Families of Hitchin systems and SCFTs of class S

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Introduction

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Joint work with Aswin Subramanian and Jacques Distler.

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In this realization from six dimensions, the Hitchin system plays an important role. Specifically, the Coulomb branch associated to the four dimensional theory can be described as the base $\mathcal{B}$ of Hitchin's integrable system associated to a simply laced Lie algebra $\mathfrak{j}$ and the UV curve $C_{g,n}$. The choice of the lie algebra j parameterizes the available 6d $(0,2)$ theories and the choice of $C_{g,n}$ determines the 2d surface on which we compactify the 6d theory (together with a partial twist). At the locations of the n punctures, we insert four dimensional defects of the 6d $(0,2)$ theory. The insertions of these defects affect the behaviour of the Hitchin system at these punctures: the Higgs fields become meromorphic.

3D, N=4 theories have a moduli space consisting of Higgs and Coulomb branches.

3D mirror symmetry is supposed to exchange Higgs and Coulomb branches.

Nakajima 2015: Towards a mathematical definition of Coulomb branches of 3-dimensional N=4 gauge theories, I

Braverman, Finkelberg, and Nakajima: Towards a mathematical definition of Coulomb branches of 3-dimensional N=4 gauge theories, II.

The physics background

We will consider mostly tame defects, where the Higgs field in the Hitchin system has a simple pole at the punctures.

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In order to obtain superconformal field theories (SCFT) using tame defects, we additionaly require that $Res(\phi) = a$ be a nilpotent element in the Lie algebra $\mathfrak{j}$. What really matters is the $\mathfrak{j}$-conjugacy class to which the element a belongs. So, it is helpful to label the Hitchin boundary condition by the nilpotent orbit $\mathcal{O}_a$ to which the element a belongs. We will sometimes call this nilpotent orbit the Hitchin orbit $\mathcal{O}_H$ associated to the defect.

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The physics background

(The absence of such discrete data for defects in type A is related to the fact that component groups of centralizers of nilpotent orbits are always trivial in type A. Let us define

$$A(\mathcal{O}_a) = C_{\mathrm{j}}(a)/C_{\mathrm{j}}^0(a)$$

to be the group of components of the centralizer of nilpotent orbit $\mathcal{O}_a$ in the adjoint group. Here, $C_{\mathrm{j}}(a)$ is the centralizer of a and $C_{\mathrm{j}}^0(a)$ is its connected component.

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The above statement is equivalent to saying that

$$A(\mathcal{O}_a) = 1$$

for every nilpotent orbit in type A. In the discussion below, we confine ourselves to examples from type A Hitchin systems.)

Weakly coupled gauge groups

An important feature of this geometric realization from six dimensions is that the space of marginal parameters associated to the SCFT is identified with the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ of complex structures on $C_{g,n}$. Restricting to a stratum of (complex) codimension one in $\overline{\mathcal{M}}_{g,n}$ corresponds to the appearance of a weakly coupled gauge group with an associated gauge coupling that is related to plumbing fixture parameter q by $q = e^{2\pi i\tau}$.

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Restricting to a codimension k boundary stratum corresponds to a locus where k simple factors in the gauge group become weak. The 0-dimensional strata are given by pants decompositions of $C_{g,n}$: the class-S theory is presented as a gauge theory whose semisimple group is the product of $3g-3+n$ simple factors (corresponding to the nodes) coupled to $2g-2+n$ isolated SCFTs corresponding to 3-punctured spheres.

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Good, Bad and Ugly theories

Gaiotto, Witten introduced a Good, Bad and Ugly trichotomy for 3d N = 4 theories using properties of their Higgs branch. Lifts to 4d N = 2 theories with Coulomb branches described by the tame Hitchin system. This trichotomy can also be understood purely in terms of the Hitchin base, B.

Higgs bundles and integrable systems

Higgs bundles and integrable systems

The Hitchin system is a global version of the quotient map $\mathfrak{g} \to \mathfrak{t}/W$, where $\mathfrak{t}$ is the Cartan of the Lie algebra $\mathfrak{g}$ of the group G, and W is the Weyl group. When $G = GL(N)$, this map sends a matrix to its spectrum, or unordered set of eigenvalues (keeping track of multiplicities).

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Given C, G and a line bundle L on C, the total space $Higgs_{C,G,L}$ is the moduli space of L-valued G-Higgs bundles on the curve C. For $G = GL(N)$, this means pairs (V, ϕ) where V is a rank N vector bundle and $\phi : V \to V \otimes L$ is the Higgs field. For general G, replace V by a principal G-bundle P over C, and $\phi \in \Gamma(C, ad(P) \otimes L)$. (You can impose a stability condition and get a moduli space, or not, and get a stack.)

Higgs bundle and integrable systems

The base $\mathcal{B}$ is $\Gamma(C, L \otimes \mathfrak{t}/W)$. A point $b \in B$ determines a W-Galois cover, called the cameral cover $\widetilde{C} \to C$, namely the inverse image in $\Gamma(C, K_C \otimes \mathfrak{t})$ of the corresponding section. The choice of a representation ρ of G maps the cameral cover to the spectral cover $\widetilde{C}_\rho$ sitting in the total space of L and parametrizing the spectra of $\rho(\phi)$ over points of C. For faithful ρ, the cameral and spectral data are equivalent data.

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Explicitly the base is $\mathcal{B} = \sum_{i=1}^{r} \Gamma(C, (L)^{\otimes d_i})$, where $r = dim(\mathfrak{t})$ is the rank of G and the d_i are the degrees of the G-invariant polynomials, e.g. for $GL(N)$ we have $r = N$ and $d_i = i$.

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The quotient map applied to the Higgs field ϕ determines a family parametrized by C of L-valued points of $\mathfrak{t}/W$. This is the Hitchin map $h : Higgs \to \mathcal{B}$: it sends ϕ to its cameral cover, or spectral cover, or the family of its spectra.

Hitchin and Markman systems

For $L = K_C$, Hitchin proved that *Higgs* is (holomorphically) symplectic and the map to $\mathcal{B}$ is Lagrangian. In the meromorphic case, i.e. for $L = K_C(D)$, where D is an effective divisor on C, Markman and Bottacin proved that *Higgs* is Poisson. The residue map $Res_D : Higgs_{C,G,L} \to (\mathfrak{g}/G)^D$ sends a meromorphic Higgs bundle to (the conjugacy class of) its residue. The symplectic leaves are the fibers. The closures of regular (i.e. generic) symplectic leaves are parametrized by $\Gamma(D, O_D \otimes \mathfrak{t}/W)$, and are the fibers of the composition:

$$Higgs_{C,G,K(D)} \xrightarrow{h} \mathcal{B} = \Gamma(C, K(D) \otimes \mathfrak{t}/W) \xrightarrow{res} \Gamma(D, O_D \otimes \mathfrak{t}/W)$$

.

Class $S[j, C_{0,4}]$ theories

It turns out that Class $\mathcal{S}$ theories associated to four punctured spheres are quite rich, so we will focus on those. Let the locations of four punctures be z_1, z_2, z_3, z_4 and let λ be their cross ratio

$$\lambda = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

Let $\widetilde{\mathbb{CP}}^2$ be the blowup of $\mathbb{CP}^2$ at 4 points : $E_1 \to (1,0,0)$, $E_2 \to (0,1,0)$, $E_3 \to (0,0,1)$, $E_4 \to (1,1,1)$.
We realize $\overline{\mathcal{M}}_{0,4}$ as a family of curves in $\widetilde{\mathbb{CP}}^2$ using the following pencil of conics

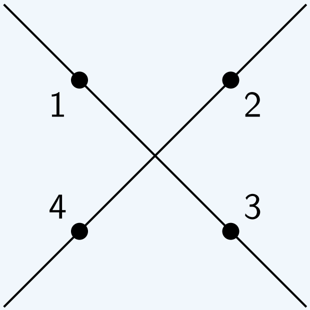
$$C_\lambda = \{\lambda_1 x(y - z) + \lambda_2 y(z - x) = 0\}$$

where λ_1, λ_2 are homogenous co-ordinates on $\overline{\mathcal{M}}_{0,4} = \mathbb{CP}^1$. We identify the cross ratio $\lambda = \lambda_1/\lambda_2$.

Class $S[j, C_{0,4}]$ theories

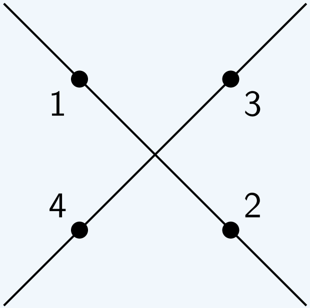
At the three boundary points of $\overline{\mathcal{M}}_{0,4}$, corresponding to $\lambda = 0, 1, \infty$, the conic C_λ degenerates into a pair of lines:

$$C_0 = \{y(z-x) = 0\}, \quad \text{(with the node at } n_0 = (1,0,1)\,)$$



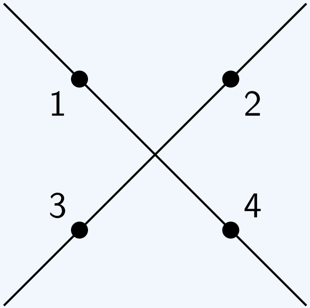
Class $S[j, C_{0,4}]$ theories

$C_1 = \{z(x-y)=0\}$, (with the node at $n_1 = (1,1,0)$)



Class $S[j, C_{0,4}]$ theories

$$C_\infty = \{x(y-z)=0\}, \quad \text{(with the node at } n_\infty = (0,1,1)\,)$$



Class $S[j, C_{0,4}]$ theories

Let us now try to describe the Higgs field Φ on $C_{0,4}$.

We pick the following line bundle on $\widetilde{\mathbb{CP}}^2$

$$L = \mathcal{O}_{\widetilde{\mathbb{CP}}^2}(-1)(\sum_i E_i + C_\infty)$$

The spectral curve $\Sigma \to C$ is given by the equation

$$0 = \det(w\mathbf{1} - \Phi) = w^N - \sum_{k=2}^{N} \phi_k w^{N-k}$$

where the ϕ_k are holomorphic sections

$$\phi_k \in H^0(\widetilde{\mathbb{CP}}^2, L^k(-\sum_i \chi_i^k E_i))$$

and the $4 \times (N-1)$ matrix χ_i^k, $i = 1, 2, 3, 4$, $k = 2, 3, 4, \ldots, N$ encodes zero orders of ϕ_k at each of the four punctures.

The standard node

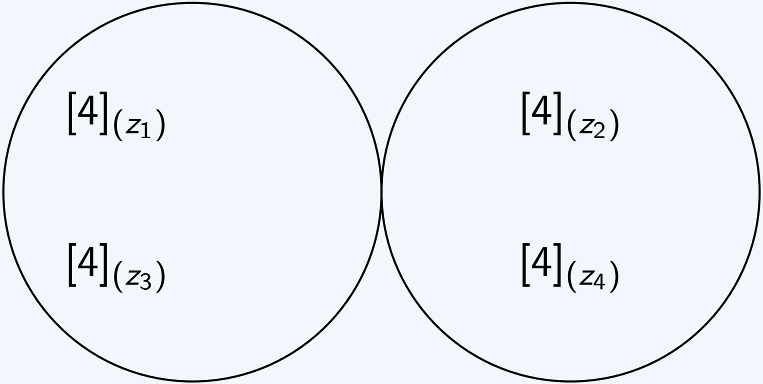
When the Hitchin orbits at the four punctures are sufficiently big, we get a node that is unrestricted. We call such a node the "standard" node. When $\mathfrak{j} = A_1$ or if $g \geq 1$, then every node is a standard node. On a four punctured sphere, a (separating) standard node occurs when the Hitchin orbits are sufficiently big. Let us understand this separating standard node using an example from $\mathfrak{j} = A_3 = SL_4$. Consider the four punctured sphere where at each puncture, the residue of the Higgs field, $Res(\phi) = a \in [4]$, the principal nilpotent orbit[1] of SL_4. This corresponds to requiring that ϕ_k has *simple* zeros at each of the four punctures $E_i, i = 1, 2, 3, 4$. So, the Higgs field is valued in

$$\phi_k \in H^0(\widetilde{\mathbb{CP}}^2, L^k(-\sum E_i))$$

[1]We label nilpotent orbits of $\mathfrak{sl}_N$ using partitions of N where the parts denote the sizes of the Jordan blocks.

The standard node

Recall that the spectral curves are parameterized by points of the Hitchin base $\mathcal{B}$ and the fibers of the integrable system are their (compactified) Picards. What happens to the family of spectral curves (or equivalently, the base $\mathcal{B}$) when $\lambda = 1$ so the curve is $z(x - y) = 0$?



The standard node

After an elementary computation, we get the following family of spectral curves:

$$0 = Det(\Phi - w\mathbf{1}) =$$
$$w^4 - y(z-x)\big[u_{2,C}w^2 + (u_{3,C}y + u_{3,L}(x-y) + u_{3,R}z)w$$
$$+(u_{4,C}xy + u_{4,L}(x-y)^2 + u'_{4,L}y(x-y) + u_{4,R}z^2 + u'_{4,R}yz)\big]$$

And we have following graded base dimensions $d_k^{L,C,R}, k = 2,3,4$:

- $d_k^L = \{0,1,2\}$
- $d_k^C = \{1,1,1\}$
- $d_k^R = \{0,1,2\}$

If we choose to focus on just the right component (the line x-y=0), we get the following spectral curve

$$w^4 - y(z-x)[u_{2,C}w^2 + (u_{3,C}y + u_{3,R}z)w + (u_{4,C}xy + u_{4,R}z^2 + u'_{4,R}yz)] = 0$$

The standard node

We define $\mathcal{B}_{R+C}$ to be a subspace of the Hitchin base $\mathcal{B}$ that is parameterized by the free parameters in the spectral curve for the right component. The subscript $R+C$ is chosen because this also includes the parameters that we would assign to the center. Note that $dim(\mathcal{B}_{R+C}) = \sum_k (d_k^R + d_k^C) = 6$. Over it we get a Poisson subsystem (the base is 6 dimensional, the fibers 3 dimensional) of the Hitchin system on the singular curve. It is a special case of Markman's meromorphic systems: two regular nilpotents at the marked points, and a regular element at the node. The three center parameters are Casimirs: they parametrize the eigenvalues of the residue at the node. Cases where the weakly coupled group is smaller than $SL(4)$ correspond to subsystems where the residue at the node is constrained to some irregular classes.

Restricted SL_4 nodes

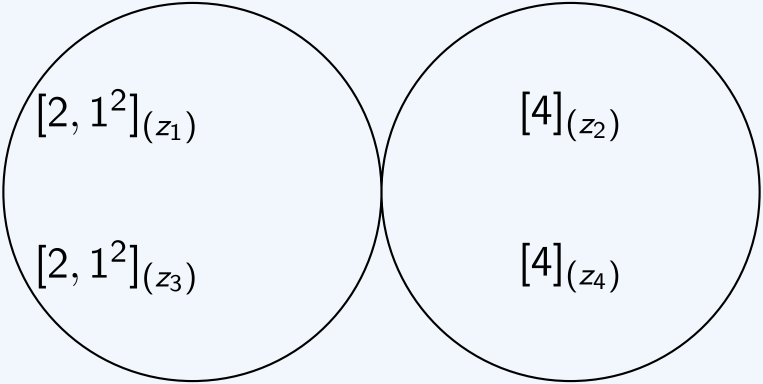
Before proceeding to the restricted nodes in SL_4, it is helpful to recall the zero orders of ϕ_k corresponding to different Hitchin (nilpotent) orbits.

Nahm $\mathcal{O}_N$	Nahm WDD	Hitchin $\mathcal{O}_H$	ϕ_2	ϕ_3	ϕ_4	Sheet Levi
$[1^4]$	0-0-0	$[4]$	1	1	1	0
$[2,1^2]$	1-0-1	$[3,1]$	1	1	2	A_1
$[2^2]$	0-2-0	$[2^2]$	1	2	2	$A_1 + A_1$
$[3,1]$	2-0-2	$[2,1^2]$	1	2	3	A_2

We have included the (unique) sheet Levi corresponding to each defect in anticipation of the SL_N discussion below. They can be ignored for the pure SL_4 discussion.

We will continue to use the Hitchin labels to identify the defects at the four punctures.

Restricted SL_4 nodes



E_i	$\mathcal{O}_H$	ϕ_2	ϕ_3	ϕ_4
E_1	$[2, 1^2]$	1	2	3
E_2	$[4]$	1	1	1
E_3	$[2, 1^2$	1	2	3
E_4	$[4]$	1	1	1

Restricted SL_4 nodes

In this case, we get the following family of spectral curves

$$w^4 - y(z-x)[u_{2,C}w^2 + u_{3,R}zw + u_{4,R}z^2]$$

and we deduce the graded base dimensions $d_k^{L,C,R}, k = 2,3,4$:

- $d_k^L = \{0,0,0\}$
- $d_k^C = \{1,0,0\}$
- $d_k^R = \{0,1,1\}$

Restricting to the sub-integrable system on the right component leads to a Hitchin system on $C_{0,3}^R$ with $Res(\phi)_{z_2,z_4} \in [4]$ as before **but** we get $Res(\phi)_{z'} \in [3,1]$, where z' is the third puncture on $C_{0,3}^R$ (the right component in the normalization of the nodal curve). This is the new feature of this example.

Nilpo orbits and spectral curves: local story, smooth curve

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We work with the Hitchin system for $G = SL(n)$ on a smooth curve C with marked points in a reduced divisor $D = \sum_k p_k$. A nilpotent orbit for $SL(n)$ is specified by its Hitchin partition or the dual Nahm partition.

Nilpo orbits and spectral curves: local story, smooth curve

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Nilpo orbits and spectral curves: local story, smooth curve

Note that the orbit O determines a generic form of the spectral cover $\tilde{C}_O$. The actual cover could be any specialization of the generic form, i.e. the orders of vanishing of the coefficients are allowed to go up but not down.

Nilpo orbits and spectral curves: local story, smooth curve

Note that the orbit O determines a generic form of the spectral cover $\tilde{C}_O$. The actual cover could be any specialization of the generic form, i.e. the orders of vanishing of the coefficients are allowed to go up but not down. Conversely, a given spectral cover $\tilde{C}$ determines a smallest orbit $O_{\tilde{C}}$. The actual orbit obtained from some sheaf on $\tilde{C}$ may be any orbit containing $O_{\tilde{C}}$ in its closure. E.g. the regular nilpotent corresponds to line bundles on $\tilde{C}$. The smallest orbit $O_{\tilde{C}}$ corresponds to direct images on $\tilde{C}$ of line bundles on the normalization of $\tilde{C}$.

The coefficient a_i is a section of a line bundle of degree:

$$d^i := i(-2 + deg(D)) - \sum_{k \in D} \chi^i_k = -2i + \sum_{k \in D} \pi^i_k.$$

The space of all such sections is a vector space of dimension: $b^i := max(d^i + 1, 0)$.
In full generality, the coefficient a_i is a section of a line bundle of degree:

$$d^i := i(2g - 2 + deg(D)) - \sum_{k \in D} \chi^i_k = (2g - 2)i + \sum_{k \in D} \pi^i_k.$$

The space of all such sections is a vector space of dimension: $b^i \geq max(d^i - g + 1, 0)$.
We say the system is good if equality holds, i.e. if these line bundles have at most one non vanishing cohomology.

Integrability

Let $Disc \subset \mathcal{B}$ be the discriminant divisor, parametrizing singular spectral curves. A general point of $d \in Disc$ corresponds to a spectral curve $\overline{C}_d$ with a single node. Let $N \to \overline{C}_d$ be the normalization. The Jacobian of $\overline{C}_d$ is a C^*-bundle over $Jac(N)$, and the Hitchin fiber is its compactification. The union over $Disc$ of the $Jac(N)$ forms a new integrable system, where both the base and fiber dimensions have dropped by 1. Indeed, let *disc* be the equation of *Disc*. This new system is just the Marsden-Weinstein reduction of the Hitchin system along *disc*: the corresponding Hamiltonian flows along the C^* fibers. An analogous result holds for the closure of the locus of spectral curves with a given number of nodes, or more generally, an imposed collection of n_i-tuple points.

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Resolution of singularities

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A smooth spectral cover can arise only from regular Higgs fields. For nilpotent residues, the Hitchin partition can have only 1 part, i.e. the Nahm partition starts with a 1.

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Degenerations and collisions of marked points

Hitchin system on a Gorenstein curve

Replace the smooth base curve C by a Gorenstein curve: it can be singular, as long as there is still a good canonical line bundle K_C. Any curve that is a divisor in a smooth surface will do: the canonical line bundle is given by the adjunction formula. This includes any curve whose only singularities are nodes: sections of the canonical bundle are 1-forms on the normalization with first order poles allowed at the (inverse images of) the nodes, with opposite residues at the two inverse images of each node.

Hitchin system on a Gorenstein curve

As in the smooth case, the Hitchin system for C and a reductive group G is the space *Higgs* of (isomorphism classes of) K_C-valued G-Higgs bundles on C. A G-Higgs bundle is a pair (V, ϕ) where V is a principal G-bundle on C and $\phi \in H^0(C, ad(V) \otimes K_C)$. For now we will focus on the case $G = GL(n)$, so V is a vector bundle and $\phi : V \to V \otimes K_C$; or $G = SL(n)$, where $det(V)$ is required to be O_C and the trace of ϕ is required to vanish . As in the smooth case, one can consider a GIT version where the Higgs bundles are subject to a stability condition; or one can allow all Higgs bundles and work with the resulting stack.

Hitchin system on a Gorenstein curve

Also as in the smooth case, the spectral curve of (V, ϕ) is the curve in the total space of K_C defined by the vanishing of the characteristic polynomial of the endomorphism ϕ. The Hitchin base B is defined to be the space of all spectral curves. This can be identified with the vector space:
$B := \oplus_i H^0(C, (K_C)^i)$
where i runs over the degrees of the G-invariants. For $G = SL(n)$, these degrees are $i = 2, \ldots, n$. The Hitchin map $h : Higgs \to B$ sends (V, ϕ) to (the coefficients of) the characteristic polynomial of ϕ. There is a natural analog of the above allowing poles on a divisor D consisting of distinct smooth points of C.

Results

- Higgs is Poisson
- Its symplectic leaves are obtained by restricting the residues at each marked point to a coadjoint orbit.
- The Hitchin map h is weakly Lagrangian. In particular, its restriction to a generic symplectic leaf is a symplectic integrable system.

The proofs follow the corresponding proofs for smooth C in Markman. (One new feature is that properness of the Hitchin map fails in the singular case. There is some literature discussing this and proposing appropriate partial compactifications of Higgs (for $GL(N)$) to which the Hitchin map h extends as a proper map: Chervov and Talalaev; Balaji, Barik and Nagaraj.)

Hitchin system on a reducible nodal curve

Further results:

The splitting of the base: $B = B_L \times B_C \times B_R$.

The nodal limit

The spectral curves

The fibers

Central Hamiltonians vs non-compactness of fibers.

Global structure of the bundle of Hitchin bases.

Degeneration of Hitchin systems

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The case of immediate interest is that of a one-parameter family of smooth rational curves C_t degenerating to a reducible limit C_0 consisting of two components $C_l \cup C_r$ intersecting transversally in a node $p = C_l \cap C_r$, with the marked points D splitting into divisors D_l, D_r on the two components.

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The case of immediate interest is that of a one-parameter family of smooth rational curves C_t degenerating to a reducible limit C_0 consisting of two components $C_l \cup C_r$ intersecting transversally in a node $p = C_l \cap C_r$, with the marked points D splitting into divisors D_l, D_r on the two components. For example, when $deg(D) = 4, deg(D_l) = deg(D_r) = 2$, this can be realized explicitly by the pencil of conics through 4 base points in $\mathbb{P}^2$ specializing to a pair of lines, as we saw in the physics section.

Degeneration of Hitchin systems

Sufficient condition for flatness: If the basic line bundle $K_{C_0}(D)$ is very ample on the limiting base curve, then the (Poisson) Hitchin system for (C_0, D, G) is a flat limit of the Hitchin systems for (C_t, D, G). In particular, the same holds for each symplectic leaf. For example, if the family of curves is the pencil of conics through 4 points in $\mathbb{P}^2$ as above, the condition is simply that $K_{C_0}(D)$ has positive degree on each of the two components of C_0. This simply requires at least two marked points on each of the two components. This holds in our Examples 1-5.

Degeneration of B: L, C, R.

For our degenerating Hitchin systems, we see that the restriction to one component of the limit can be a **proper** subspace of $H^0(P^1, O(d_r^i))$. This happens only when the line bundle $L_{i,l}$ has higher cohomology, i.e. when its degree d_l^i satisfies $d_l^i \leq -2$. (The condition is necessary and sufficient when L_0 itself has no higher cohomology.)

Degeneration of B: L, C, R.

As we saw, the coefficient a_i on a smooth C_t is a section of a line bundle $L_{i,t}$ of degree: $d^i := -2i + \sum_{k \in D} \pi^i_k$.
Let $L_{i,0}$ be the limiting line bundle on the reducible curve C_0, let $L_{i,l}$ and $L_{i,r}$ be its restrictions to the left or right component of C_0, and let d^i_l, d^i_r be their degrees. These are given by analogous formulas, except that the sum on the left is now over $k \in D_l \cup \{p\}$, with $\pi^i_p := i - 1$, and similarly for the sum on the right. (The pole orders $\pi^i_p = i - 1$ allow for arbitrary (e.g. regular) nilpotents at the node p.)
In terms of these degrees, the dimensions of the spaces of sections are:
$b^{i'}_l := max(d_l, 0)$
$b^{i'}_r := max(d_r, 0)$
$b^i_c = 1$ if $d_l \geq 0$ and $d_r \geq 0$, otherwise, $b^i_c = 0$.
The question is whether these dimensions equal their unprimed counterparts b^i_l, b^i_r arising as limits from the Hitchin systems on the smooth curves.

Degeneration of B: L, C, R.

We may as well consider the following:
General setup : we have a family of smooth rational curves C_t degenerating to a 2-component reducible rational curve C_0, and a line bundle $L_t = O_{C_t}(d)$ specializing to $L_0 = (O(d_l), O(d_r))$ with $d_l + d_r = d$. What is the limit of the vector space of sections $H^0(P^1, O(d))$?
If $d_r \geq 0, d_l < 0$, the limit gives a proper subspace of $H^0(P^1, O(d_r))$. The codimension is $-d_l$, which equals $h^1(P^1, O(d_l)) + 1 = h^1(P^1, O(d_l - 1))$. One sees immediately that this limiting subspace is the kernel of the coboundary map: $H^0(P^1, O(d_r)) \to H^1(P^1, O(d_l - 1))$, coming from the restriction SES:
$0 \to O_{P^1_l}(d_l - 1) \to L_0 \to O_{P^1_r}(d_r) \to 0.$

Back to example 5

$\pi = 1, 1, 1$ and $1, 1, 1$ on left, $1, 2, 3$ and $1, 2, 3$ on right.
$\chi = 1, 2, 3$ and $1, 2, 3$ on left, $1, 1, 1$ and $1, 1, 1$ on right.
(The three entries for each quantity correspond to $i = 2, 3, 4$.)
On the smooth curves:
degrees $= 0, 0, 0$
dims $= 1, 1, 1$
On the limiting nodal curve:
$d_l = 0, -1, -2$ (note the occurrence of -2!)
$d_r = 0, 1, 2$
$b_l = 0, 0, 0$
$b'_r = 0, 1, 2$ (This is the dim on right side at $t = 0$, ignoring the limit.)
$b_c = 1, 0, 0$

Back to example 5

For $i = 4$, the SES for restricting from the nodal curve to its right component is:
$0 \to O_{P^1_l}(-3) \to L_{4,0} \to O_{P^1_r}(2) \to 0.$
On cohomology this gives:
$0 \to H^0(L_{4,0}) \to H^0(P^1_r, O(2)) \to H^1(P^1_l, O(-3)) \to 0,$
so: $b^4_r = h^0(L_{4,0}) = 1$. This is the first example when $d^i_l \leq -2$ and consequently $b^i_r < b^{i'}_r$.
For a regular nilpotent, we saw that the vanishing orders of the a_i are $1, 1, 1$. This is what we get from the $b^{i'}_r$. The discrepancy between these and b^i_r for $i = 4$ means that on the right component, the orders of vanishing are $1, 1, 2$. This corresponds to the subregular orbit of SL(4), with Hitchin partition $(3, 1)$.

Conclusion

A key puzzle for physicists is that Coulomb branches of S-class theories decompose as hyperKahler quotients of products of Coulomb branches of limiting components. The group you need to divide by is often the full gauge group, but sometimes a proper subgroup. We give an algorithm for determining which is the case and what the weakly coupled gauge subgroup is. There is also a realization due to BFN which - counter to physics intuition - is a HK quotient by the full group. We are trying to explain this alternative.

Conclusion

Other approaches to Higgs bundles on nodal curves:
Some recent progress on Higgs bundles on nodal curves: Bhosle 2013, V. Balaji, P. Barik, and D. S. Nagaraj 2016, J. Swoboda 2017, M. Logares 2018.
However, for our purposes, it is important to understand how the integrable system behaves in the nodal limit and this appears to not have been addressed previously in the mathematical literature. So, we develop this from scratch.

Conclusion

Another direction in which our results could be applied is in the study of the character variety. It is related to the moduli space of Higgs bundles through the nonabelian Hodge correspondence.
Unlike the geometry of Higgs bundles, the geometry of the character variety is independent of the choice of a complex structure on C. In particular, this means we could choose to work with any complex structure on C and then use the non-abelian Hodge correspondence to obtain the character variety.
Analogue of WDDV or Verlinde formula: extract information from behavior near a degeneration.
Another possible application: the study of natural Darboux co-ordinates on the character variety and their behaviour under different choices of pants decompositions of the underlying Riemann surface. Higher Fenchel-Nielsen co-ordinates.

THANK YOU !!!

3D mirror symmetry

Dimofte:

Three-dimensional N=4 supersymmetric gauge theories admit two topological "A" and "B" twists. They are analogous to the more familiar A and B twists in two dimensions, and they are exchanged by a physical duality known as "3d mirror symmetry." Many mathematical consequences of 3d mirror symmetry, especially at the homological/categorical level, are just beginning to be explored. I'll discuss a proposed identification of categories of line operators in the A and B twists, some partial structure of the 2-category of boundary conditions, and an application to HOMFLY-PT knot homology.

3D mirror symmetry

Wikipedia:
In theoretical physics, 3D mirror symmetry is a version of mirror symmetry in 3-dimensional gauge theories with N=4 supersymmetry, or 8 supercharges. It was first proposed by Kenneth Intriligator and Nathan Seiberg in their 1996 paper, Mirror symmetry in three-dimensional gauge theories[1], as a relation between pairs of 3-dimensional gauge theories, such that the Coulomb branch of the moduli space of one is the Higgs branch of the moduli space of the other. It was demonstrated using D-brane cartoons by Amihay Hanany and Edward Witten 4 months later,[2] where they found that it is a consequence of S-duality in type IIB string theory.

3D mirror symmetry

Four months later 3D mirror symmetry was extended to N=2 gauge theories resulting from supersymmetry breaking in N=4 theories.[3] Here it was given a physical interpretation in terms of vortices. In 3-dimensional gauge theories, vortices are particles. BPS vortices, which are those vortices that preserve some supersymmetry, have masses which are given by the FI term of the gauge theory. In particular, on the Higgs branch, where the squarks are massless and condense yielding nontrivial vacuum expectation values (VEVs), the vortices are massive. On the other hand, Intriligator and Seiberg interpret the Coulomb branch of the gauge theory, where the scalar in the vector multiplet has a VEV, as being the regime where massless vortices condense. Thus the duality between the Coulomb branch in one theory and the Higgs branch in the dual theory is the duality between squarks and vortices.