}^{d\cdot[Y^*]} ([Y^*] + k) \;\; O_{X^*}(d) \times a_0^{-d\cdot[Y^*]-[Y^*]-1} \prod_i a_i^{d\cdot D_i^* + D_i^*}.$$

  Here $M_+(X^*)$ is the lattice points in the Mori cone, the D_i^* are the T-divisors, and $[Y^*]$ the anticanonical divisor of X^*; $O_{X^*}(d)$ are certain combinatorially defined cohomology class of X^*.
- This was obtained by 'renormalizing' the formal Γ-series solutions of [GKZ '90].

30. Monodromy

- By analytic continuation (monodromy), the formula determines a local system $\mathbb{H}^\bullet(X^*)$, with general fiber $H^\bullet(X^*)$. By the chain integral map, we have the canonical isomorphism (over B):

$$H^n(X - Y_b, D - Y_b) \simeq \mathbb{H}^\bullet(X^*)_b.$$

Hyperplane Conjecture: [Hosono-L-Yau '95] The period sheaf $\mathbf{\Pi}$ of $\mathcal{Y}$ is isomorphic to the local system

$$[Y^*] \cup \mathbb{H}^\bullet(X^*) \simeq \mathbb{H}^\bullet(X^*)/Ann([Y^*]).$$

- The conjecture is known to hold for $X = \mathbb{P}^n$ [L- M.Zhu '15]. Work on general case is in progress.

31. Monodromy

- Note that the leading term of $B_X(a)$, namely the $H^0(X^*)$ component, is invariant under *local* monodromy on $\mathcal{U}_\infty$. This invariant period is the period corresponding to the Lagrangian torus of the SYZ fibration of Y_b near the standard LCSL.

- A conjecture due to [Kontsevich '96] gives an explicit description of the monodromy of the period sheaf. Namely

$$\pi_1(B) \to \mathrm{Aut}\, H^\bullet(X^*)$$

is generated by e^J, $J \in H^2(X^*)/Ann([Y^*])$, together the operator $\mathcal{T}(\gamma) = \gamma - \int_{Y^*} Td(Y^*) \cup \gamma$.

32. Some open questions

- **Question:** Given X, L, G, for which characters β does $\tau = \tau(X, L, G, \beta)$ have nontrivial solutions?

  For $X = G/P$, $L = -K_X$, and the symmetry semisimple group G, it can be shown that τ has no nontrivial solutions unless the $\hat{\mathfrak{g}}$-character has $\beta(e) = 0, 1$ (and $\beta(\mathfrak{g}) = 0$ obviously) [Fu-Huang-L-Yau '15].

- Contrast this with GKZ system for toric varieties: τ_{GKZ} has same dimension for solution space for generic character β.

- A more detailed analysis for general a G-variety X suggests that the general answer depends in a subtle way on the geometry of the Whitney stratification given by the G-orbits.

33. Some open questions

- **Problem:** Construct a natural ring structure on the local system $H_0^{Lie}(\hat{\mathfrak{g}}, R_X e^f)^\vee \simeq \mathrm{Hom}(\tau, \mathcal{O}_{V^\vee})$. This is expected to agree with $QH^\bullet$ on the A-side.
- **Question:** Can we describe a finite family of Lagrangian cycles $\gamma_1, \gamma_2, \ldots$ in Y_b such that their periods $(\int_{\gamma_i} \mathrm{Res}\, \Omega/f)_b$ generate $\mathbf{\Pi}_b$? There is an explicit correspondence proposed by Hosono between certain 3-cycles on Y_b, and objects in the derived category of the mirror CY, using the function $B_X(a)$. We expect that this correspondence coincides with the HMS mirror functor in this case.
- **Problem:** When the symmetry group G of τ is reductive, compute periods in terms of G-invariant polynomials à la Luna-Schwarz theorem invariant theory. E.g. For cubic curves, periods have been computed explicitly in terms of the Aronhold invariants.
- **Question:** When τ is complete, i.e. determines the periods on the B-side, hence the mirror map and GW invariants on the A-side near LCSL, can we recover τ directly by HMS?