Homological Mirror Symmetry for higher dimensional pairs-of-pants

Sasha Polishchuk

January 28, 2020

This is joint work with Yanki Lekili

The goal is to prove the equivalence of the wrapped Fukaya category of n dimensional pairs-of-pants with the derived category of coherent sheaves on $x_1 x_2 \dots x_{n+1} = 0$.

Inspired by Auroux's calculation of the partially wrapped Fukaya category of the symmetric powers of punctured surfaces.

Main idea: introduce stops to simplify the endomorphism algebra of the set of generators. Identify corresponding nc resolution of $x_1 x_2 \dots x_{n+1} = 0$ on the B-side.

There exist other approaches (Gammage-Nadler, Auroux).

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Pair-of-pants

Let Σ be the 3-punctured sphere with the set of two stops Λ.

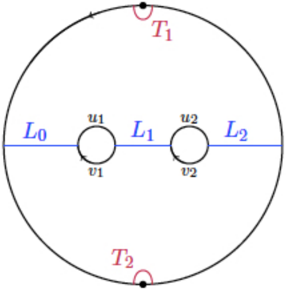
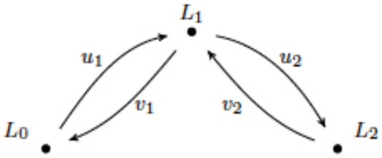


FIGURE 1. Pair-of-pants

The partially wrapped Fukaya category $\mathcal{W}(\Sigma, \Lambda)$ is generated by the Lagrangians L_0, L_1, L_2.

Endomorphism algebra



$$u_2 u_1 = 0 = v_1 v_2$$

FIGURE 2. Endomorphism algebra of a generating set

There exists a unique grading structure (given by the line field on Σ) such that the endomorphism algebra is concentrated in degree 0.

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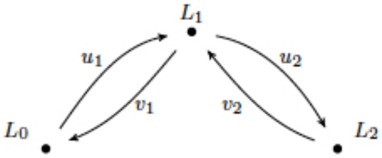

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Auslander order

On the B-side, we consider the algebra of the node

$$R = \mathbf{k}[x_1, x_2]/(x_1 x_2).$$

The Auslander order is given by

$$A = \mathrm{End}_R(R/(x_1) \oplus R/(x_2) \oplus R).$$

It is easy to see that A is isomorphic to the algebra associated with the above quiver with relations, so we get an equivalence

$$\mathrm{Perf}(A) \simeq \mathcal{W}(\Sigma, \Lambda).$$

In general, the Auslander order of a nodal curve C is $\mathcal{E}nd(\mathcal{I} \oplus \mathcal{O}_C)$, where $\mathcal{I}$ is the ideal sheaf of the nodes. The above equivalence generalizes to Auslander orders over nodal chains and rings.

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Localization

On the A-side, removing the stops corresponds to taking the quotient by the subcategory generated by the objects T_1, T_2 supported near the stops.

We can express them in terms of L_0, L_1, L_2 as follows:

$$T_1 \simeq \{L_0 \xrightarrow{u_1} L_1 \xrightarrow{u_2} L_2\}$$

$$T_2 \simeq \{L_2 \xrightarrow{v_2} L_1 \xrightarrow{v_1} L_0\}$$

Can identify corresponding objects on the B-side: we get simple modules at vertices L_0 and L_2 of the quiver. As a corollary, get an equivalence

$$\mathcal{W}(\Sigma) \simeq D^b(R).$$

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Symmetric products

Consider Π_n, the complement to $n+2$ generic hyperplanes in $\mathbb{P}^n$, as an exact symplectic manifold. Since $\mathbb{P}^n = \mathrm{Sym}^n(\mathbb{P}^1)$, have an identification

$$\Pi_n = \mathrm{Sym}^n(\mathbb{P}^1 \setminus \{p_0, p_1, \dots, p_{n+1}\}).$$

More generally, we consider

$$M_{n,k} = \mathrm{Sym}^n(\Sigma_k), \text{ where } \Sigma_k = \mathbb{P}^1 \setminus \{p_0, p_1, \dots, p_k\})$$

(for $k \geq n$). Away from a small neighborhood of the diagonal, the symplectic form can be arranged to be induced by one on the surface.

We fix two points q_1, q_2 on one of the boundary components of the punctured sphere, and consider the induced hypersurfaces $\Lambda_i = q_i \times \mathrm{Sym}^{n-1}(\Sigma_k)$. We will use either Λ_1 or $\Lambda = \Lambda_1 \cup \Lambda_2$ as stops.

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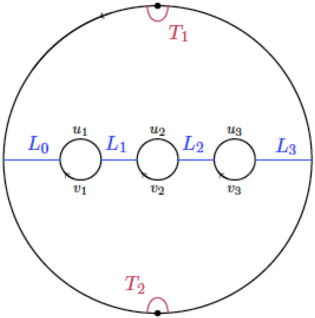
Generating Lagrangians

We start with the same collection of Lagrangians on Σ_k as before:



By Auroux's theorem, the products $L_S := L_{i_1} \times \ldots \times L_{i_n}$, for $S = \{i_1 < \ldots < i_n\} \subset [0, k]$, generate $\mathcal{W}(M_{n,k}, \Lambda)$.

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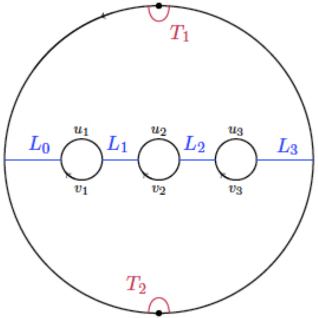

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Computation on the A-side. I

We can compute (cohomology of) morphism spaces between generating objects in $\mathcal{W}(M_{n,k}, \Lambda)$.

For every proper subinterval $[i, j] \subset [0, k]$, set

$$\mathcal{A}_{[i,j]} = \begin{cases} \mathbf{k}[x_i, \dots, x_{j+1}]/(x_i \dots x_{j+1}) & \text{if } i > 0, j < k, \\ \mathbf{k}[x_1, \dots, x_{j+1}] & \text{if } i = 0, j < k, \\ \mathbf{k}[x_i, \dots, x_k] & \text{if } i > 0, j = k, \end{cases} .$$

Proposition. For $S = [i_1, j_1] \sqcup [i_2, j_2] \sqcup \dots \sqcup [i_r, j_r]$ with $j_s + 1 < i_{s+1}$, one has

$$\operatorname{End}(L_S) \simeq \mathcal{A}(S, S) := \mathcal{A}_{[i_1,j_1]} \otimes \mathcal{A}_{[i_2,j_2]} \otimes \dots \otimes \mathcal{A}_{[i_r,j_r]},$$

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Computation on the A-side. II

The subsets $S, S' \subset [0,k]$ are called close if there exists a bijection $g : S \to S'$ with $g(i) \in \{i-1, i, i+1\}$. In this case there exists a decomposition

$$S = S_0 \sqcup \bigsqcup_a I_a \sqcup \bigsqcup_b J_b,$$

where I_a and J_b are subintervals, such that

$$S' = S_0 \sqcup \bigsqcup_a (I_a + 1) \sqcup \bigsqcup_b (J_b - 1).$$

Proposition. One has

$$\mathrm{Hom}(L_S, L_{S'}) \simeq \begin{cases} 0, & S, S' \text{ not close,} \\ \mathcal{A}(S_0, S_0) \otimes \bigotimes_a \mathcal{A}'_{I_a} \otimes \bigotimes_b \mathcal{A}'_{J_b}, & S, S' \text{ close,} \end{cases}$$

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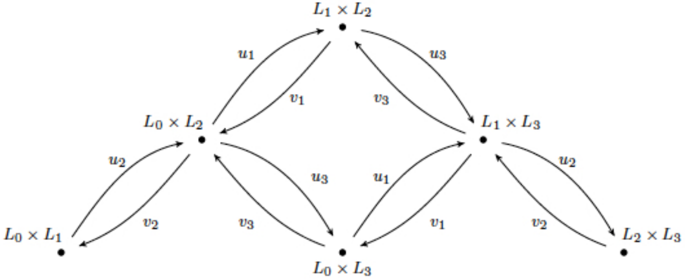
Computation on the A-side. III

Can compute compositions as well.
Example: $n = 2$, $k = 3$ (Sym^2 of 4-punctured sphere).
Get quiver with relations over $R = \mathbf{k}[x_1, x_2, x_3]/(x_1 x_2 x_3)$



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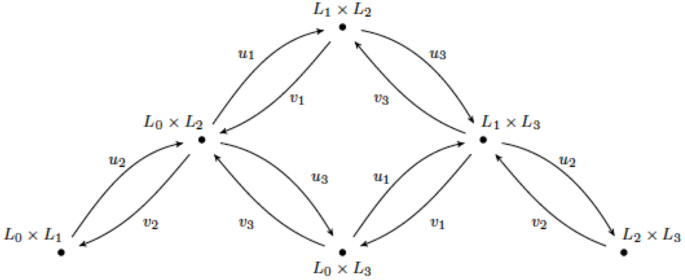

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Connection with Ozsváth-Szabó bordered algebras

Our algebra $\mathcal{A}$ of endomorphisms turns out to be the same as the algebra $\mathcal{B}(k, n)$ defined combinatorially in

[Ozsváth,Szabó], *Kauffman states, bordered algebras and a bigraded knot invariant*

They use bimodules over such algebras to define a categorification of the Alexander polynomial of a knot.

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$\mathbb{Z}$-gradings

Since $c_1(M_{n,k}) = 0$, the symplectic manifold $M_{n,k}$ can be equipped with a $\mathbb{Z}$-grading structure. The grading structures naturally form a torsor over $H^1(M_{n,k}, \mathbb{Z}) \simeq \mathbb{Z}^k$.

All our Lagrangians L_S are contractible, so they can be graded (uniquely up to a shift by $\mathbb{Z}$).

Proposition. For any assignment of degrees, $\deg(x_i) = d_i \in \mathbb{Z}$, $i = 1, \ldots, k$, there is a unique $\mathbb{Z}$-grading on the algebra

$$\mathcal{A} = \bigoplus_{S,S'} \mathrm{Hom}(L_S, L_{S'})$$

coming from some choices of $\deg(f_{S,S'}) = d_{S,S'} \in \mathbb{Z}$, for S, S' close, up to a transformation of the form $d_{S,S'} \mapsto d_{S,S'} + d_{S'} - d_S$.

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B-side

Let $R = R_{[1,k]} = \mathbf{k}[x_1, \dots, x_k]/(x_1 \dots x_k)$. We construct an nc-resolution of R.

$$\mathcal{B} = \mathcal{B}_{[1,k]} := \mathrm{End}_R\Big(\bigoplus_{I \subset [1,k], I \neq \emptyset} R/(x_I)\Big),$$

where the summation is over all nonempty subintervals of $[1, k]$, $x_I = \prod_{i \in I} x_i$.
E.g, for $k = 2$, this is precisely the Auslander order.

For each subinterval $I \subset [1, k]$, denote by P_I the corresponding projective module over $\mathcal{B}$. Note that $\mathrm{End}_{\mathcal{B}}(P_I) = R/(x_I)$. So we have a fully faithful embedding

$$i_R^{\mathcal{B}} : \mathrm{Perf}(R) \to \mathrm{Perf}(\mathcal{B}) : R \mapsto P_{[1,k]}$$

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Localization on the B-side

We also have the right adjoint functor to $i_R^{\mathcal{B}}$,

$$r_R^{\mathcal{B}} : D^b(\mathcal{B}) \to D^b(R) : M \mapsto \mathrm{Hom}_{\mathcal{B}}(P_{[1,k]}, M).$$

For a pair of nonempty disjoint subintervals $I, J \subset [1, k]$, such that $I \sqcup J$ is also a subinterval, can define a $\mathcal{B}$-module $M\{I, J\}$, so that we have an exact sequence

$$0 \to P_I \to P_{I \sqcup J} \to P_J \to M\{I, J\} \to 0.$$

Proposition. Assume $\mathbf{k}$ is regular. Then $r_R^{\mathcal{B}}$ induces an equivalence

$$D^b(\mathcal{B})/\ker(r_R^{\mathcal{B}}) \simeq D^b(R),$$

and $\ker(r_R^{\mathcal{B}})$ is generated by the modules $(M\{[i], [i+1, j]\}, M\{[j], [i, j-1]\})_{i<j}$.

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Matching the A-side with the B-side

For $n = k - 1$ the Lagrangians L_S are numbered by subsets $S \subset [0, k]$ with $|S| = k - 1$. Now we define the correspondence between such L_S and subintervals $I \subset [1, k]$ by

$$L_{[0,k]\setminus\{i,j\}} \leftrightarrow [i + 1, j],$$

where $0 \leq i < j \leq k$.

Theorem. This extends to an isomorphism of algebras $\mathcal{A} \simeq \mathcal{B}$, so that we get an equivalence of categories

$$\mathcal{W}(\Pi_{k-1}, \Lambda) \simeq \mathrm{Perf}(\mathcal{B}_k).$$

The $\mathbb{Z}$-grading on the left is the unique one with $\deg(x_i) = 0$.

Furthermore, the subcategory corresponding to stops matches with $\ker(r_R^{\mathcal{B}})$, so for $\mathbf{k}$ regular, we deduce

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Additional features

1. Can similarly identify the nc resolution of $R = \mathbf{k}[x_1, \dots, x_k]/(x_1 \dots x_k)$ corresponding to $\mathcal{W}(\Pi_{k-1}, \Lambda_1)$ (only one stop). It is given by

$$\mathcal{B}^\circ := \mathrm{End}_R(R/(x_1) \oplus R/(x_{[1,2]}) \oplus \dots \oplus R/(x_{[1,k-1]}) \oplus R).$$

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Abelian covers

For $k > 2$ we have a natural isomorphism $\pi_1(\Pi_{k-1}) \simeq \mathbb{Z}_k$. Fix a homomorphism

$$\phi : \pi_1(\Pi_{k-1}) \simeq \mathbb{Z}_k \to \Gamma,$$

where Γ is a finite abelian group, and let

$$\pi : M \to \Pi_{k-1}$$

be the corresponding finite covering.

Let $G = \mathrm{Hom}(\Gamma, \mathbb{G}_m)$ denote the dual abelian group scheme to Γ, and let

$$G \to \mathbb{G}_m^k$$

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Theorem. For **k** regular, we have an equivalence

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Punctured Milnor fibers for invertible polynomial

Let $\mathbf{w} = \sum_{i=1}^{k} \prod_{j=1}^{k} x_j^{a_{ij}}$ be an invertible polynomial described by the matrix of exponents (a_{ij}).

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$$M_{\mathbf{w}} := \{(x_1, \ldots, x_k) \in (\mathbb{C}^*)^{\times k} \mid \mathbf{w}(x_1, \ldots, x_k) = 1\}$$

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The group of deck transformations of this covering map is

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which is exactly the group of diagonal symmetries of $\mathbf{w}$.
Let $G = \mathrm{Hom}(\Gamma, \mathbb{G}_m)$ be the dual abelian group.

Corollary. For $\mathbf{k}$ regular we have an equivalence $\mathcal{W}(M_{\mathbf{w}}) \simeq D^b_G(\mathbf{k}[x_1, \ldots, x_k]/(x_1 \ldots x_k))$.

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