The pushforward theorem and applications

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Homological Mirror Symmetry and Topological Recursion
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Outline

- joint with Dima Arinkin and Bertrand Toën
- Understand symplectic structures along the stalks of sheaves of derived stacks over a topological space.
- Apply to the geometry of the moduli of Stokes filtered local systems on a smooth variety over $\mathbb{C}$:
    - construct (shifted) Poisson structures;
    - describe their symplectic leaves.

Sheaves of derived stacks

Problem: Define and construct symplectic structures on the stalks of a sheaf of derived stacks $\mathscr{F}$ over a space S.

Note:

- For this to make sense the sections of the sheaf $\mathscr{F}$ will have to satisfy representability conditions.
- The non-degeneracy condition on a stalkwise symplectic form will have to involve some notion of duality for complexes of sheaves of $\mathbb{C}$-vector spaces on S.

Closed forms and symplectic structures

Recall: [P-Toën-Vaquié-Vezzosi] ([PTVV])

- If F is derived Artin locally f.p. over $\mathbb{C}$ we have a **complex of closed p-forms** $\mathcal{A}^{p,cl}(F)$ on F.
- When $F = \mathbf{Spec}\, A$, then $\mathcal{A}^{p,cl}(F)$ corresponds to the module $\mathrm{tot}^{\prod}(F^p(A)[p])$.
- An n-cocycle ω in the complex $\mathcal{A}^{2,cl}(F)$ is a **closed n-shifted 2-form**.
- ω is an **n-shifted symplectic structure** if the contraction $\omega^\flat : \mathbb{T}_F \longrightarrow \mathbb{L}_F[n]$ with the induced element in $H^n(F, \bigwedge^2 \mathbb{L})$ is a quasi-iso.

Structures on maps

Let $f : F \to F'$ be a morphism in $\mathbf{dSt}_{\mathbb{C}}$, then

- An n-shifted **isotropic structure** on f is a pair (ω, h), where ω is an $(n+1)$-shifted symplectic structure on F', and h is a homotopy between $f^*(\omega)$ and 0 inside the complex $\mathcal{A}^{2,cl}(F)$.
- An isotropic structure (ω, h) is **Lagrangian** if the induced morphism $h^\flat : \mathbb{T}_f \xrightarrow{\sim} \mathbb{L}_F[n]$ is a quasi-isomorphism.

Note:

• An n-shifted Lagrangian structure $(0, h)$ on $f : F \to \operatorname{Spec} \mathbb{C}$ is simply an n-shifted symplectic structure on F.

• Non-degeneracy: a duality between the **stacky** (positive degrees) and the **derived** (negative degrees) parts of $\mathbb{L}_X$.

Easy examples

- If $G/\mathbb{C}$ is reductive any non-degenerate $\kappa \in (\mathrm{Sym}^2\,\mathfrak{g}^\vee)^G$ gives rise to a canonical 2-shifted symplectic form ω_κ on BG whose underlying 2-shifted 2-form is $\mathbb{C}\to(\mathbb{L}_{BG} \wedge \mathbb{L}_{BG})[2] \simeq (\mathfrak{g}^\vee[-1] \wedge \mathfrak{g}^\vee[-1])[2] = \mathrm{Sym}^2\,\mathfrak{g}^\vee$ given by κ.
- The ***n*-shifted cotangent bundle** $T^\vee X[n] := \mathrm{Spec}_X(\mathrm{Sym}(\mathbb{T}_X[-n]))$ has a canonical n-shifted symplectic form.

Diagrams of sheaves and stacks

Fix

- $\mathbf{I}$ - a finitely presentable ∞-category;
- C - any category with finite limits.

Notation:

- $F_\bullet$ - a **diagram of shape $\mathbf{I}$ in C**.

Diagrams of sheaves and stacks

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a functor $F_\bullet : \mathbf{I} \to C$,
$F_\bullet = \{F_\alpha | \alpha \in \mathbf{I}\}$

Diagrams of sheaves and stacks

Fix

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Notation:

- $F_\bullet$ - a **diagram of shape I in** C.
- $F_{\mathbf{I}} = \lim_{\alpha \in \mathbf{I}} F_\alpha$ - **global sections of the diagram** $F_\bullet$.

Diagrams of sheaves and stacks

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- $\mathbf{I}$ - a finitely presentable ∞-category;
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Notation:

- $F_\bullet$ - a **diagram of shape $\mathbf{I}$ in C**.
- $F_{\mathbf{I}} = \lim_{\alpha\in\mathbf{I}} F_\alpha$ - **global sections of the diagram $F_\bullet$**.
- $\mathbf{I}^{\mathrm{tw}}$ - the category of **twisted arrows in $\mathbf{I}$**.

ob($\mathbf{I}^{tw}$): maps $x \xrightarrow{\gamma} y \in \mathrm{mor}(I)$;

$\mathrm{Hom}_{\mathbf{I}^{tw}}\begin{pmatrix} x_1 & , & x_2 \\ \downarrow\gamma_1 & & \downarrow\gamma_2 \\ y_1 & & y_2 \end{pmatrix}$: commutative diagrams $\begin{array}{ccc} x_1 & \xleftarrow{u} & x_2 \\ \gamma_1\downarrow & & \downarrow\gamma_2 \\ y_1 & \xrightarrow[v]{} & y_2 \end{array}$

Diagrams of sheaves and stacks

Fix

- $\mathbf{I}$ - a finitely presentable ∞-category;
- C - any category with finite limits.

Notation:

- $F_\bullet$ - a **diagram of shape I in C**.
- $F_{\mathbf{I}} = \lim_{\alpha\in\mathbf{I}} F_\alpha$ - **global sections of the diagram** $F_\bullet$.
- $\mathbf{I}^{\text{tw}}$ - the category of **twisted arrows in I**.
- $(t, s) : \mathbf{I}^{\text{tw}} \to \mathbf{I} \times \mathbf{I}^{op}$ - the natural functor.

Closed forms on diagrams of stacks (i)

Fix $E_\bullet : \mathbf{I} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$, and $\mathscr{F}_\bullet : \mathbf{I} \longrightarrow \mathrm{dSt}_{\mathbb{C}}$.

Consider the functor

$$\begin{aligned} \mathcal{A}^{p,cl}_{\mathbf{I}}(\mathscr{F}_\bullet)_{E_\bullet} : \quad \mathbf{I}^{tw} & \longrightarrow \mathrm{Vect}_{\mathbb{C}} \\ \gamma & \longrightarrow \mathcal{A}^{p,cl}(\mathscr{F}_{s(\gamma)}) \otimes E_{t(\gamma)}, \end{aligned}$$

and define the **complex of closed p-forms on $\mathscr{F}_\bullet$ with values in $E_\bullet$** as the complex

$$\boxed{\mathbb{A}^{p,cl}_{\mathbf{I}}(\mathscr{F}_\bullet)_{E_\bullet} = \lim_{\gamma \in \mathbf{I}^{tw}} \mathcal{A}^{p,cl}(\mathscr{F}_{s(\gamma)}) \otimes E_{t(\gamma)}.}$$

Closed forms on diagrams of stacks (iii)

The **space of closed p-forms on $\mathscr{F}_\bullet$ with values in $E_\bullet$** is defined as

$$\mathbf{A}_{\mathsf{I}}^{p,cl}(\mathscr{F}_\bullet)_{E_\bullet} = \left|\mathbf{DK}\left(\tau^{\leqslant 0}\mathbb{A}_{\mathsf{I}}^{p,cl}(\mathscr{F}_\bullet)_{E_\bullet}\right)\right|$$

and an **$E_\bullet$-valued closed p-form on $\mathscr{F}_\bullet$** is an element in $\pi_0\mathbf{A}_{\mathsf{I}}^{p,cl}(\mathscr{F}_\bullet)_{E_\bullet} = H^0(\mathbb{A}_{\mathsf{I}}^{p,cl}(\mathscr{F}_\bullet)_{E_\bullet})$.

Note: The space of forms comes equipped with a natural **global sections morphism**

$$\Gamma : \mathbb{A}_{\mathsf{I}}^{p,cl}(\mathscr{F}_\bullet)_{E_\bullet} \to \mathcal{A}^{p,cl}(\mathscr{F}_{\mathsf{I}}) \otimes E_{\mathsf{I}}.$$

Cospecialization and gluing (i)

Suppose

- X - nice topological space (e.g. a CW complex);
- $\imath : Z \hookrightarrow X$ - closed subspace;
- $\jmath : U \hookrightarrow X$ - the complementary open subspace.
- $\mathscr{C}$ -an ∞-category with all small limits and colimits.

For any $F \in \mathrm{Sh}(X, \mathscr{C})$ write $F_{|Z} = \imath^* F$ and $F_{|U} = \jmath^* F$. Applying $\imath^*$ to the unit of the adjunction $\jmath^* \dashv \jmath_*$ yields a **cospecialization map** in $\mathrm{Sh}(Z, \mathscr{C})$:

$$\mathrm{cosp}_Z : F_{|Z} \to \imath^* \jmath_* \left(F_{|U}\right).$$

Cospecialization and gluing (ii)

The assignment $F \longrightarrow \left(F_{|U}, F_{|Z}, \mathrm{cosp}_Z\right)$ provides an equivalence of ∞-categories:

$$\mathrm{Sh}(X, \mathscr{C}) \xrightarrow{\cong} \mathrm{lax}^{\mathrm{op}} \lim \left[\mathrm{Sh}(U, \mathscr{C}) \xrightarrow{\imath^* \jmath_*} \mathrm{Sh}(Z, \mathscr{C}) \right],$$

i.e. $\mathrm{Sh}(X, \mathscr{C})$ can be viewed as the lax$^{\mathrm{op}}$ limit of the functor $\imath^* \jmath_*$.

Key observation: Applying this gluing description to strata in a stratification leads to a combinatorial picture for closed forms and symplectic structures on constructible sheaves of stacks over a space.

Sheaves on a stratified space (i)

Suppose

- X is a **good stratified space**.
- I is the finite poset labeling the strata of X.
- $X_\alpha \subset X$ - the stratum labeled by $\alpha \in I$.
- $\mathrm{Sh}^{\mathrm{str}}(X)$ - sheaves F of spaces, constructible for the given stratification.
- $F_\alpha \in \mathrm{Sh}^{\mathrm{str}}(X_\alpha)$ - the restriction of F to X_α.

Sheaves on a stratified space (ii)

Construction: Fix $\alpha \in I$, $F \in \mathrm{Sh}^{\mathrm{str}}(X)$, and let

- $Z \subset X$ - a closed subset s.t. $X_\alpha \subset Z$ is open.
- $U \subset X$ - the complementary open to Z.
- $\overset{\circ}{F}_\alpha := \left(\imath^* \jmath_* \left(F_{|U}\right)\right)_{|X_\alpha}$.
- $\mathrm{cosp}_\alpha : F_\alpha \to \overset{\circ}{F}_\alpha$ - the restriction of cosp_Z to X_α.

Note:

- $\overset{\circ}{F}_\alpha$ and cosp_α depend only on α and not on Z.
- $\overset{\circ}{F}_\alpha$ is the sheaf of nearby (co) cycles of F along X_α, and cosp_α is the integral of cospecialization maps over nearby points.

Sheaves on a stratified space (iii)

Note: As with gluing, nearby cycles, and integrated cospecializations make sense for constructible sheaves with values in any category C that admits finite limits.

Let $\mathbf{I}$ be the **∞-category of exit paths** for the stratification on X. Then

- If I is good, then $\mathbf{I}$ is finitely presentable.
- $\mathrm{Sh}^{\mathrm{str}}(X) = \mathrm{Fun}(\mathbf{I}, \mathrm{SSets})$.
- For any category C with finite products we define

$$\mathrm{Sh}^{\mathrm{str}}(X, C) = \mathrm{Fun}(\mathbf{I}, C),$$

  i.e. C-valued constructible sheaves on X are $\mathbf{I}$-shaped diagrams in C.

Sheaves on a stratified space (iii)

Notation: Fix $F \in \mathrm{Sh}^{\mathrm{str}}(X, C)$. Then:

- Every $x \in X$ gives an object $\boldsymbol{\iota}(x) \in \mathbf{I}$. The value $F(\boldsymbol{\iota}(x)) \in C$ is called the **stalk of F at x** and is denoted by F_x.
- $\Gamma(X, F) := \lim_{\boldsymbol{\sigma} \in \mathbf{I}} F_{\boldsymbol{\sigma}} \in C$ is the **global sections** object of F. We have a natural **evaluation map** $\mathrm{ev}_x : \Gamma(X, F) \to F_x$ for every $x \in X$.
- For $\alpha \in I$ the fundamental groupoid $\Pi_1(X_\alpha)$ of the stratum X_α embeds in $\mathbf{I}$ as the full subcategory $\mathbf{I}_\alpha \subset \mathbf{I}$ spanned by $\boldsymbol{\iota}(x)$ for all $x \in X_\alpha$. We define the **value of F on X_α** as $F_\alpha := F_{|\mathbf{I}_\alpha} \in \mathrm{Sh}^{\mathrm{str}}(X_\alpha, C)$.

Sheaves on a stratified space (iv)

Notation: Fix $\alpha \in I$, and $F \in \mathrm{Sh}^{\mathrm{str}}(X, C)$. Then consider:

- $\mathbf{I}_\alpha^{\geqslant} = \{a' \to a \mid a' \in \mathbf{I}, a \in \mathbf{I}_\alpha\}$
- $\mathbf{I}_\alpha^{>} \subset \mathbf{I}_\alpha^{\geqslant}$ - the full subcategory consisting of all $a' \to a$ which are **not** isomorphisms.
- $\mathring{F}_\alpha \in \mathrm{Sh}^{\mathrm{str}}(X_\alpha, C)$ - the right Kan extension of $F_{|\mathbf{I}_\alpha^{>}}$ along the natural functor $\mathbf{I}_\alpha^{>} \to \mathbf{I}_\alpha$.
- $\mathrm{cosp}_\alpha : F_\alpha \to \mathring{F}_\alpha$ - the map guaranteed by the universal property of the right Kan extension.

Terminology: $\mathring{F}_\alpha$ is the **sheaf of nearby cycles** of F at X_α, and cosp_α is the (integrated) **cospecialization map**.

Sheaves on a stratified space (v)

Given a nice stratified space X with strata labeled by a poset I, and an exit path category $\mathbf{I}$, and
$E \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{Vect}_{\mathbb{C}}) = \mathrm{Fun}(\mathbf{I}, \mathrm{Vect}_{\mathbb{C}})$,
$\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}}) = \mathrm{Fun}(\mathbf{I}, \mathrm{dSt}_{\mathbb{C}})$, we get

- a **complex** $\mathbb{A}_X^{p,cl}(\mathscr{F})_E := \mathbb{A}_{\mathbf{I}}^{p,cl}(\mathscr{F}_\bullet)_{E_\bullet}$ and a **space** $\mathbf{A}_X^{p,cl}(\mathscr{F})_E := \mathbf{A}_{\mathbf{I}}^{p,cl}(\mathscr{F}_\bullet)_{E_\bullet}$ **of global closed E-valued p-forms on $\mathscr{F}$**;
- a natural **global sections map** of complexes

$$\Gamma : \mathbb{A}_X^{p,cl}(\mathscr{F})_E \to \mathcal{A}^{p,cl}(\Gamma(X, \mathscr{F})) \otimes \Gamma(X, E).$$

Sheaves on a stratified space (vi)

Let $\mathscr{F}$ be a constructible sheaf of locally f.p. derived Artin stacks. For any point $z \in \Gamma(X, \mathscr{F})$ the relative tangent complex $\mathbb{T}_{\mathscr{F},z} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{Vect}_{\mathbb{C}})$ is a constructible complex of vector spaces on X.

Given a point $z \in \Gamma(X, \mathscr{F})$, any closed form $\omega \in \mathbf{A}^{p,cl}_X(\mathscr{F})_E$ defines a map of constructible complexes on X

$$\omega_z^\flat : \bigwedge^p \mathbb{T}_{\mathscr{F},z} \to E.$$

Definition: The induced map $\Gamma(\omega_z^\flat)$ on global sections is the **value of ω at z**.

Non-degeneracy (i)

Setup:

- X - nicely stratified space with equidimensional strata.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ is a constructible sheaf of locally f.p. derived Artin stacks (or just locally formally representable derived stacks).
- $E = K_X[n] \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{Vect}_{\mathbb{C}})$, where $n \in \mathbb{Z}$ and K_X is the Verdier's dualizing complex of X.

Non-degeneracy (ii)

Definition:

(a) The **complex and space of relative n-shifted closed p-forms on $\mathscr{F}$** are defined to be

$$\mathbb{A}_X^{p,cl}(\mathscr{F}, n) := \mathbb{A}_X^{p,cl}(\mathscr{F})_{K_X[n]},$$
$$\mathbf{A}_X^{p,cl}(\mathscr{F}, n) := \mathbf{A}_X^{p,cl}(\mathscr{F})_{K_X[n]}.$$

(b) A closed relative n-shifted 2-form $\omega \in \mathbf{A}_X^{p,cl}(\mathscr{F}, n)$ is **symplectic** if for every $U \subset X$ and any point $z \in \Gamma(U, \mathscr{F})$, the map

$$\omega_z^\flat : \ \mathbb{T}_{\mathscr{F}_{|U},z} \longrightarrow \mathbb{L}_{\mathscr{F}_{|U},z} \otimes K_U[n]$$

is a quasi-isomorphism.

Criterion for non-degeneracy

$\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ - a sheaf of l.f.p. derived Artin stacks;
$\omega \in \mathbf{A}^{2,cl}_X(\mathscr{F}, n)$ - a relative closed n-shifted 2-form on $\mathscr{F}$.

Then for every $\alpha \in I$ we get

$\overline{\omega}_\alpha$ - an absolute $n + 1 + \dim X_\alpha$ shifted closed 2-form on $\mathring{\mathscr{F}}_\alpha$;

h_α - an isotropic structure on $\mathrm{cosp}^{\mathscr{F}}_\alpha : \mathscr{F}_\alpha \to \mathring{\mathscr{F}}_\alpha$.

Theorem: **[Arinkin-P-Toën]** The realtive form ω is symplectic if and only if for every α we have:

(a) $\overline{\omega}_\alpha$ is $n + 1 + \dim X_\alpha$ shifted symplectic on $\mathring{\mathscr{F}}_\alpha$;.

(b) $\mathrm{cosp}^{\mathscr{F}}_\alpha : (\mathscr{F}_\alpha, h_\alpha) \to (\mathring{\mathscr{F}}_\alpha, \overline{\omega}_\alpha)$ is Lagrangian.

Push-forwards (i)

Pushing forward also makes sense in this setting. Suppose

- $f : X \to Y$ is a stratified map of nicely stratified spaces;
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$;
- $E \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{Vect}_{\mathbb{C}})$;

We have push-forwards

$$f_*\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(Y, \mathrm{dSt}_{\mathbb{C}}) \quad \text{and} \quad f_*E \in \mathrm{Sh}^{\mathrm{str}}(Y, \mathrm{Vect}_{\mathbb{C}})$$

computed via the right Kan extensions along the functor between exit path categories induced from f.
The pushforward of $\omega \in \mathbf{A}_X^{p,cl}(\mathscr{F})_E$ then is a closed relative form:

$$\boxed{f_*\omega \in \mathbf{A}_Y^{p,cl}(f_*\mathscr{F})_{f_*E}.}$$

Push-forwards (ii)

Theorem: [Arinkin-P-Toën] Suppose that $f : X \to Y$ is proper and that $\omega \in \mathbf{A}_X^{2,cl}(\mathscr{F})_{K_X[n]}$ is symplectic. Then the pushforward

$$\mathrm{tr}\, f_*\omega \in \mathbf{A}_Y^{2,cl}(f_*\mathscr{F})_{K_Y[n]}$$

is symplectic as well.

Remark:

- Here $\mathrm{tr} : f_*K_X \to K_Y$ denotes the canonical trace map.
- Most of the standard constructions of shifted symplectic structures arise as special cases of the above theorem.

Deligne-Malgrange-Stokes sheaves (i)

Suppose X is a smooth surface underlying a quasi-projective complex algebraic curve and stratified with the following strata:

- a connected open stratum X_{in};
- a (not necessarily connected) open stratum X_{out};
- arcs X_e, $e \in E$;
- endpoints X_v, $v \in V$.

We require that the strata satisfy the following conditions:

(1) exactly two arcs meet at each endpoint;

(2) each arc separates X_{in} and X_{out};

(3) X_{out} retracts onto its boundary.

Deligne-Malgrange-Stokes sheaves (ii)

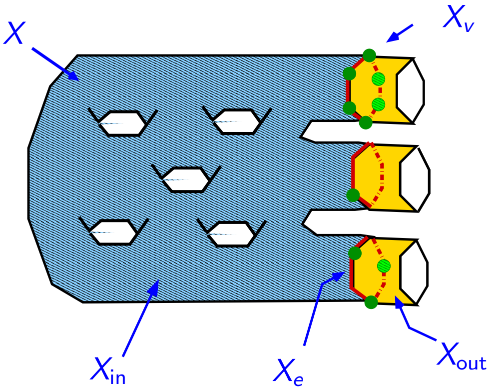


Figure: Stratified surface

Deligne-Malgrange-Stokes sheaves (iii)

Setup: Consider $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ - a constructible sheaf of stacks satisfying:

$\boxed{\text{codim} = 0}$ Locally on X_{in} and on X_{out}, $\mathscr{F}$ is isomorphic to BG for some reductive G.

Deligne-Malgrange-Stokes sheaves (iii)

Setup: Consider $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ - a constructible sheaf of stacks satisfying:

codim $= 0$ Locally on X_{in} and on X_{out}, $\mathscr{F}$ is isomorphic to BG for some reductive G.

typically the groups will be different on different connected components of strata

Deligne-Malgrange-Stokes sheaves (iv)

$\boxed{\text{codim} = 1}$ For each point $x \in X_e$, $e \in E$ the specialization of strata

$$X_{\text{in}} \rightsquigarrow X_e \leftsquigarrow X_{\text{out}}$$

leads to a cospecialization diagram of stacks:

$$\mathscr{F}_{\text{in}} \longleftarrow \mathscr{F}_e \longrightarrow \mathscr{F}_{\text{out}} \cdot$$

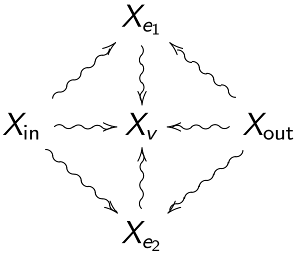
We require that this diagram must be isomorphic to

$$BG \longleftarrow BP \longrightarrow BL ,$$

where G is a reductive group, $P \subset G$ is a parabolic subgroup, and $P \to L$ is the Levi quotient.

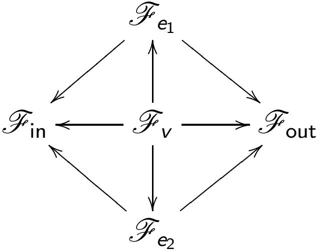
Deligne-Malgrange-Stokes sheaves (iv)

codim $= 2$ In a neighborhood of a point $\{b\} = X_v$, $v \in V$, the specialization of strata



Deligne-Malgrange-Stokes sheaves (iv)

codim $= 2$ In a neighborhood of a point $\{b\} = X_v$, $v \in V$, the specialization of strata leads to a diagram of stacks



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Deligne-Malgrange-Stokes sheaves (v)

$\boxed{\text{codim} = 2}$ We require that

$$\begin{array}{ccccc} & & \mathscr{F}_{e_1} & & \\ & \swarrow & \uparrow & \searrow & \\ \mathscr{F}_{\text{in}} & \longleftarrow & \mathscr{F}_{v} & \longrightarrow & \mathscr{F}_{\text{out}} \\ & \nwarrow & \downarrow & \nearrow & \\ & & \mathscr{F}_{e_2} & & \end{array} \quad \cong \quad \begin{array}{ccccc} & & BP_1 & & \\ & \swarrow & \uparrow & \searrow & \\ BG & \longleftarrow & B(P_1 \cap P_2) & \longrightarrow & BL \\ & \nwarrow & \downarrow & \nearrow & \\ & & BP_2 & & \end{array} ,$$

where G is a reductive group, $P_1, P_2 \subset G$ are parabolic subgroups that admit a common Levi, and $P_{1,2} \to L$ are the Levi quotients.

Deligne-Malgrange-Stokes sheaves (vi)

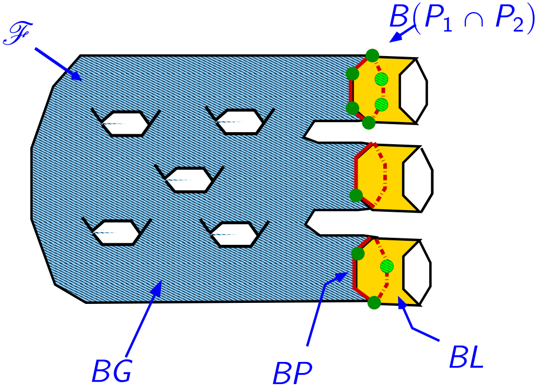

Figure: DMS sheaf

Deligne-Malgrange-Stokes sheaves (vii)

Theorem: [Arinkin-P-Toën] Let $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ be a DMS sheaf. Then

(1) The restriction map $\mathbf{A}^{2,cl}(\mathscr{F})_{K_X} \to \mathbf{A}^{2,cl}(\mathscr{F}_{\mathrm{in}})_{K_{X_{\mathrm{in}}}}$, is a homotopy equivalence.

(2) The extension ω of a form ω_{in} is non-degenerate if and only if ω_{in} is non-degenerate.

Deligne-Malgrange-Stokes sheaves (ix)

Remarks:

- $\mathscr{F}_{\text{in}}$ is a local system of stacks with fiber BG. Hence $\mathbb{T}_{\mathscr{F}_{\text{in}}}$ is a local system of complexes on X_{in} with fiber $\mathfrak{g}[1]$.
- In particular $\begin{pmatrix} \text{a } \Pi_1(X_{\text{in}}) \times G\text{-invariant} \\ \text{symmetric pairing } \kappa \text{ on } \mathfrak{g} \end{pmatrix}$

$$\Updownarrow$$

$$\begin{pmatrix} \text{a } K_{X_{\text{in}}}\text{-valued relative symplectic} \\ \text{form } \omega_{\text{in}} = \omega_\kappa \text{ on } \mathscr{F}_{\text{in}} \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{pmatrix} \text{a } K_X\text{-valued relative symplectic} \\ \text{form } \omega \text{ on } \mathscr{F}. \end{pmatrix}$$

- When $\mathscr{F}_{\text{in}}$ is **constant**, the form κ exists automatically since G is assumed to be reductive.

Deligne-Malgrange-Stokes sheaves (ix)

Suppose

- $\mathfrak{X}$ is a smooth projective curve/$\mathbb{C}$;
- $X = \mathfrak{X} - \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\}$;
- $\mathscr{I} = \{\mathscr{I}_1, \ldots, \mathscr{I}_k\}$, $\mathscr{I}_i$ an irregular type at $\boldsymbol{x}_i$.

Then:

• $\mathscr{I}$ can be recorded equivalently in $\mathrm{DMS}_{G,\mathscr{I}} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$;
• $\mathrm{DMS}_{G,\mathscr{I}}$ classifies Stokes data on $\mathfrak{X}$ of irregular type $\mathscr{I}$, in the sense that

$$Loc_G(X, \mathscr{I}) = \Gamma\left(\widehat{\mathfrak{X}}, \mathrm{DMS}_{G,\mathscr{I}|\widehat{\mathfrak{X}}}\right).$$

Here $\widehat{\mathfrak{X}} \subset X$ denotes the real oriented blow-up of $\mathfrak{X}$ at the points x_i.

Deligne-Malgrange-Stokes sheaves (x)

Note:

- The sheaf $\mathrm{DMS}_{G,\mathscr{I}}$ tautologically satisfies the $\mathrm{codim} = 0, 1, 2$ properties.
- Since the underlying G-local systems are untwisted $\mathrm{DMS}_{G,\mathscr{I}}$ comes with a canonical relative symplectic form which depends only on a choice of a non-degenerate $\kappa \in (\mathrm{Sym}^2 \mathfrak{g}^\vee)^G$.

Because of this we introduce the following:

Definition: A **DMS sheaf** on X is a sheaf $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ that satisfies the $\mathrm{codim} = 0, 1, 2$ properties and admits a relative K_X-valued symplectic structure.

Stokes data in dimension one (i)

Suppose

$\mathscr{F}$ is a DMS sheaf of stacks equipped with a K_X-valued relative symplectic form ω.

$f : X \to (0,1]$ is the stratified map which collapses $X - X_{\text{out}}$ to 1 and projects each cylinder component of X_{out} onto its ruling $(0,1)$.

Then f is a proper stratified map and

Pushforward theorem $\Longrightarrow$ $\operatorname{tr} f_*\omega$ is a relative $K_{(0,1]}$-valued symplectic structure on $f_*\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$.

Stokes data in dimension one (ii)

Hence

- tr $f_*\omega$ defines a 1-shifted symplectic structure on $\Gamma(X_{\text{out}}, \mathscr{F}_{\text{out}})$.
- tr $f_*\omega$ defines a 0-shifted Lagrangian structure on the cospecialization map

$$\Gamma(X - X_{\text{out}}, \mathscr{F}) \to \Gamma(X_{\text{out}}, \mathscr{F}_{\text{out}}).$$

Stokes data in dimension one (iii)

In the special case when $\mathscr{F} = \mathrm{DMS}_{G,\mathscr{I}}$ we get

$$\begin{aligned} Loc_G(X, \mathscr{I}) &= \Gamma(X - X_{\mathrm{out}}, \mathscr{F}) \\ Loc_{\widetilde{L}}(\partial X) &= \Gamma(X_{\mathrm{out}}, \mathscr{F}_{\mathrm{out}}) \end{aligned}$$

where $\widetilde{L}$ is the local system of Levi subgroups on X_{out} for which $\mathscr{F}_{\mathrm{out}} = B\widetilde{L}$.

Moreover in this setting the map

$$r_{\mathscr{I}} : Loc_G(X, \mathscr{I}) \to Loc_{\widetilde{L}}(\partial X)$$

assigns to each Stokes filtered local system its formal monodromy at ∞.

Stokes data in dimension one (iv)

Since $\widetilde{L}$ is a locally constant sheaf we again have that fixing a flat section $\lambda \in \Gamma(\partial X, \widetilde{L})$ gives a Lagrangian map

$$\prod_{i=1}^{k} BG_{\lambda_i} \to \prod_{i=1}^{k} [G/G] = Loc_{\widetilde{L}}(\partial X).$$

The intersection

$$Loc_G(X, \mathscr{I}; \lambda)$$

of this Lagrangian with the Lagrangian map $r_{\mathscr{I}}$ is the moduli of Stokes data of type $\mathscr{I}$ having a fixed formal monodromy at ∞ and is therefore symplectic.

Extraction (i)

M - topological space, $N \subset M$ - a closed subspace.
Half open cylinder of the inclusion $N \hookrightarrow M$:

$$M^+ := (N \times (0,1]) \bigsqcup_N M,$$

where

- $N \to (0,1] \times N$ identifies N with the base $N \times \{1\}$;
- $M \to M^+$ identifies M with a closed subset of M^+, with complement $M - M^+ = N \times (0,1)$.

Note: For us M will be a topological manifold with boundary and $N = \partial M$. In this case M^+ is a topological manifold as well.

Extraction (ii)

Given $F \in \mathrm{Sh}^{\mathrm{str}}(M, C)$, $F^e \in \mathrm{Sh}^{\mathrm{str}}(N, C)$, $F_{|N} \to F^e$, define the **extract** $F^+ \in \mathrm{Sh}^{\mathrm{str}}(M^+, C)$ of (F, F^e) as the unique sheaf s.t.:

- The restriction $F^+|_M$ equals F;
- The restriction $F^+|_{N\times(0,1)}$ equals $\pi^* F^e$;
- The cospecialization map $F \to \imath^* \jmath_*(\pi^* F^e)$ corresponds to

$$F \to \iota_*(F|_N) \to \iota_*(F^e)$$

under the identification $\imath^* \jmath_*(\pi^* F^e) = \iota_* F^e$.

Here $\imath : M \hookrightarrow M^+$, $\jmath : N \times (0,1) \hookrightarrow M^+$, $\iota : N \hookrightarrow M$ are the natural inclusions, while $\pi : N \times (0,1) \twoheadrightarrow N$ is the projection.

Local model (i)

Group theoretic data:

- G - a complex reductive group, $T \subset G$ - a maximal torus.
- $\Lambda = \text{cochar}(T) = \text{Hom}(\mathbb{G}_m, T)$, $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.

Local model (i)

Constructions:

- $\xi \in \Lambda$ defines a parabolic $P(\xi) \subset G$ with $T \subset P(\xi)$.
- a Levi $L \supset T$ defines a subspace

$$\Lambda_{\mathbb{R}}^{L} = \{\xi \in \Lambda_{\mathbb{R}} \,|\, P(\xi) \supset L\}\,.$$

- a continuous map $\xi : Z \to \Lambda_{\mathbb{R}}^{L}$ defines a subsheaf of groups $P_{\xi} \subset G_Z$ via

$$\Gamma(U, P_{\xi}) := \left\{ g : U \to G \;\middle|\; \begin{array}{l} g \text{ is locally constant and} \\ g(z) \in P(\xi(z)) \text{ for all } z \in U \end{array} \right\}.$$

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Remark: the stalk of P_{ξ} at $z \in Z$ is not equal to $P_{\xi(z)}$: $P_{\xi(z)}$ depends upper-semicontinuously on z, while the stalk is lower-semicontinuous and in general not parabolic.

Local model (ii)

Setup:

- Y - a manifold with corners, $\partial Y \subset Y$ - its boundary.
- $\pi : Z \to Y$ a smooth fibration.
- $\xi : \partial Z \to \Lambda_{\mathbb{R}}$ an analytic function such that for any root

$\alpha : \Lambda_{\mathbb{R}} \to \mathbb{R}$, if $\alpha(\xi(z)) = 0$, then either $\alpha \circ \xi = 0$ identically, or the derivative of $\alpha \circ \xi$ is non-zero on the vertical tangent space $T_{\pi,z}$

These data give rise a constructible subsheaf $P_{\xi} \subset G_Z$ defined by essentially the same formula:

$$\Gamma(U, P_{\xi}) := \left\{ g : U \to G \;\middle|\; \begin{array}{l} g \text{ is locally constant and} \\ g(z) \in P(\xi(z)) \text{ for all} \\ z \in U \cap \partial Z \end{array} \right\}.$$

Local model (iii)

Consider $L \subset G$ - the Levi whose root system consists of all roots α such that $\alpha \circ \xi = 0$ identically. Then for all $z \in \partial Z$ we have $L \subset P(\xi(z))$ and hence we get an embedding of sheaves

$$L_{\partial Z} \subset P_{\xi}|_{\partial Z}$$

which has a natural splitting:

Proposition [Arinkin-P-Töen]: There is a retract

$$r : P_{\xi}|_{\partial Z} \to L_{\partial Z}$$

that is a stalk-wise Levi quotient: the germ of r at every point of z identifies L with the maximal reductive quotient of $(P_{\xi})_{z}$.

Local model (iv)

Apply extraction to $\partial Z \subset Z$ and the map of sheaves $r : P_\xi|_{\partial Z} \to L_{\partial Z}$. Then:

- we get a constructible sheaf of groups P_ξ^+ on the topological manifold Z^+.
- Z^+ contains $\overset{\circ}{Z} = Z - \partial Z$ as an open and $BP_\xi^+|_{\overset{\circ}{Z}} = BG_{\overset{\circ}{Z}}$.

Local model (iv)

With this notation we then have:

Theorem [Arinkin-P-Töen]:

(1) Any $K_Z^\circ[2-\dim(Z)]$-valued closed 2-form $\mathring{\omega}$ on BG_Z° extends to a form ω on $B(P_\xi^+)$, and the space of such extensions is contractible, i.e. the map

$$\mathbf{A}_Z^{2,cl}(B(P_\xi^+))_{K_Z[2-\dim(Z)]} \to \mathbf{A}_{\overset{\circ}{Z}}^{2,cl}(BG_Z^\circ)_{K_Z^\circ[2-\dim(Z)]}$$

is a homotopy equivalence.

(2) The extension ω is non-degenerate iff the form $\mathring{\omega}$ is non-degenerate.

Deligne-Malgrange-Stokes gerbes (i)

Setup:

- $\mathfrak{X}$ - a compact complex manifold, $\dim_{\mathbb{C}} \mathfrak{X} = d$.
- $D \subset \mathfrak{X}$ - a simple normal crossings divisor.
- $\sigma : \widehat{\mathfrak{X}} \to \mathfrak{X}$ - the real oriented blow-up of $\mathfrak{X}$ along D.

Notation:

$$\begin{aligned} \mathring{\mathfrak{X}} &:= \mathfrak{X} - D = \widehat{\mathfrak{X}} - \sigma^{-1}(D); \\ \partial\widehat{\mathfrak{X}} &:= \sigma^{-1}(D). \\ X &:= \text{the cylinder } \widehat{\mathfrak{X}}^{+} \text{ of } \widehat{\mathfrak{X}} \hookrightarrow \widehat{\mathfrak{X}}. \end{aligned}$$

Deligne-Malgrange-Stokes gerbes (i)

Setup:

- $\mathfrak{X}$ - a compact complex manifold, $\dim_{\mathbb{C}} \mathfrak{X} = d$.
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Plan: Introduce a special class of sheaves of stacks on $\widehat{\mathfrak{X}}$ - **DMS gerbes** - which support tautologically relative shifted symplectic structures.

Note: • These sheaves are constructible with respect to some stratification, which will be considered a part of the data.
• The class of DMS gerbes includes the classifying stacks for Stokes data of given irregular type (with eliminated turning points).

Deligne-Malgrange-Stokes gerbes (ii)

Definition: A **DMS gerbe** on $\widehat{\mathfrak{X}}$ is a sheaf $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(\widehat{\mathfrak{X}}, \mathrm{dSt}_{\mathbb{C}})$ such that

- $\mathscr{F}$ is locally constant on $\mathring{\mathfrak{X}}$ with fiber BG for some reductive G.
- For each point $p \in \partial\widehat{\mathfrak{X}}$ there exists a reductive group G, an open neighborhood $p \in U \subset \widehat{\mathfrak{X}}$, and a real analytic function $\xi : \partial\widehat{\mathfrak{X}} \cap U \to \Lambda_{\mathbb{R}}$ so that $\mathscr{F}|_U \cong BP_{\xi}$.
- If $\alpha : \Lambda_{\mathbb{R}} \to \mathbb{R}$ is a root such that $\alpha \circ \xi(p) = 0$, then either $\alpha \circ \xi = 0$ identically, or the derivative of $\alpha \circ \xi|_{\widehat{\mathfrak{X}}_p}$ is non-zero at p

Note: The fiber $\widehat{\mathfrak{X}}_p = \sigma^{-1}(\sigma(p))$ is a k-dimensional torus, $k =$ the number of components of D passing through p.

Deligne-Malgrange-Stokes gerbes (iii)

Proposition [Arinkin-P-Töen]: Suppose $\mathscr{F}$ is a DMS gerbe on $\widehat{\mathfrak{X}}$. Then:

- There is a locally constant sheaf of stacks $\mathcal{L}$ over $\partial\widehat{\mathfrak{X}}$ equipped with a morphism

$$r : \mathscr{F}|_{\partial\widehat{\mathfrak{X}}} \to \mathcal{L}.$$

- The pair $(\mathcal{L}, r)$ is uniquely determined (up to a contractible space of choices) by the condition that the germ of r at all points $z \in \partial\widehat{\mathfrak{X}}$ is isomorphic to a map $BH \to BL_H$, where H is a connected linear group and L_H is its maximal reductive quotient.

Deligne-Malgrange-Stokes gerbes (iv)

The extraction construction applied to $\mathscr{F}$ and the map

$$r : \mathscr{F}|_{\partial\hat{\mathfrak{X}}} \to \mathcal{L},$$

yields a constructible sheaf of stacks $\mathscr{F}^+$ on $X = \hat{\mathfrak{X}}^+$.
Note also that

- $\mathring{\mathfrak{X}} \subset X$ is open;
- X is an oriented topological manifold of dimension $2d$;
- $K_X = \mathbb{C}_X[2d]$ or equivalently $K_X[2-2d] = \mathbb{C}_X[2]$.

We can now formulate our general existence theorem

Deligne-Malgrange-Stokes gerbes (v)

Theorem [Arinkin-P-Töen]: Suppose $\mathscr{F}$ is a DMS gerbe on $\widehat{\mathfrak{X}}$. Then:

(1) Any $\mathbb{C}_{\mathring{\mathfrak{X}}}[2]$-valued closed 2-form $\mathring{\omega}$ on $\mathscr{F}^{+}|_{\mathring{\mathfrak{X}}} = \mathscr{F}|_{\mathring{\mathfrak{X}}}$ extends to a $\mathbb{C}_X[2]$-valued 2-form ω on X, and the space of such extensions is contractible. Equivalently, the restriction map

$$\mathbf{A}_X^{2,cl}(\mathscr{F}^{+})_{K_X[2-2d]} \to \mathbf{A}^{2,cl}(\mathscr{F})_{K_{\mathring{\mathfrak{X}}}[2-2d]}$$

is a homotopy equivalence.

(2) The extension ω is non-degenerate iff $\mathring{\omega}$ is non-degenerate.

Stokes data in higher dimension (i)

Suppose

$\mathscr{F}$ is a DMS gerbe equipped with a $K_{\mathring{\mathfrak{X}}[2-2d]}$-valued relative symplectic form ω.

$f : X \to (0,1]$ is the stratified map which collapses $X - \mathring{\mathfrak{X}}$ to 1 and projects each cylinder component of X onto its ruling $(0,1)$.

Then f is a proper stratified map and

Pushforward theorem $\Longrightarrow$ $\mathrm{tr}\, f_*\omega$ is a relative $K_{(0,1]}[2-2d]$-valued symplectic structure on $f_*\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$.

Stokes data in higher dimension (ii)

Hence

- tr $f_*\omega$ defines a $3-2d$-shifted symplectic structure on $\Gamma(\partial\widehat{\mathfrak{X}}, \mathscr{F})$.
- tr $f_*\omega$ defines a $2-2d$-shifted Lagrangian structure on the cospecialization map

$$\Gamma(X, \mathscr{F}^+) \to \Gamma(\partial\widehat{\mathfrak{X}}, \mathscr{F}).$$

Stokes data in higher dimension (iii)

In the special case when $\mathscr{F} = \mathrm{DMS}_{G,\mathscr{I}}$ we get

$$\begin{aligned} Loc_G(\mathring{\mathfrak{X}}, \mathscr{I}) &= \Gamma(\mathring{\mathfrak{X}}, \mathscr{F}), \\ Loc_{\mathcal{L}}(\partial\widehat{\mathfrak{X}}) &= \Gamma(X - \mathring{\mathfrak{X}}, \mathscr{F}). \end{aligned}$$

Moreover in this setting the map

$$r_{\mathscr{I}} : Loc_G(X, \mathscr{I}) \to Loc_{\mathcal{L}}(\partial X)$$

assigns to each Stokes filtered local system its formal monodromy at ∞.

p-forms

$A \in \text{cdga}_{\mathbb{C}}, \quad X = \mathbf{Spec}(A) \in \text{dSt}_{\mathbb{C}}$,
$QA \to A$ a cofibrant (quasi-free) replacement. Then:

$\oplus_{p \geqslant 0} \bigwedge_A^p \mathbb{L}_A = \oplus_{p \geqslant 0} \Omega_{QA}^p$ - a fourth quadrant bicomplex with
vertical differential $d : \Omega_{QA}^{p,i} \to \Omega_{QA}^{p,i+1}$ induced by d_{QA}, and
horizontal differential $d_{DR} : \Omega_{QA}^{p,i} \to \Omega_{QA}^{p+1,i}$ given by the de Rham differential.

Hodge filtration: $F^q(A) := \oplus_{p > q} \Omega_{QA}^p$: still a fourth quadrant bicomplex.

(shifted) closed p-forms

Motivation: If X is a smooth scheme/$\mathbb{C}$, then $\Omega_X^{p,cl} \cong \left(\Omega_X^{\geqslant p}[p], d_{DR}\right)$. Use $(\Omega_X^{\geqslant p}[p], d_{DR})$ as a model for closed p forms in general.

Definition:

- **complex of closed p-forms on $X = \mathrm{Spec}\, A$:** $\mathbf{A}^{p,cl}(A) := \mathrm{tot}^{\prod}(F^p(A))[p]$
- **complex of n-shifted closed p-forms on $X = \mathrm{Spec}\, A$:** $\mathbf{A}^{p,cl}(A;n) := \mathrm{tot}^{\prod}(F^p(A))[n+p]$
- **Hodge tower:** $\cdots \to \mathbf{A}^{p,cl}(A)[-p] \to \mathbf{A}^{p-1,cl}(A)[1-p] \to \cdots \to \mathbf{A}^{0,cl}(A)$

(shifted) closed p-forms (ii)

Explicitly an n-shifted closed p-form ω on $X = \mathbf{Spec}\, A$ is an infinite collection

$$\omega = \{\omega_i\}_{i \geqslant 0}, \qquad \omega_i \in \Omega_A^{p+i, n-i}$$

satisfying

$$d_{DR}\omega_i = -d\omega_{i+1}.$$

Note: The collection $\{\omega_i\}_{i \geqslant 1}$ is the **key** closing ω.

p-forms and closed p-forms

Note:

- The complex $\mathbf{A}^{0,cl}(A)$ of closed 0-forms on $X = \mathbf{Spec}\, A$ is exactly Illusie's derived de Rham complex of A.
- There is an underlying p-form map

$$\mathbf{A}^{p,cl}(A; n) \to \bigwedge^p \mathbb{L}_{A/k}[n]$$

inducing

$$H^0(\mathbf{A}^{p,cl}(A)[n]) \to H^n(X, \bigwedge^p \mathbb{L}_{A/k}).$$

- The homotopy fiber of the underlying p-form map can be very complicated (complex of **keys**): being closed is **not** a property but rather a list of coherent data.

Functoriality and gluing:

Globally we have:

- the *n*-shifted *p*-forms ∞-functor $\mathcal{A}^p(-;n) : \mathrm{cdga}_{\mathbb{C}} \to \mathrm{SSets} : A \mapsto |\,\Omega^p_{QA}[n]\,|$, and
- the *n*-shifted closed *p*-forms ∞-functor $\mathcal{A}^{p,cl}(-;n) : \mathrm{cdga}_{\mathbb{C}} \to \mathrm{SSets} : A \mapsto |\,\mathbf{A}^{p,cl}(A)[n]\,|$.

Note: $\mathcal{A}^p(-;n)$ and $\mathcal{A}^{p,cl}(-;n)$ are **derived stacks** for the étale topology. underlying *p*-form map (of derived stacks)

$$\mathcal{A}^{p,cl}(-;n) \to \mathcal{A}^p(-;n)$$

Notation: $|-|$ denotes $Map_{\mathbb{C}-dgMod}(\mathbb{C},-) = \mathbf{DK}\tau^{\leqslant 0}(-)$ i.e. Dold-Kan applied to the $\leqslant 0$-truncation.

global forms and closed forms (i)

Fix a derived Artin stack X (locally of finite presentation $/\mathbb{C}$)

Definition:

- $\mathcal{A}^p(X) := Map_{\mathbf{dSt}_{\mathbb{C}}}(X, \mathcal{A}^p(-))$ - **space of p-forms** on X;
- $\mathcal{A}^{p,cl}(X) := Map_{\mathbf{dSt}_{\mathbb{C}}}(X, \mathcal{A}^{p,cl}(-))$ - **space of closed p-forms** on X;
- n-shifted versions : $\mathcal{A}^p(X;n) := Map_{\mathbf{dSt}_{\mathbb{C}}}(X, \mathcal{A}^p(-;n))$ and $\mathcal{A}^{p,cl}(X;n) := Map_{\mathbf{dSt}_{\mathbb{C}}}(X, \mathcal{A}^{p,cl}(-;n))$
- an n-shifted (respectively closed) p-form on X is an element in $\pi_0\mathcal{A}^p(X;n)$ (respectively in $\pi_0\mathcal{A}^{p,cl}(X;n)$)

global forms and closed forms (ii)

Note:

1) If X is a smooth scheme there are no negatively shifted forms.
2) When $X = \mathbf{Spec}\, A$ then there are no positively shifted forms.
3) For general X shifted forms may exist for any $n \in \mathbb{Z}$.

global forms and closed forms (ii)

- **underlying *p*-form** map (of simplicial sets)

$$\mathcal{A}^{p,cl}(X;n) \to \mathcal{A}^{p}(X;n)$$

- not a monomorphism for general X, its homotopy fiber at a given p-form ω_0 is the space of **keys** of ω_0.
- If X is a smooth and proper scheme then this map is indeed a mono (homotopy fiber is either empty or contractible) $\Rightarrow$ no new phenomena in this case.

global forms and closed forms (ii)

Theorem (PTVV): for X derived Artin, then forms satisfy smooth descent:

$$\mathcal{A}^p(X;n) \simeq \mathrm{Map}_{\mathrm{L}_{\mathrm{qcoh}}(X)}(\mathcal{O}_X, (\bigwedge^p \mathbb{L}_X)[n]).$$

In particular: an n-shifted p-form on X is an element in $H^n(X, \bigwedge^p \mathbb{L}_X)$

global forms and closed forms (iii)

Remark: If $A \in cdga$ is quasi-free, and $X = \mathbf{Spec}\, A$, then

$$\begin{aligned}\mathcal{A}^{p,cl}(X;n) &= \left| \prod_{i \geqslant 0} \left(\Omega_A^{p+1}[n-i], d + d_{DR} \right) \right| \\ &= \left| \mathrm{tot}^{\Pi}(F^p(A))[n] \right| \\ &= \left| NC(A)(p)[n+p] \right|\end{aligned}$$

global forms and closed forms (iii)

Remark: If $A \in cdga$ is quasi-free, and $X = \mathbf{Spec}\, A$, then

$$\mathcal{A}^{p,cl}(X;n) = \left| \prod_{i \geqslant 0} \left(\Omega_A^{p+1}[n-i], d + d_{DR} \right) \right|$$

$$= \left| \mathrm{tot}^{\Pi}(F^p(A))[n] \right|$$

$$= \left| NC(A)(p)[n+p] \right|$$

negative cyclic complex of weight p

global forms and closed forms (iii)

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$$\mathcal{A}^{p,cl}(X;n) = \left| \prod_{i \geqslant 0} \left(\Omega_A^{p+1}[n-i], d + d_{DR} \right) \right|$$

$$= \left| \mathrm{tot}^{\Pi}(F^p(A))[n] \right|$$

$$= \left| NC(A)(p)[n+p] \right|$$

Hence

$$\boxed{\pi_0 \mathcal{A}^{p,cl}(X;n) = HC_-^{n-p}(A)(p).}$$

Back

Sheaves and functors (i)

X - locally compact Hausdorff space;
$\mathscr{U}(X)$ - the poset of opens in X;
$\mathscr{C}$ - an ∞-category which admits all small limits.

Definition: A $\mathscr{C}$-valued **sheaf** on X is a functor $F : \mathscr{U}(X)^{op} \to \mathscr{C}$ satisfying the **sheaf condition**: for every open cover $\{U_\alpha\}$ of an open set U the natural map

$$F(U) \longrightarrow \varprojlim_{V} F(V)$$

is an equivalence in $\mathscr{C}$. Here the limit is taken over all open subsets $V \subset U$ which are contained in one of the U_α.

Sheaves and functors (ii)

Remark: Equivalently a $\mathscr{C}$-valued functor $F : \mathscr{U}(X)^{op} \to \mathscr{C}$ is a **sheaf** if

- For every sequence $U_0 \subset U_1 \subset \cdots \subset U_k \subset \cdots$ of opens in X the natural map $F(\cup U_i) \to \varprojlim_i F(U_i)$ is an equivalence in $\mathscr{C}$.
- The object $F(\varnothing)$ is terminal in $\mathscr{C}$.
- For every pair of opens $U, V \subset X$ the diagram

$$\begin{array}{ccc} F(U \cup V) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(U \cap V) \end{array}$$

is a pullback square in $\mathscr{C}$.

Back

Stratifications (i)

Let I be a poset viewed as a topological space for the **upward closed topology:** $U \subset I$ is open if and only if $x \in U$ implies $y \in U$ for all $y \geqslant x$ in I.

Definition: An ***I*-stratification** of a topological space X is a continuous map $a : X \to I$. The **stratum** in X corresponding to $\alpha \in I$ is the subset $X_\alpha = a^{-1}(\alpha)$.

Note: We will only be working with stratifications that satisfy a regularity condition.

Stratifications (ii)

Notation: If I is a poset, write $I^{\lhd}$ for the poset obtained from I by adjoining a new smallest element $-\infty$.

Definition: Let $a : X \to I$ be a stratified space. The **cone over X** is the $I^{\lhd}$-stratified space $C(X)$ defined as follows:

- As a set $C(X) = \{*\} \coprod (X \times \mathbb{R}_{>0})$.
- A subset $U \subset C(X)$ is open if and only if $U \cap (X \times \mathbb{R}_{>0})$ is open, and if $* \in U$, then $X \times (0, \epsilon) \subset U$ for some positive real ϵ.
- $C(X)$ is stratified by the map $a^{\lhd} : C(X) \to I^{\lhd}$ given by $a^{\lhd}(*) = -\infty$, and $a^{\lhd}(x, t) = a(x)$ for $(x, t) \in X \times \mathbb{R}_{>0}$.

Stratifications (iii)

Definition: [Lurie] An I-stratification on X is **conical** if every point $x \in X_\alpha \subset X$ has a stratified neighborhood U_x which is stratified homeomorphic to a product $Z \times C(Y)$ where Z is a topological space, and Y is a $I_{<\alpha}$-stratified space.

Definition: An I-stratification on X is **good** if I is finite and every point $x \in X_\alpha \subset X$ has a stratified neighborhood U_x which is stratified homeomorphic to a product $\mathbb{R}^k \times C(Y)$ for some k and some compact $I_{<\alpha}$-stratified space Y.

Back

Exit paths (i)

Setup:

- X be a stratified space with strata labeled by a poset I.
- $|\Delta^n| = \left\{ (t_0, \ldots, t_n) \in [0,1]^{\times n} \;\middle|\; \sum_{i=0}^{1} = 1 \right\}$ is the standard simplex.

Definition: The **simplicial set of exit paths** of X is the simplicial subset $\mathbf{I} \subset \mathrm{Sing}(X)$ consisting of those simplices $\sigma : |\Delta^n| \to X$ that satisfy the condition

$$(*) \quad \left(\begin{array}{l} \text{There exists a chain of elements } \alpha_0 < \alpha_1 < \cdots < \alpha_n \in I \\ \text{so that for every point } (t_0, t_1, \ldots, t_i, 0, \ldots, 0) \in |\Delta^n| \\ \text{with } t_i \neq 0 \text{ we have that } \sigma(t_0, t_1, \ldots, t_i, 0, \ldots, 0) \in X_{\alpha_i}. \end{array} \right)$$

Exit paths (ii)

Theorem: [Lurie]

(a) If the stratification on X is conical, then $\mathbf{I}$ is an ∞-category.

(b) Let X be a paracompact topological space which is locally of a singular shape, and is equipped with a conical I-stratification. Then the ∞-category of I-constructible sheaves of spaces on X is equivalent to the ∞-category $\mathrm{Fun}(\mathbf{I}, \mathrm{SSets})$.

Back

Non-degeneracy and nearby cycles (i)

Let $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$, $E = K_X[n]$, $\alpha \in I$.

- $\mathscr{F}_\alpha$ and $\mathring{\mathscr{F}}_\alpha$ are local systems of derived stacks on X_α, i.e. are derived stacks equipped with an action of $\Pi_1(X_\alpha) = \mathbf{I}_\alpha$.
- The fiber of $\mathrm{cosp}_\alpha^E : E_\alpha \to \mathring{E}_\alpha$ is equal to the !-restriction of $K_X[n]$ to X_α and so we have an exact triangle

$$(*) \qquad E_\alpha \xrightarrow{\mathrm{cosp}_\alpha^E} \mathring{E}_\alpha \longrightarrow \mathbb{C}\left[n + 1 + \dim X_\alpha\right].$$

Note: When $\alpha \in I$ is maximal we have $\mathring{E}_\alpha = 0$ and so $E_\alpha = \mathbb{C}\left[n + \dim X_\alpha\right]$.

Non-degeneracy and nearby cycles (ii)

Let $\omega \in \mathbf{A}^{2,cl}_X(\mathscr{F}, n) = \mathbf{A}^{2,cl}_X(\mathscr{F})_E$ be a relative n-shifted closed 2-form on $\mathscr{F}$. Then ω induces

- ω_α a relative closed E_α-valued 2-form on $\mathscr{F}_\alpha$, i.e. $\omega_\alpha \in \mathbf{A}^{2,cl}_{X_\alpha}(\mathscr{F}_\alpha)_{E_\alpha}$.
- $\mathring{\omega}_\alpha$ a closed $\mathring{E}_\alpha$-valued 2-form on $\mathring{\mathscr{F}}_\alpha$, i.e. $\mathring{\omega}_\alpha \in \mathbf{A}^{2,cl}_{X_\alpha}(\mathring{\mathscr{F}}_\alpha)_{\mathring{E}_\alpha}$.
- $\overline{\omega}_\alpha$ a closed $(n+1)$-shifted 2-form on $\mathring{\mathscr{F}}_\alpha$, i.e.

$$\overline{\omega}_\alpha \in \mathbf{A}^{2,cl}_{X_\alpha}(\mathring{\mathscr{F}}_\alpha, n+1) = \mathbf{A}^{2,cl}_{X_\alpha}(\mathring{\mathscr{F}}_\alpha)_{\mathbb{C}[n+1+\dim X_\alpha]}.$$

Non-degeneracy and nearby cycles (iii)

Note:

- $\overline{\omega}_\alpha$ is the pushout of $\mathring{\omega}_\alpha$ by the map
$$\mathring{E}_\alpha \to \mathbb{C}\left[n+1+\dim X_\alpha\right].$$
- Viewing $\mathring{\mathscr{F}}_\alpha$ as a constant derived Artin stack equipped with a $\Pi_1(X_\alpha)$ action, then we can view $\overline{\omega}_\alpha$ equivalently as an **absolute** $(n+1+\dim X_\alpha)$-shifted 2-form, i.e.
$$\overline{\omega}_\alpha \in \mathcal{A}^{2,cl}(\mathring{\mathscr{F}}_\alpha, n+1+\dim X_\alpha).$$

Non-degeneracy and nearby cycles (iv)

Key observation: Consider the $\Pi_1(X_\alpha)$-equivariant cospecialization map

$$\mathrm{cosp}^{\mathscr{F}}_\alpha : \mathscr{F}_\alpha \to \mathring{\mathscr{F}}_\alpha$$

of derived Artin stacks. The exact triangle

$$E_\alpha \xrightarrow{\mathrm{cosp}^E_\alpha} \mathring{E}_\alpha \longrightarrow \mathbb{C}\left[n+1+\dim X_\alpha\right].$$

yields a natural path h_α between $\mathrm{cosp}^{\mathscr{F}*}_\alpha(\overline{\omega}_\alpha)$ and 0 in the space $\mathbf{A}^{2,cl}_{X_\alpha}(\mathscr{F}_\alpha, n+1)$ or equivalently a path h_α between $\mathrm{cosp}^{\mathscr{F}*}_\alpha(\overline{\omega}_\alpha)$ and 0 in the space $\mathcal{A}^{2,cl}(\mathscr{F}_\alpha, n+1+\dim X_\alpha)$.

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Constant sheaves (i)

Setup:

- X - a topological space = a stratified space with a single stratum.
- $F \in \mathrm{dSt}_{\mathbb{C}}$ - a derived Artin stack, locally of finite presentation.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ - the constant sheaf on X with fiber F.
- $\omega \in \mathcal{A}^{2,cl}(F, n)$ - an (absolute) n-shifted symplectic form on F.

Constant sheaves (ii)

Note:

- ω corresponds to a relative $\mathbb{C}_X[n]$-valued closed 2-form $\omega_X \in \mathbf{A}^{2,cl}_X(\mathscr{F})_{\mathbb{C}[n]}$ on $\mathscr{F}$.
- If X is an oriented manifold of pure dimension d, then $K_X = \mathbb{C}_X[d]$ and so $\omega_X \in \mathbf{A}^{2,cl}_X(\mathscr{F})_{K_X[n-d]}$.

$$\begin{pmatrix}\omega \text{ is non-degenerate as} \\ \text{an } n\text{-shifted absolute} \\ \text{form on } F\end{pmatrix} \Longleftrightarrow \begin{pmatrix}\omega_X \text{ is non-degenerate} \\ \text{as a } K_X[n-d]\text{-valued} \\ \text{relative form on } \mathscr{F}.\end{pmatrix}$$

Pushforward theorem $\Longrightarrow$ $\mathrm{tr}\left(\Gamma(\omega_X)\right)$ is an $(n-d)$-shifted symplectic structure on $\Gamma(X, \mathscr{F}) = Map_{\mathrm{dSt}_{\mathbb{C}}}(X, F)$.

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$$\left(\begin{array}{l}\omega \text{ is non-degenerate as} \\ \text{an } n\text{-shifted absolute} \\ \text{form on } F\end{array}\right) \Leftrightarrow \left(\begin{array}{l}\omega_X \text{ is non-degenerate} \\ \text{as a } K_X[n-d]\text{-valued} \\ \text{relative form on } \mathscr{F}.\end{array}\right)$$

Pushforward theorem $\Longrightarrow$ $\operatorname{tr}(\Gamma(\omega_X))$ is an $(n-d)$-shifted symplectic structure on $\Gamma(X, \mathscr{F}) = Map_{\mathrm{dSt}_{\mathbb{C}}}(X, F)$.
This is the mapping stack theorem from **[PTVV]**.

Lagrangian maps

Setup:

- $X = (0,1]$ is the half-open interval stratified by the strata $X_{\mathrm{in}} = (0,1)$ and $X_1 = \{1\}$, labeled by $I = \{\mathrm{in}, 1\}$ with $1 < \mathrm{in}$.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}}) \Longrightarrow$ specified by $\mathscr{F}_1, \mathscr{F}_{\mathrm{in}} \in \mathrm{dSt}_{\mathbb{C}}$, and one cospecialization map $\mathscr{F}_1 \to \mathscr{F}_{\mathrm{in}}$.

Note: K_X = (extension by zero of $\mathbb{C}_{X_{\mathrm{in}}}[1]$ from X_{in} to X). Thus a relative n-shifted symplectic structure on $\mathscr{F}$ consists of:

- an $(n+1)$-shifted symplectic structure on $\mathscr{F}_{\mathrm{in}}$;
- an n-shifted Lagrangian structure on $\mathscr{F}_1 \to \mathscr{F}_{\mathrm{in}}$.

Lagrangian intersections

Setup:

- $X = [0,1]$ stratified by $X_{\mathrm{in}} = (0,1)$, $X_0 = \{0\}$, and $X_1 = \{1\}$, where $I = \{0, 1, \mathrm{in}\}$ with $0 < \mathrm{in}$, $1 < \mathrm{in}$, while 0 and 1 are incomparable.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}}) \Longrightarrow$ specified by $\mathscr{F}_0, \mathscr{F}_1, \mathscr{F}_{\mathrm{in}} \in \mathrm{dSt}_{\mathbb{C}}$, and two cospecialization maps $\mathscr{F}_0 \to \mathscr{F}_{\mathrm{in}}$ and $\mathscr{F}_1 \to \mathscr{F}_{\mathrm{in}}$.

Note: $K_X = $ (extension by zero of $\mathbb{C}_{X_{\mathrm{in}}}[1]$ from X_{in} to X). A relative n-shifted symplectic structure on $\mathscr{F}$ consists of:

- an $(n+1)$-shifted symplectic structure on $\mathscr{F}_{\mathrm{in}}$;
- n-shifted Lagrangian structures on $\mathscr{F}_0 \to \mathscr{F}_{\mathrm{in}}$ and $\mathscr{F}_1 \to \mathscr{F}_{\mathrm{in}}$.

Lagrangian intersections

Setup:

- $X = [0,1]$ stratified by $X_{\mathrm{in}} = (0,1)$, $X_0 = \{0\}$, $X_1 = \{1\}$.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}}) \Longrightarrow$ specified by $\mathscr{F}_0, \mathscr{F}_1, \mathscr{F}_{\mathrm{in}} \in \mathrm{dSt}_{\mathbb{C}}$, and two cospecialization maps $\mathscr{F}_0 \to \mathscr{F}_{\mathrm{in}}$ and $\mathscr{F}_1 \to \mathscr{F}_{\mathrm{in}}$.

A relative n-shifted symplectic structure ω on $\mathscr{F}$ consists of:

- an $(n+1)$-shifted symplectic structure on $\mathscr{F}_{\mathrm{in}}$;
- n-shifted Lagrangian structures on $\mathscr{F}_0 \to \mathscr{F}_{\mathrm{in}}$ and

$\mathscr{F}_1 \to \mathscr{F}_{\mathrm{in}}$.

Pushforward theorem $\Longrightarrow$ $\mathrm{tr}\left(\Gamma(\omega)\right)$ is an n-shifted symplectic structure on $\Gamma(X, \mathscr{F}) = \mathscr{F}_0 \times^h_{\mathscr{F}_{\mathrm{in}}} \mathscr{F}_1$.

Lagrangian intersections

Setup:

- $X = [0,1]$ stratified by $X_{\text{in}} = (0,1)$, $X_0 = \{0\}$, $X_1 = \{1\}$.
- $\mathscr{F} \in \mathsf{Sh}^{\text{str}}(X, \mathsf{dSt}_{\mathbb{C}}) \Longrightarrow$ specified by $\mathscr{F}_0, \mathscr{F}_1, \mathscr{F}_{\text{in}} \in \mathsf{dSt}_{\mathbb{C}}$, and two cospecialization maps $\mathscr{F}_0 \to \mathscr{F}_{\text{in}}$ and $\mathscr{F}_1 \to \mathscr{F}_{\text{in}}$.

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This is the Lagrangian intersection theorem from **[PTVV]**.

Hamiltonian reduction (iii)

Setup:

- $X = [0,1]$ with strata $X_{\text{in}} = (0,1)$, $X_0 = \{0\}$, $X_1 = \{1\}$.
- G - a linear algebraic group$/\mathbb{C}$, and $\mathbb{O} \subset \mathfrak{g}^\vee$ - a coadjoint orbit.
- (M, ω) - an algebraic symplectic manifold equipped with a Hamiltonian G-action.
- $\mu : M \to \mathfrak{g}^\vee$ - a G-equivariant moment map.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ given by

$$\mathscr{F}_0 = [\mathbb{O}/G], \quad \mathscr{F}_1 = [M/G], \quad \mathscr{F}_{\text{in}} = [\mathfrak{g}^\vee/G] = T^\vee_{BG}[1]$$

and maps $[\mathbb{O}/G] \hookrightarrow [\mathfrak{g}^\vee/G]$ and $\mu : [M/G] \to [\mathfrak{g}^\vee/G]$.

Hamiltonian reduction (iv)

Note: The **Kirillov-Kostant-Souriau** form on $\mathfrak{g}^\vee$ induces a relative 0-shifted symplectic structure ω on $\mathscr{F}$.

Pushforward theorem $\Longrightarrow$ $\mathrm{tr}\,(\Gamma(\omega))$ is a 0-shifted symplectic structure on $\Gamma(X, \mathscr{F}) = [R\mu^{-1}(\mathbb{O})/G]$.

Hamiltonian reduction (iv)

Note: The **Kirillov-Kostant-Souriau** form on $\mathfrak{g}^\vee$ induces a relative 0-shifted symplectic structure ω on $\mathscr{F}$.

Pushforward theorem $\Longrightarrow$ $\mathrm{tr}\,(\Gamma(\omega))$ is a 0-shifted symplectic structure on $\Gamma(X, \mathscr{F}) = [R\mu^{-1}(\mathbb{O})/G]$.

This is the **Marsden-Weinstein** Hamiltonian reduction theorem.

Quasi-Hamiltonian reduction (iii)

Setup:

- $X = S^2$ with strata $X_{\text{in}} = S^2 - \{S, N\}$, $X_0 = \{S\}$, $X_1 = \{N\}$.
- G - a complex reductive group, $\mathbf{C} \subset G$ - a conjugacy class.
- (M, ω) - an algebraic symplectic manifold equipped with a quasi-Hamiltonian G-action.
- $\mu : M \to G$ - a G-equivariant group valued moment map.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ given by

$$\mathscr{F}_0 = [\mathbf{C}/G], \quad \mathscr{F}_1 = [M/G], \quad \mathscr{F}_{\text{in}} = BG$$

and maps $[\mathbf{C}/G] \hookrightarrow [G/G]$ and $\mu : [M/G] \to [G/G]$.

Quasi-Hamiltonian reduction (iv)

Note:

- $K_X = $ (extension by zero of $\mathbb{C}_{X_{\text{in}}}[2]$ from X_{in} to X).
- The standard 2-shifted symplectic form ω_κ on BG extends to a natural relative 0-shifted symplectic form $\omega_X \in \mathbf{A}^{2,cl}_X(\mathscr{F})_{K_X}$ on $\mathscr{F}$.

Pushforward theorem $\Longrightarrow$ $\operatorname{tr}(\Gamma(\omega_X))$ is a 0-shifted symplectic structure on $\Gamma(X, \mathscr{F}) = [R\mu^{-1}(\mathbf{C})/G]$.

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Quasi-Hamiltonian reduction (iv)

Note:

- K_X = (extension by zero of $\mathbb{C}_{X_{\text{in}}}[2]$ from X_{in} to X).
- The standard 2-shifted symplectic form ω_κ on BG extends to a natural relative 0-shifted symplectic form $\omega_X \in \mathbf{A}^{2,cl}_X(\mathscr{F})_{K_X}$ on $\mathscr{F}$.

Pushforward theorem $\Longrightarrow$ $\operatorname{tr}(\Gamma(\omega_X))$ is a 0-shifted symplectic structure on $\Gamma(X, \mathscr{F}) = [R\mu^{-1}(\mathbf{C})/G]$.

This is the **Alexeev-Malkin-Meinrenken** quasi-Hamiltonian reduction theorem.

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