Cohomological Hall algebras, Nekrasov instantons, and Coulomb branches of $3d$ $\mathcal{N} = 4$ gauge theories

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My talk is devoted to the geometric representation theory in the spirit of the papers of Nakajima in the 90's (see his “Lectures on Hilbert schemes"). We know after the work of many people including Atiyah, Drinfeld, Hitchin, Manin, Donaldson, Nakajima that the moduli space of framed $U(r)$ instantons of charge n on $S^4 = \mathbb{R}^4 \cup \{\infty\}$ can be identified with the moduli space $\mathcal{M}(r, n)$ of torsion-free sheaves of rank r and $c_2 = n$ on $\mathbb{CP}^2$ together with a fixed trivialization on the given line $l_\infty \simeq \mathbb{CP}^1 \subset \mathbb{CP}^2$. The conjecture of Alday-Gaiotto-Tachikawa (see arXiv: 0906.3212) predicts that on the direct sum over n of equivariant Borel-Moore homology groups of $\mathcal{M}(r, n)$ there is an action of the W-algebra of $gl(r)$. This is a generalization of the result of Nakajima for $r = 1$. It is also an incarnation of the so-called BPS/CFT correspondence of Nekrasov, which predicts that various structures of 4d gauge theories correspond in a non-trivial way to certain structures of 2d CFT's (another point of view was suggested by Gaiotto).

One of my goals is to discuss a generalization of the AGT conjecture (now theorem after Maulik-Okounkov, Schiffmann-Vasserot). This generalization is related to the recent works of Nekrasov, Gaiotto, Rapcak and others in physics and my joint work with Rapcak, Yang, Zhao, Creutzig, Chuang, Diaconescu in mathematics (see arXiv:1810.10402, arXiv:1907.13005). Differently from the previous work on the subject, we treat the problem from the perspective of $3CY$ categories rather than $2CY$ categories. In our approach main part is played by the Cohomological Hall algebra (COHA) introduced in my joint paper with Maxim Kontsevich (see arXiv: 1006.2706) .

If time permits, in the second part of the talk I will discuss the relation of COHA to $3d$ $\mathcal{N} = 4$ gauge theories. Besides of the relationship to COHA the topic of $3d$ $\mathcal{N} = 4$ theories seems to be related to my joint project with Kontsevich on Holomorphic Floer Theory and generalized Riemann-Hilbert correspondence. This part of the talk will be very speculative, since I am not an expert in $3d$ $\mathcal{N} = 4$ theories.

COHA: reminder

For a quiver Q endowed with potential W (=cyclically invariant polynomial in arrows) the Cohomological Hall algebra as a graded vector space is given by:

$$\mathcal{H}^{(Q,W)} = \oplus_{\gamma\in\mathbb{Z}^I_{\geq 0}} H^{BM}_{G_\gamma}(M_\gamma(Q), \phi_{Tr(W)}).$$

Here $M_\gamma(Q)$= space of representations of Q of dimensional vector $\gamma = (\gamma^i)_{i\in I}$ (I=set of vertices of Q). The gauge group G_γ is the product of general linear groups $GL(\gamma^i, \mathbb{C})$, it acts by changing the basis. Summands are the equivariant Borel-Moore homology of M_γ with coefficients in the sheaf of vanishing cycles of the function $Tr(W)$ (BM homology:=dual to compactly supported cohomology). The associative algebra structure is defined by considering short sequences of representations of a Q and utilizing Thom-Sebastiani theorem for vanishing cycles.

Quivers and potentials for today

a) Quiver Q_3: one vertex, three loops B_1, B_2, B_3, potential $W_3 = B_3[B_1, B_2]$. Representation of (Q_3, W_3)=critical locus of $Tr(W_3)$ = variety of commuting matrices: $[B_i, B_j] = 0$.

b) More interesting is the framed quiver Q_3^{fr}:

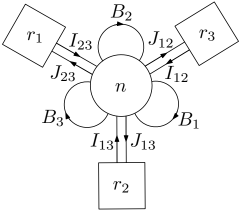

Figure: Framed quiver for the spiked instantons moduli space

$$W_3^{fr} := B_3([B_1, B_2] + I_{12}J_{12}) + B_2([B_1, B_3] + I_{13}J_{13}) + B_1([B_2, B_3] + I_{23}J_{23}) = W_3 + B_1 I_{23}J_{23} + B_2 I_{13}J_{13} + B_3 I_{12}J_{12}.$$

Spiked instantons of Nekrasov

We call a representation of (Q_3^{fr}, W_3^{fr}) stable if there is no non-trivial $V \subset \mathbb{C}^n$, which is invariant with respect to $B_i, i = 1, 2, 3$, and $I_{ab}(\mathbb{C}^{r_c}) \subset V, \{c\} = \{1, 2, 3\} - \{a, b\}$. Stable representations form the moduli space $\mathcal{M}(r_1, r_2, r_3, n)$. This moduli space turns out to be isomorphic to the moduli space of spiked instantons introduced by Nekrasov in arXiv:1608.07272. If only B_1, B_2, I_{12}, J_{12} are present on the figure we arrive to the classical description of the moduli space $\mathcal{M}(r, n) = \mathcal{M}(0, 0, r, n)$ of rank r instantons mentioned before. Differently from this special case there is no pure geometric description of $\mathcal{M}(r_1, r_2, r_3, n)$ in general. It should be the moduli space of coherent sheaves on $\mathbb{C}^3$ supported on the union of coordinate planes in $\mathbb{C}^3$, which are pure of rank r_c on the coordinate plane $x_c = 0, c = 1, 2, 3$ and satisfying some additional stability condition. The moduli space $\mathcal{M}(r, n)$ should be thought of as the one associated with the divisor $\mathbb{C}^2 \subset \mathbb{C}^3$.

Let $V_{r_1,r_2,r_3} = \oplus_{n\geq 0} H^{BM}_{GL(n)\times GL(r_1)\times GL(r_2)\times GL(r_3)\times \mathbf{T}_2}(\mathcal{M}(r_1,r_2,r_3,n),\phi_{W_3^{fr}})$, where the torus $\mathbf{T}_2=(\mathbb{C}^*)^2\subset(\mathbb{C}^*)^3$ acts by $(B_i,I_{ab},J_{ab})\mapsto(t_iB_i,I_{ab},t_at_bJ_{ab})$ subject to the Calabi-Yau condition $t_1t_2t_3=1$. Various versions of COHA for (Q_3,W_3) act on V_{r_1,r_2,r_3}. The most interesting for us is the following one. Consider equivariant spherical COHA $\mathcal{SH}^{(Q_3,W_3)}$ which is defined as follows: first we define the equivariant version of COHA adding the action of $\mathbf{T}_2$ on the loops B_i, then we take the subalgebra of the equivariant COHA generated by the graded component with $\gamma=1$. In arXiv:1810.10402 we proved the following result.

Theorem

The algebra $\mathcal{SH}^{(Q_3,W_3)}$ acts on V_{r_1,r_2,r_3} by correspondences. There is a Hopf algebra structure on $\mathcal{SH}^{(Q_3,W_3)}$. The Drinfeld double Hopf algebra $D(\mathcal{SH}^{(Q_3,W_3)})$ acts on V_{r_1,r_2,r_3} as well.

Further comments

a) The Drinfeld double of equivariant spherical COHA is isomorphic to the affine Yangian of $gl(1)$. This observation is based on the dimensional reduction theorem for COHA proved by M.K.+Y.S. in 1006.2706. This reduction also explains how previous approaches to the AGT conjecture are related to $3CY$ categories.

b) As a corollary of the above Theorem we proved that the recently introduced by Gaiotto and Rapcak vertex algebra at the corner W_{r_1,r_2,r_3} (see arXiv:1703.00982) acts on V_{r_1,r_2,r_3}.

c) From physics perspective we study the action of BPS algebra of $D0$ branes on the Calabi-Yau 3-fold $\mathbb{C}^3$ on the space of $D0-D4$ bound states. There are other interpretations as well.

d) Conjecturally the results can be generalized to any toric Calabi-Yau 3-folds defined by a toric diagram with faces colored by non-negative integers r_c (e.g. decorated dimer model).

e) Another class of representations arises when we consider the Hilbert scheme of points of the non-reduced singular divisor $x_1^{r_1}x_2^{r_2}x_3^{r_3}=0$ of $\mathbb{C}^3$. In the above-mentioned paper with Chuang, Creutzig and Diaconescu arXiv:1907.13005 we studied the simplest case of the “fat divisor" $x_3^{r_3}=0$. This is a generalization of the Nakajima result for $r_3=1$. Our story is more difficult, since the structure of Hilbert scheme is not well-studied. Main result: the BM homology gives the vacuum representation of the W-algebra of $gl(r_3)$. We use the relation of the Hilbert scheme to the moduli space of Higgs bundles on $\mathbb{CP}^2$ with nilpotent Higgs field, which comes from the multiplication by x_3. The corresponding quiver is obtained from the one in the Nakajima book by adding an additional loop at the framing vertex and fixing the conjugacy class of the loop. This is the end of the first part of the talk.

Origin of $3d$ $\mathcal{N}=4$ mirror symmetry

Since this is the workshop on mirror symmetry I will start with the very incomplete reminder on the mirror symmetry for $3d$ $\mathcal{N}=4$ theories.
Recall that the origin of HMS was the superstring theory in $10d$ compactified by a Calabi-Yau 3-fold. Compactification on mirror dual CY 3-folds exchanges the moduli spaces of vector and hyper multiplets for the corresponding $4d$ theories. Intrilligator and Seiberg in 1996 suggested to consider the $4d$ theories on $\mathbb{R}^3 \times S^1_R$ and then further compactify on the circles of mutually inverse radii R and $1/R$. The moduli space of vacua of the arising $3d$ $\mathcal{N}=4$ theories $\mathcal{T}$ and $\mathcal{T}^\vee$ have hyperkähler components called Coulomb $\mathcal{C}$ and Higgs $\mathcal{H}$ branches which get interchanged, e.g. $\mathcal{M}_\mathcal{C}(\mathcal{T}) \simeq \mathcal{M}_\mathcal{H}(\mathcal{T}^\vee)$. This duality was called by I-S the mirror symmetry for $3d$ $\mathcal{N}=4$ theories.

Three levels of MS

The Higgs branch does not change when one compactifies the $4d$ to $3d$, while Coulomb branch receives instanton corrections. Physicists have defined A and B twists of $3d$ $\mathcal{N} = 4$ theories without reference to superstrings. Twists give rise to TQFTs of A and B types. For details see large papers of Tudor Dimofte with coauthors, e.g. arXiv:1908.00013. They argue that there are three levels of mirror symmetry in $3d$ $\mathcal{N} = 4$ theories: 2-category *Bdy* of boundary conditions, symmetric monoidal 1-category of line operators Lines and (shifted) Poisson algebra of local operators *Ops*. The Coulomb branch of the theory is an affine Poisson variety with open symplectic leaf (typically symplectic cone). The Coulomb branch $\mathcal{M}_\mathcal{C}$ is the spectrum of the commutative algebra $\mathbb{C}[\mathcal{M}_\mathcal{C}] = Hom(\mathbf{1}, \mathbf{1}) = Ops$ of the monoidal unit in Lines. (Cf: two levels for $2d$ $\mathcal{N} = (2, 2)$: Bdy=FS-category and Ops=(shifted) Poisson algebra of its Hochschild cohomology).

Quantized Coulomb branch

One expects that the above structures can be quantized (in physics they speak about the theory in the presence of Ω-background). Then one gets the quantized algebra $\mathbb{C}[\mathcal{M}_{\mathcal{C}}]_{\hbar}$ and can ask for a reasonable category of modules over it. If $\mathcal{M}_{\mathcal{C}}$ is a symplectic conic singularity (this is not the most general case), one takes equivariant symplectic resolution $\widehat{\mathcal{M}_{\mathcal{C}}} \to \mathcal{M}_{\mathcal{C}}$, and gets the algebra $\mathbb{C}[\widehat{\mathcal{M}_{\mathcal{C}}}]_{\hbar}$. Example: $\mathcal{M}_{\mathcal{C}}$=nilpotent cone $\mathcal{N}_{\mathfrak{g}}$ of $\mathfrak{g} = \mathit{Lie}(G)$, $\widehat{\mathcal{M}_{\mathcal{C}}}$ is Springer resolution $T^*(G/B) \to \mathcal{N}_{\mathfrak{g}}$. Then the quantization gives $\hbar - D$-modules on G/B. There are interesting subcategories of $\mathbb{C}[\widehat{\mathcal{M}_{\mathcal{C}}}]_{\hbar} - \mathit{mod}$ which can be obtained if one assumes that $\mathcal{M}_{\mathcal{C}}$ is a symplectic conic singularity. E.g. there is an analog of the category $\mathcal{O}$ of Bernstein-Gelfand-Gelfand introduced by Ben Webster with coauthors. See e.g. arXiv:1407.0964.

Mathematical definition of the Coulomb branch

Mathematical definition of the Coulomb branch for a big class of $3d$ $\mathcal{N}=4$ theories was proposed by Braverman-Finkelberg-Nakajima. Those theories are parametrized by pairs $(G, \mathbf{M})$, where G is a complex reductive group and $\mathbf{M}$ its symplectic representation. According to BFN the algebra of functions on $\mathcal{M}_\mathcal{C}$ is given by the equivariant Borel-Moore homology $H_\bullet^{G(\mathcal{O})}(\mathcal{R}), \mathcal{O}=\mathbb{C}[[t]]$ of the space $\mathcal{R}$ closely related to the affine Grassmannian Gr_G. Dimofte with coauthors argue that $Hom(\mathbf{1},\mathbf{1})$ is isomorphic to the algebra defined by BFN. Physics also implies that for a good boundary condition $\mathbf{B}$ in the $3d$ $\mathcal{N}=4$ theory there is a module $V_\mathbf{B}=Hom(\mathbf{1},\mathbf{B})$ over $\mathbb{C}[\mathcal{M}_\mathcal{C}]$ which is in the infrared limit becomes a coherent sheaf on $\mathcal{M}_\mathcal{C}$ with complex Lagrangian support L.

Quantized Coulomb branch

Quantization in BFN story: take $G(\mathcal{O})\times\mathbb{C}^*$ equivariant BM homology. Assuming symplectic resolution, a boundary condition gives rise to a holonomic DQ-module over $\mathbb{C}[\widehat{M_{\mathcal{C}}}]_\hbar$. Via the generalized Riemann-Hilbert correspondence of Kontsevich and myself holonomic DQ-modules in the quantized category correspond to objects of the partially wrapped Fukaya category of $\widehat{\mathcal{M}_{\mathcal{C}}}$ with $\omega+iB=\omega^{2,0}$ (at least for $G=GL(n,\mathbb{C})$).

Question

What are these objects?

E.g. if $\mathcal{M}_{\mathcal{C}}=\mathcal{N}_{\mathfrak{g}}$, the (complex) Lagrangian support of holonomic DQ-modules belongs to the union of conormals to Schubert cells in G/B. Assuming also symplectic $\mathbb{C}^*$-action one can define complex Lagrangian skeleton. This is the support for "good" holonomic DQ-modules.

COHA and Coulomb branch: an example

We have seen in the first part that $D(\mathcal{SH}^{(Q_3,W_3)}) \simeq Y(\widehat{gl}(1))$. Consider now the $3d$ $\mathcal{N} = 4$ gauge theory $\mathcal{T}_{Jordan}$ corresponding to $G = GL(n)$ and the symplectic representation $\mathbf{M} = \mathbf{N} \oplus \mathbf{N}^*$ where $\mathbf{N} = gl(n) \oplus (\mathbb{C}^n)^{\oplus l}$ (adjoint representation of $gl(n)$ plus l copies of the standard one). According no Kodera-Nakajima (see arXiv:1608.00875) a certain deformation of the subalgebra Y_l of $Y(\widehat{gl}(1))$ is mapped surjectively to the quantized Coulomb branch $\mathbb{C}[\mathcal{M}_{\mathcal{C}}]_\hbar$ of $\mathcal{T}_{Jordan}$. The algebra Y_l is known as shifted Yangian. It admits explicit description by generators and relations. Conjecturally it can be realized in terms of representations of COHA of the CY 3-fold $T^*\mathbb{P}^1 \times \mathbb{C}$ in cohomology of the moduli space of stable pure coherent sheaves supported on a certain toric divisor. This will be discussed in my joint project with Rapcak, Yang and Zhao.

Framed quiver gauge theories

More general class of examples is related to framed quiver gauge theories of *ADE* type. In order to fix the theory one fixes the dimension vector v (=positive root of the corresponding quiver Q) as well as the framing dimension w (dominant weight of the gauge group G). Let Gr_w be the closure of the $G(\mathcal{O})$-orbit in Gr_G corresponding to w and Gr_w^v be the transversal slice. According to BFN the quantized algebra of functions on Gr_w^v is the quantized Coulomb branch for the corresponding framed quiver gauge theory. BFN also proved that this algebra is isomorphic to the shifted Yangian of $\mathfrak{g}=Lie(G)$, where the shift is determined by w and truncation by v.

Preliminary computations made by Yang and Zhao show that there are two homomorphisms of the equivariant preprojective algebra of Q to this quantized Coulomb branch. The equivariant preprojective algebra itself is the dimensional reduction of the equivariant COHA associated with the tripled quiver Q (tripled=add opposite for each arrow and add new loop at each vertex) endowed with a cubic potential $\sum_{i,a} l_i[a, a^*]$, where l_i is a loop at the vertex i and a (resp. a^*) is an arrow (resp. opposite arrow). Hopefully these two homomorphisms can be combined into a homomorphism of the double of COHA which we discussed in the first part of the talk.Then we obtain a homomorphism of the double of equivariant spherical COHA to the quantized Coulomb branch of a framed quiver gauge theory.

Holomorphic CS, monopoles and COHA

One can ask why COHA appears in the story with Coulomb branches. Recall that COHA is defined as the convolution algebra for the BM-homology of the stack of objects of a $3CY$ category. In the first part of the talk it was the $3CY$ category associated with the pair (Q, W). In our paper arXiv:1006.2706 we speculated about bigger framework in which COHA (or its derived version) could be defined. In particular, one can try to think of COHA of a compact CY 3-fold X endowed with the holomorphic CS functional $CS_{\mathbb{C}}(A) = \int_X Tr(\overline{\partial} A + \frac{1}{3}A^3)\Omega^{3,0}$, where A is a $(0,1)$-connection and $\Omega^{3,0}$ is a holomorphic volume form. Critical locus of $CS_{\mathbb{C}}$ consists of holomorphic vector bundles. We can think of the corresponding COHA as of $H_{\bullet}^{BM}(Coh_X, \phi_{CS_{\mathbb{C}}})$, where Coh_X is the stack of coherent sheaves. In the case of the complexified Chern-Simons this was made more precise in the last section of arXiv:1006.2706.

In arXiv:1503.03676 Nakajima proposed to define $\mathbb{C}[M_{\mathcal{C}}]$ by using the BM-homology with coefficients in the sheaf of vanishing cycles of the CS-type functional and then use the ideas of motivic Donaldson-Thomas theory (see arXiv:0811.2435, arXiv:1006.2706) in order to study invariants of Coulomb branches. His analog of the holomorphic CS functional depends on some choices: pair $(G, \mathbf{M})$ as before, compact Riemann surface C with fixed spin structure $K_C^{1/2}$, principal G-bundle P. Then, the Nakajima's functional depends on the fields which are: $\overline{\partial}+A$=the $(0,1)$-connection on P, and the Higgs field $\Phi \in \Gamma(C, K_C^{1/2} \otimes (P \otimes_G \mathbf{M}))$. The actual formula $CS_{Nak} = CS_{Nak}(A, \Phi) = \frac{1}{2}\int_C \omega_{\mathbf{M}}^{2,0}((\overline{\partial}+A)(\Phi) \wedge \Phi)$. Here the wedge product is multiplied by the holomorphic symplectic form on the vector space $\mathbf{M}$. This gives a C^∞ section of $(T^{0,1})^* \otimes K_C = (T^{1,1})^*$, which can be integrated over C.

Critical points

Critical points of CS_{Nak} are solutions to the system of equations

$$(\bar{\partial} + A)(\Phi) = 0,$$

$$\mu(\Phi) = 0,$$

where μ is the moment map for the symplectic reduction of **M** with respect to the symplectic action of G. Borel-Moore homology (with certain infinite shifts) of the critical set or set of all fields with coefficients in the sheaf of vanishing cycles $\phi_{CS_{Nak}}$ should be $\mathbb{C}[M_{\mathcal{C}}]$ if we take $C = \mathbb{P}^1$.

Comments

a) (derived considerations) Nakajima explains the upgrade of this approach to the derived framework. For that one drops the moment map condition in the definition of the critical set and considers the moment map as the map of the stack $\mathcal{X}$ of solutions to the derived stack $Higgs_G(C)$. By a result of Ginzburg-Rozenblyum this map is Lagrangian (in the derived sense). Another derived Lagrangian map is $Bun_G(C) \to Higgs_G(C)$. The derived fiber product of these two Lagrangians in the 0-shifted symplectic stack is the stack $Crit(CS_{Nak})$. It is (-1)-shifted symplectic by Pantev-Toën-Vaqué-Vezzosi in agreement with the expectation of physicists.

b) In the definition of $\mathcal{M}_\mathcal{C}$ proposed by BFN the geometry is closely related to the geometry of singular monopoles in 3d. This is the geometry of an affine 3d manifold with the affine 1d foliation and transversal complex structure.

Then one can try to connect the topic of Coulomb branches to parabolic Fukaya categories for complex surfaces (my last year talk) and to elliptic spaces (Kontsevich, 90's). The latter is an adequate language for categorical description of foliated manifolds with transversal complex structure (adequate structure for description of partially holomorphic twists in physics). The former appears when we consider partially wrapped Fukaya categories of non-compact complex symplectic surfaces. Then real blow-ups of compactifying boundary divisors are $3d$ manifolds with affine structure and affine $1d$ foliation. Transversal complex structure is automatic, since the boundary divisors are complex curves.

c) Nakajima's proposal cannot be taken literally since his definition involves infinite-dimensional spaces of solutions and infinite-dimensional gauge groups (hence infinite shifts in the BM homology grading of the corresponding stack). Also as in any definition which involves the sheaf of vanishing cycles one should choose the so-called orientation data, the notion introduced in our arXiv:0811.2435. The latter (probably) can be constructed using methods developed by Joyce with collaborators (see arXiv:1811.01096, 1908.03524). As for the former (more conceptual) difficulty, a solution can be to factorize the infinite-dimensional neighborhood of the critical set into the product of a finite-dimensional manifold and the "transversal" infinite-dimensional slice along which the CS_{Nak} has quadratic behavior. At least this idea works in the case of holomorphic CS and complexified CS functionals. Question whether multiplication is well-defined is open. We hope that the ideas of arXiv:0811.2435, 1006.27 06 (and others) will help to answer it.

Examples

1. Springer resolution for a semisimple group G is symplectically dual to the one for the Langlands dual $G^\vee$.
2. Affine type A quiver variety is symplectically dual to another affine type A quiver variety.
3. Finite type ADE quiver variety is symplectically dual to a slice in the affine Grassmannian of the Langlands dual group.
4. Hilbert scheme $Hilb_r(\widehat{\mathbb{C}^2/\Gamma})$, where $\Gamma = \mathbb{Z}/n\mathbb{Z}$ is symplectically dual to the moduli space $\mathcal{M}(r,n)$ of $rk = r, c_2 = n$ torsion-free framed sheaves on $\mathbb{CP}^2$ mentioned at the beginning of the talk.
5. Hypertoric variety is symplectically dual to another hypertoric variety.

6. Coulomb branch of the GMN theory of class $\mathcal{S}$ compactified on S^1 (such $3d, \mathcal{N} = 4$ theories are called Sicilian theories, and $\mathcal{M}_{\mathcal{C}}$ is a sort of moduli space of regular singular Higgs bundles on a curve) is mirror dual to the quiver variety associated with an appropriate $*$-shaped quiver.