Interplay between Higgs bundles, opers and TQFT
Lecture I

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Mirror Symmetry and Related Topics 2018

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Lecture 1: Interplay between Quantum curves, Higgs bundles and opers.
Lecture 2: Invitation to 2d TQFT
Lecture 3: Edge Contraction axioms and CohFT (work in progress with Motohico Mulase)
Outline of Lecture 1

1. Introduction
2. On the other side of the rainbow
3. Introduction to Higgs bundles and connections
4. Higgs bundles and quantum curves
5. Quantization of Airy function
6. The methamorphosis of quantum curves into opers
7. General theory of Hitchin systems for the Lie group $G = SL_r(\mathbb{C})$
8. Opers
A Story of Quantum Curves

What they are, and what they do

- The simplest example
- A general geometric construction for the rank 2 case
- Geometry of Higgs fields
Primary and secondary rainbow

The color pattern of a rainbow is sunlight, displaced by reflection and dispersed by refraction in raindrops. It is seen by an observer with his or her back to the sun.
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Primary and secondary rainbow
There is an infinite sequence of arches
The rainbow integral

1. **Rene Descartes** in 1637 used reflection and refraction in spherical raindrops to explain formation of rainbow.

2. White light consists of rainbow colors was not known at the time. **Isaac Newton** explained the colors of the rainbow.

3. **Sir George Airy** devised a formula, ”the rainbow integral”, in his attempt of explaining the rainbow phenomena in terms of wave optics.

### The rainbow integral of Sir George Biddell Airy

The Airy function

$$Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} e^{ip^3/3} dp$$

explains the color diffraction pattern and the multiple arches of the rainbow.
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- \( x < 0 \) is the rainbow side.
- On the other side of the rainbow \( (x > 0) \), \( Ai(x) \) decays exponentially. We see no rainbows under the brightest one.
What awaits us on the other side of the infinite rainbow sequence?
Apply integration by parts, taking care of the boundary contributions in oscillatory integral to \( Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} e^{i\frac{p^3}{3}} dp \). Then

\[
\frac{d^2}{dx^2} Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-p^2) e^{ipx} e^{i\frac{p^3}{3}} dp
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} \left( i \frac{d}{dp} e^{i\frac{p^3}{3}} \right) dp
\]

\[
= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left( i \frac{d}{dp} e^{ipx} \right) e^{i\frac{p^3}{3}} dp
\]

\[
= \frac{1}{2\pi} x \int_{-\infty}^{\infty} e^{ipx} e^{i\frac{p^3}{3}} dp
\]

\[
= x \cdot Ai(x).
\]
The simplest example of a quantum curve

So Airy function satisfies the differential equation

\[
\left( \frac{d^2}{dx^2} - x \right) \cdot Ai(x) = 0.
\]

Introduce the Planck constant \( \hbar \) a positive real parameter for now.

\[
Ai(x, \hbar) := Ai\left( \frac{x}{\hbar^{2/3}} \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ip\frac{x}{\hbar^{2/3}}} e^{i\frac{p^3}{3}} dp.
\]

Then

The Airy differential equation = the simplest quantum curve

\[
(\hbar^2 \frac{d^2}{dx^2} - x) \cdot Ai(x, \hbar) = 0.
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The Airy differential equation = the simplest quantum curve

\[
\left( \hbar^2 \frac{d^2}{dx^2} - x \right) \cdot Ai(x, \hbar) = 0.
\]
The Airy function $Ai(x, \hbar)$ is entire in $x \in \mathbb{C}$.

1. (Dante Aligheri). "The path to paradise begins in hell."
2. (Riemann). The nature of holomorphic functions is concentrated at their singularities.

The Airy function has an essential singularity at $x = \infty$. How do we analyze the behavior of a holomorphic function at its essential singularity?

1. holomorphic functions are studied via Taylor expansions.
2. poles are studied via Laurent expansions.
3. essential singularities are studied via asymptotic expansions.
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The asymptotic expansion

From a textbook we learn

\[ Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} e^{i\frac{p^3}{3}} \, dp \]

\[ \sim \frac{1}{2\sqrt{\pi}} \frac{1}{x^{1/4}} \exp \left( -\frac{2}{3} x^{3/2} \right). \]

for \( x \gg 0 \) and \( x \) real.
A better asymptotic expansion

We can do even better:

\[
Ai(x, \hbar) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ip\frac{x}{\hbar^{2/3}}} e^{i\frac{p^3}{3}} dp \\
\sim \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{1}{\hbar} \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{4} \log x - \hbar \frac{5}{48} x^{-\frac{3}{2}} + \hbar^2 \frac{5}{64} x^{-3}\right).
\]

1. How far can we go?
2. What are these rational coefficients that approximate the Airy integral?
“Somewhere over the rainbow, dreams really do come true.”

What awaits us on the other side of the infinite rainbow sequence?

Intersection numbers on $\overline{\mathcal{M}}_{g,n}$
"Somewhere over the rainbow, dreams really do come true."

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Intersection numbers on $\overline{\mathcal{M}}_{g,n}$
The exact asymptotics

We have a closed formula for the exact asymptotics!

\[
Ai(x, \hbar) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i p \frac{x}{\hbar^{2/3}}} e^{i \frac{p^3}{3}} dp = \frac{1}{2\sqrt{\pi}} \exp \left( \sum_{m=0}^{\infty} \hbar^{m-1} S_m(x) \right),
\]

\[
S_m(x) = \frac{x^{\frac{3(m-1)}{2}}}{2^{m-1}} \cdot \sum_{2g-2+n=m-1} \frac{(-1)^n}{n!} \sum_{d_1+\cdots+d_n=3g-3+n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \prod_{i=1}^{n} |2d_i - 1|!!,
\]

\[
\langle \tau_{d_1} \cdots \tau_{d_n} \rangle := \int_{\overline{M}_{g,n}} c_1(\mathcal{L}_1)^{d_1} \cdots c_1(\mathcal{L}_n)^{d_n}
= \text{cotangent class intersection number on } \overline{M}_{g,n}
\]
Corollary (Rainbow formula 2015)

The cotangent class intersection numbers of \( \overline{M}_{g,n} \) satisfy the following relation for every \( m \geq 1 \):

\[
\sum_{g \geq 0, n > 0} \frac{1}{n!} \sum_{d_1 + \cdots + d_n = g \geq 0, 2g - 2 + n = m} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} \prod_{i=1}^{n} |2d_i - 1|!! = (\frac{1}{288})^m \sum_{\lambda \vdash m} (-1)^{\ell(\lambda) - 1} \frac{(\ell(\lambda) - 1)!}{|\text{Aut}(\lambda)|} \prod_{i=1}^{\ell(\lambda)} \frac{(6\lambda_i)!}{(2\lambda_i)!(3\lambda_i)!}.
\]

For example, \( m = 2 \), we have

\[
\frac{1}{6} \langle \tau_0^3 \tau_1 \rangle_{0,4} + \frac{1}{2} \langle \tau_1^2 \rangle_{1,2} + 3 \langle \tau_0 \tau_2 \rangle_{1,2} = \frac{5}{16}.
\]

This can be verified by evaluating

\[
\langle \tau_0^3 \tau_1 \rangle_{0,4} = 1, \quad \langle \tau_1^2 \rangle_{1,2} = \langle \tau_0 \tau_2 \rangle_{1,2} = \frac{1}{24}.
\]
$\frac{d^2}{dx^2} - x$ is the Lax operator of the celebrated Witten-Kontsevich solution to the KdV equation that generates all cotangent class intersection numbers on $\overline{M}_{g,n}$.

The relation between the Airy function and the intersection numbers was discovered by Kontsevich.

Note that in (2) $g$ and $n$ are summed for a fixed value $2g - 2 + n$. 
Semi-classical limit of Airy differential equation

Note that the $\hbar \to 0$ limit does not make sense for

$$\left( \hbar^2 \frac{d^2}{dx^2} - x \right) \text{Ai}(x, \hbar) = 0.$$ 

The semi-classical limit is the $\hbar \to 0$ limit of

$$\left[ e^{-\frac{1}{\hbar} S_0(x)} \left( \hbar^2 \frac{d^2}{dx^2} - x \right) e^{\frac{1}{\hbar} S_0(x)} \right] \exp \left( \sum_{m=1}^{\infty} \hbar^{m-1} S_m(x) \right) = 0.$$ 

$$\left( (S_0'(x))^2 - x + (\hbar \frac{d}{dx})^2 + 2\hbar S_0'(x) \frac{d}{dx} + \hbar S_0''(x) \right) \exp \left( \sum_{m=1}^{\infty} \hbar^{m-1} S_m(x) \right) \to 0.$$ 

It gives $(S_0'(x))^2 - x = 0$ or $S_0(x) = -\frac{2}{3} x^{\frac{3}{2}}$. For $y := S_0'(x)$, the semi-classical limit is called Airy curve.

$$y^2 - x = 0.$$
Semi-classical limit of Airy differential equation

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$$\left[ e^{-\frac{1}{\hbar} S_0(x)} \left( \hbar^2 \frac{d^2}{dx^2} - x \right) e^{\frac{1}{\hbar} S_0(x)} \right] e^{\sum_{m=1}^{\infty} \hbar^{m-1} S_m(x)} = 0.$$

$$[(S'_0(x))^2 - x + (\hbar \frac{d}{dx})^2 + 2\hbar S'_0(x) d/dx + \hbar S''_0(x)] e^{\sum_{m=1}^{\infty} \hbar^{m-1} S_m(x)} = 0. \text{ It gives } (S'_0(x))^2 - x = 0 \text{ or } S_0(x) = -\frac{2}{3} x^{\frac{3}{2}}. \text{ For } y := S'_0(x), \text{ the semi-classical limit is called Airy curve.}$$

$$y^2 - x = 0.$$
We developed a formalism so that by starting from the Airy curve

\[ y^2 - x = 0 \]

we can reconstruct the Airy differential equation and the asymptotic expansion of the Airy integral.

1. Consider parabola \( y^2 - x = 0 \) as a double cover of a line in \( T^* \mathbb{P}^1 \).

2. Let \( u := \frac{1}{x} \) be the infinity chart on \( \mathbb{P}^1 \). By a change of coordinates \( y \cdot dx = v \cdot du \) and \( \frac{1}{v} = w \), the Airy curve becomes singular.

\[
\begin{align*}
y &= v \cdot \frac{du}{dx} = v \cdot \frac{d}{dx} \left( \frac{1}{x} \right) = v \cdot \left( -\frac{1}{x^2} \right) = -v \cdot u^2
\end{align*}
\]

3. The Airy curve \( x = y^2 \) becomes singular \( w^2 = u^5 \) at infinity.

The singularity of the Airy curve reflects the essential singularity of \( Ai(x, \hbar) \) at infinity. Intersection numbers and their recursions appear after resolving the singularity of the Airy curve.
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\[
\frac{dy}{dx} = v \cdot \frac{du}{dx} = v \cdot \frac{d}{dx} \left( \frac{1}{x} \right) = v \cdot \left( -\frac{1}{x^2} \right) = -v \cdot u^2
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The singularity of the Airy curve reflects the essential singularity of \( Ai(x, \hbar) \) at infinity. Intersection numbers and their recursions appear after resolving the singularity of the Airy curve.
Let $K_{\mathbb{P}^1}$ be the sheaf of holomorphic 1-forms on $\mathbb{P}^1$. Since $u = \frac{1}{x}$, $du = -x^{-2}dx$ so $K_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$.

$$K_{\mathbb{P}^1}^\frac{1}{2} = \mathcal{O}_{\mathbb{P}^1}(-1) \text{ and } K_{\mathbb{P}^1}^{-\frac{1}{2}} = \mathcal{O}_{\mathbb{P}^1}(1).$$

A rank 2 vector bundle on a curve $C = \bigcup_\alpha U_\alpha$ is defined by holomorphic fn $f_{\alpha,\beta} : U_\alpha \cap U_\beta \to \text{Gl}_2(\mathbb{C})$ with $f_{\alpha,\beta} \cdot f_{\beta,\gamma} = f_{\alpha,\gamma}$.

Cocycle $f_{x,u} := \begin{pmatrix} x & 0 \\ 0 & \frac{1}{x} \end{pmatrix}$ defines vector bundle $E|_{\mathbb{P}^1} := K_{\mathbb{P}^1}^\frac{1}{2} \oplus K_{\mathbb{P}^1}^{-\frac{1}{2}}$.

Define a meromorphic Higgs field $\phi : E \to E \otimes K_{\mathbb{P}^1}(\ast)$ as $\phi_x := \begin{pmatrix} 0 & x \\ 1 & 0 \end{pmatrix} dx$. Then $\phi_u = f_{x,u} \cdot \phi_x \cdot f_{x,u}^{-1}$.

$\Sigma$, the spectral curve of $\phi$ inside $\overline{T^*\mathbb{P}^1} = \mathbb{F}_2$ is locally at the $(0,0)$ chart

$$\det((ydx)I_2 - \phi_x) = (y^2 - x)(dx)^2 = 0$$

and quintic cusp at $(\infty, \infty)$ chart: $(u^5 - w^2) \cdot (du)^2 = 0$.
Let $K_{\mathbb{P}^1}$ be the sheaf of holomorphic 1-forms on $\mathbb{P}^1$. Since $u = \frac{1}{x}$, $du = -x^{-2}dx$ so $K_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$. $K_{\mathbb{P}^1}^{\frac{1}{2}} = \mathcal{O}_{\mathbb{P}^1}(-1)$ and $K_{\mathbb{P}^1}^{-\frac{1}{2}} = \mathcal{O}_{\mathbb{P}^1}(1)$.

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The Airy differential equation \( \left( \hbar^2 \frac{d^2}{dx^2} - x \right) Ai(x, \hbar) = 0 \) corresponds to a linear system of ODE

\[
\hbar \nabla^\hbar \begin{bmatrix} Ai' \\ Ai \end{bmatrix} = 0
\]

where

\[
\nabla^\hbar = d - \frac{1}{\hbar} \begin{pmatrix} 0 & x \\ 1 & 0 \end{pmatrix} dx.
\]

Define vector bundle \( V_\hbar \) by transition functions \( g_{x,u} := \begin{pmatrix} x & \hbar_x \\ 0 & \frac{1}{x} \end{pmatrix} \).

Vector bundles \( V_\hbar \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \) for every \( \hbar \neq 0 \).

Then \( (\nabla^\hbar, V_\hbar) \) is a family of connections, namely

\[
\frac{1}{\hbar} \phi_u = g_{x,u} \cdot \left( \frac{1}{\hbar} \phi_x \right) \cdot g_{x,u}^{-1} - d g_{x,u} \cdot g_{x,u}^{-1}
\]
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\]
Motivation to study quantization process in moduli spaces

Spectral curve $\xrightarrow{Q} \text{Quantum curve}$

$y^2 - x = 0 \xrightarrow{Q} \left(\hbar^2 \frac{d^2}{dx^2} - x\right) Ai(x, \hbar) = 0.$

What is the geometry for the passage from the spectral curve to its quantization?

$Ai$ is generating function of quantum invariants:
- Gromov-Witten invariants
- Seiberg-Witten invariants
- Knot polynomials

$\Downarrow$

moduli space of spectral curves $\xrightarrow{Q}$ moduli space of $\mathcal{D}$-opers
Let $C$ be a smooth projective curve, $K_C$ canonical bundle, $E$ and $V$ be holomorphic rank $r$ vector bundle on $C$.

**Definition**

- A **holomorphic Higgs bundle** is a pair $(E, \phi)$ where $\phi : E \to E \otimes K_C$ is a $\mathcal{O}_C$-module homomorphism i.e. $\phi(sf) = f\phi(s)$, $\forall f \in \mathcal{O}_C, s \in E$.

- A **connection** is a pair $(V, \nabla)$, $\nabla : V \to V \otimes K_C$ is a $\mathbb{C}$-homomorphism s.t. $\nabla(fs) = df \cdot s + f \cdot \nabla(s)$, $\forall f \in \mathcal{O}_C, s \in V$.

**Example (rank two Higgs filed examples)**

$$(K_C^2 \oplus K_C^{-2}, \phi = \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix})$$

and $\phi : K_C^2 \oplus K_C^{-2} \to K_C^3 \oplus K_C^1$ for $q \in H^0(C, K_C^2)$.

- Locally on $C$, $\nabla|_U = d + A|_U$ where $A : V \to V \otimes K_C$.
- If $\{f_{\alpha\beta}\}, \{g_{\alpha\beta}\}$ are transition functions for $E$ and $V$ respectively then $\phi_\alpha = f_{\alpha\beta} \phi_\beta f_{\alpha\beta}^{-1}$ while $A_\alpha = g_{\alpha\beta} A_\beta g_{\alpha\beta}^{-1} - g_{\alpha\beta}^{-1} dg_{\alpha\beta}$.
Let $C$ be a smooth projective curve of arbitrary genus, $K_C$ canonical bundle. Let $E$ a holomorphic rank 2 vector bundle on $C$. For $\phi : E \to E \otimes K_C(\ast)$, the pair $(E, \phi)$ Higgs bundle of rank 2.

- In [DM ’13] $\phi$ is holomorphic, Hitchin constructed $\Sigma \hookrightarrow T^* C$

- In [DM ’14] $\phi$ is meromorphic

\[ \begin{array}{ccc}
\Sigma & \xrightarrow{i} & B\overline{T^* C} \\
\downarrow & & \downarrow \text{blow-up} \\
\Sigma & \xrightarrow{i} & \overline{T^* C}
\end{array} \]

**Theorem (D-Mulase ’13, ’14)**

For rank 2 Higgs bundle $(E, \phi)$, $x \in C$ and $\hbar \in \mathbb{C}$. Locally, we construct a function $\psi(x, \hbar)$ and a 2nd order differential operator $P(x, \hbar d/dx)$ whose semi-classical limit recovers $\Sigma$, s.t. $P(x, \hbar d/dx)\psi(x, \hbar) = 0$. 
Let $C$ be a projective algebraic curve of arbitrary genus, $K_C$ canonical bundle. Let $E$ a holomorphic rank 2 vector bundle on $C$. For $\phi : E \to E \otimes K_C(\ast)$, the pair $(E, \phi)$ Higgs bundle of rank 2.

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  $\Sigma \xrightarrow{i} B\overline{\Omega T^* C}$

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  $\Sigma \xrightarrow{i} \overline{T^* C}$

- Gromov-Witten theory $\xrightarrow{T.R}$ Hitchin Theory

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Let $C$ be a projective algebraic curve of arbitrary genus, $K_C$ canonical bundle. Let $E$ a holomorphic rank 2 vector bundle on $C$. For $\phi : E \to E \otimes K_C(\ast)$, the pair $(E, \phi)$ Higgs bundle of rank 2.

- In [DM ’13] $\phi$ is holomorphic, Hitchin constructed $\Sigma \hookrightarrow T^* C$
  - Limit oper of Gaiotto’s conjecture
    
    $$\tilde{\Sigma} \xrightarrow{i} B\text{IT}^* C$$

- In [DM ’14] $\phi$ is meromorphic
  
  $$\Sigma \xrightarrow{i} \text{IT}^* C$$
  
  - Gromov-Witten theory $\xrightarrow{T.R.} \text{Hitchin Theory}$

**Theorem (D-Mulase ’13, ’14)**

For rank 2 Higgs bundle $(E, \phi)$, $x \in C$ and $\hbar \in \mathbb{C}$. **Locally**, we construct a function $\psi(x, \hbar)$ and a 2nd order differential operator $P(x, \hbar \text{d}/\text{d}x)$ whose semi-classical limit recovers $\Sigma$, s.t. $P(x, \hbar \text{d}/\text{d}x)\psi(x, \hbar) = 0$. 
Airy curve \( \equiv \) local spectral curve of a Higgs bundle

- The curve \( C = \mathbb{P}^1 \), the vector bundle \( E = K_{\mathbb{P}^1}^{\frac{1}{2}} \oplus K_{\mathbb{P}^1}^{-\frac{1}{2}} \)
- The meromorphic Higgs field \( \phi : E \to E \otimes K_C(\ast) \) is locally
  \[
  \phi = \begin{pmatrix}
  0 & x(dx)^2 \\
  1 & 0
  \end{pmatrix}
  \]

\( \Sigma \), the spectral curve of \( \phi \) inside \( \overline{T^*\mathbb{P}^1} = \mathbb{F}_2 \) is locally at the \((0, 0)\) chart

\[
det(\phi - (ydx)I_2) = (y^2 - x)(dx)^2 = 0
\]

with a quintic cusp at \((\infty, \infty)\) chart: \( u^2 = w^5 \)

- The spectral curve \( \Sigma \to C = \mathbb{P}^1 \) double cover.
- Take a resolution of singularity of curve \( \Sigma \) by blowing up \( \mathbb{F}_2 \). The proper transform \( \tilde{\Sigma} \) of \( \Sigma \) becomes a rational curve.
Airy curve \equiv local spectral curve of a Higgs bundle

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The methamorphosis of quantum curves into opers

Projective coordinate system

Let $C$ be a Riemann surface of genus at least two. Its universal covering is the upper half-plane

$$
\mathbb{H} = \{z \in \mathbb{C} | \text{Im} \, z > 0\}
$$

The global coordinate on $\mathbb{H}$ induces by the quotient map $\mathbb{H} \rightarrow C$ a particular coordinate system on the Riemann surface $C$.

Definition (Gunning 1967)

A projective coordinate system on $C$ is a coordinate system on which transition function is given by Möbius transformation.

$$
C = \bigcup_{\alpha} U_\alpha, \quad z_\alpha \in U_\alpha, \quad z_\alpha = \frac{a_{\alpha\beta}z_\beta + b_{\alpha\beta}}{c_{\alpha\beta}z_\beta + d_{\alpha\beta}}, \quad \begin{bmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{bmatrix} \in \text{SL}(2, \mathbb{C}).
$$

Note $dz_\alpha = \frac{dz_\beta}{(c_{\alpha\beta}z_\beta + d_{\alpha\beta})^2}$ so $K_\frac{\alpha}{C}^\frac{1}{2}$ is given by $\zeta_{\alpha\beta} = \pm (c_{\alpha\beta}z_\beta + d_{\alpha\beta})$. 
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Note $dz_{\alpha} = \frac{dz_{\beta}}{(c_{\alpha\beta}z_{\beta} + d_{\alpha\beta})^2}$ so $K_C^{\frac{1}{2}}$ is given by $\zeta_{\alpha\beta} = \pm (c_{\alpha\beta}z_{\beta} + d_{\alpha\beta})$. 
**Importance of Gunning’s definition:**

On a projective coordinate system of \( C \), *our quantum curve* is globally defined!

**Definition (intuitive)**

An oper, \( \nabla^{\text{oper}} \), is a globally defined differential operator.

The quantum curve \( P(x, \hbar d/dx)|_{\hbar=1} \) is an oper. Interpret globally

\[
P(x, \hbar d/dx) \psi = 0 \quad \text{as} \quad \nabla^{\hbar} \begin{bmatrix} \psi' \\ \psi \end{bmatrix} = 0
\]

**Example:** The quantum curve of the Higgs field \( E = K^{1/2}_{\mathbb{P}^1} \oplus K^{-1/2}_{\mathbb{P}^1} \),

\[
\phi = \begin{pmatrix} 0 & x(dx)^2 \\ 1 & 0 \end{pmatrix}
\]

is \( (\hbar^2 \frac{d^2}{dx^2} - x) \) \( Ai(x, \hbar) = 0 \). It corresponds to

\[
\nabla^{\hbar} \begin{bmatrix} Ai' \\ Ai \end{bmatrix} = 0, \quad \text{where} \quad \nabla^{\hbar} = d - \frac{1}{\hbar} \begin{pmatrix} 0 & x(dx) \\ dx & 0 \end{pmatrix}
\]

on \( V_{\hbar} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \).
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What are the corresponding vector bundles for the family of connections $(?, \nabla^{\hbar})$?

**Example:** The quantum curve of the Higgs field $E = K_{\mathbb{P}^1}^{1/2} \oplus K_{\mathbb{P}^1}^{-1/2}$, 

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where $\nabla^{\hbar} = d - \frac{1}{\hbar} \begin{pmatrix} 0 & x(dx) \\ dx & 0 \end{pmatrix}$ on $V_{\hbar} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$. 
Interpret $\hbar \in \mathbb{C} = \text{Ext}^1(K_C^{-\frac{1}{2}}, K_C^{\frac{1}{2}})$. Planck constant $=$ extension class!

**Theorem (Gunning)**

*For any $\hbar \in \mathbb{C}$, $\exists!$ rank 2 vector bundle, $V_{\hbar}$, such that*

$$0 \to K_C^{\frac{1}{2}} \to V_{\hbar} \to K_C^{-\frac{1}{2}} \to 0$$

- **proof:** $V_{\hbar}$ is given by transition fn $\{f_{\alpha\beta}^{\hbar}\}$, $f_{\alpha\beta}^{\hbar} := \left(\begin{array}{c}
\zeta_{\alpha\beta} \\
\hbar \cdot \frac{d\zeta_{\alpha\beta}}{dz_\beta}
\end{array}\right)$

- $V_0 \cong K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}}$

- For $\hbar \neq 0$ the vector bundles $V_{\hbar}$ and *their complex structure* are isomorphic. Denote $V := V_{\hbar}|_{\hbar = 1}$.

- **Gunning:** $H^0(C, K_C^2) \cong \{\text{moduli space of pairs } (V, \nabla^{\text{oper}})\}$

- $0 \leftrightarrow \nabla^{\text{unif}} = \text{origin}$

- $(K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}}, \begin{bmatrix} 0 & P(x)(dx)^2 \\ 1 & 0 \end{bmatrix}) \overset{Q}{\to} (V_{\hbar}, d - \frac{1}{\hbar} \begin{bmatrix} 0 & P(x)dx \\ dx & 0 \end{bmatrix})$

Our quantum curve $\hbar \nabla^{\hbar}$ is an $\hbar$ – deformation family
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Our quantum curve $\hbar \nabla^\hbar$ is an $\hbar$ – deformation family
\( E, V \) be a holomorphic vector bundles of rank \( r \) of degree 0  
\( \phi : E \to E \otimes K_C \) a traceless holomorphic Higgs field.  
\( \nabla : V \to V \otimes K_C \) an irreducible holomorphic connection.  
\[\mathcal{M}_{\text{Dol}} := \{ \text{moduli space of rank } r \text{ stable Higgs bundles } (E, \phi) \}\]
\[\downarrow \text{[Hitchin-Simpson]}\]
\[\mathcal{M}_{\text{deR}} := \{ \text{moduli space of rank } r \text{ irreducible connections } (V, \nabla) \}\]
\[\downarrow \text{[Riemann-Hilbert]}\]
\[\mathcal{M}_B := \text{Hom}(\pi_1(C), G) \sslash G, G = SL_r(\mathbb{C})\]

Let \( \eta \) tautological 1-form on \( T^*C \). The Spectral curve of \( \phi \) is
\[\text{det}(\eta \cdot I_r - \phi) = 0 \subset T^*C\]

\( \mathcal{M}_{\text{Dol}} \) is a fibration of abelian varieties, by the Hitchin map
\[\mathcal{M}_{\text{Dol}} \ni (E, \Phi)\]
\[\downarrow \text{H}\]
\[B := \bigoplus_{i=2}^{r} H^0(C, K_C^i) \ni ((-1)^i \text{tr}(\wedge^i \phi))\]
• $E, V$ be a holomorphic vector bundles of rank $r$ of degree 0
• $\phi : E \rightarrow E \otimes K_C$ a traceless holomorphic Higgs field.
• $\nabla : V \rightarrow V \otimes K_C$ an irreducible holomorphic connection.

$M_{Dol} := \{\text{moduli space of rank } r \text{ stable Higgs bundles } (E, \phi)\}$

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Hitchin theory in rank 1 and $G = GL_1(\mathbb{C}) = \mathbb{C}^*$

\[
\begin{array}{ccccccc}
0 & 0 \\
\downarrow & \downarrow \\
0 & \mathbb{Z} & \mathbb{Z} & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathbb{C} & \mathcal{O}_C & \xrightarrow{d} & K_C & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathbb{C}^* & \mathcal{O}_C^* & \xrightarrow{d \log} & K_C & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 \\
\end{array}
\]

gives the cohomology exact sequence
Hitchin theory in rank 1 and $G = GL_1(\mathbb{C}) = \mathbb{C}^*$

\[
\begin{array}{ccccccc}
H^0(\mathcal{C}, \mathcal{K}_C) & \xrightarrow{\gamma} & H^1(\mathcal{C}, \mathbb{C}^*) & \xrightarrow{u} & H^1(\mathcal{C}, \mathcal{O}_C^*) & \xrightarrow{d \log} & H^1(\mathcal{C}, \mathcal{K}_C) \\
\downarrow & & \downarrow & & \downarrow c_1 & & \downarrow \\
0 & \longrightarrow & H^2(\mathcal{C}, \mathbb{Z}) & \xrightarrow{=} & H^2(\mathcal{C}, \mathbb{Z}) & \longrightarrow & 0 \\
\end{array}
\]

$H^1(\mathcal{C}, \mathbb{C}^*) = \{(\mathcal{L}, \nabla)\} = \mathcal{M}_{\text{deR}}$ (Deligne, Bloch)

$H^1(\mathcal{C}, \mathcal{O}_C^*) = \{\mathcal{L}\}$ moduli space of all line bundles

Holomorphic connections in a line bundle

tells us that $\{\sigma_{\alpha\beta}\}$, corresponding to the image of $\{\zeta_{\alpha\beta}\}$ via the $d \log$ map, gives a nontrivial extension class.

\[
H^1(\mathcal{C}, \mathcal{O}_C^*) \xrightarrow{d \log} H^1(\mathcal{C}, \mathcal{K}_C).
\]
Riemann Hilbert correspondence in rank 1 and $GL_1(\mathbb{C}) = \mathbb{C}^*$

Moreover,

$$M_B = \text{Hom}(\pi_1(C), GL_1(\mathbb{C})) \sslash GL_1(\mathbb{C}) = \text{Hom}\left(\frac{\pi_1(C)}{[\pi_1(C), \pi_1(C)]}, \mathbb{C}^*\right)$$

$$= \text{Hom}_\mathbb{Z}(H_1(C, \mathbb{Z}), \mathbb{C}^*)$$

(3)

Indeed, Riemann Hilbert correspondence is the analytic isomorphism between $M_{\text{deR}} \cong M_B$. Further, Universal Coefficient Theorem gives

$$H^1(C, \mathbb{C}^*) = \text{Hom}_\mathbb{Z}(H_1(C, \mathbb{Z}), \mathbb{C}^*) \oplus \text{Ext}^1(H_0(C, \mathbb{Z}), \mathbb{C}^*)$$

Since $H_0(C, \mathbb{Z}) = \mathbb{Z}$ then $\text{Ext}^1(H_0(C, \mathbb{Z}), \mathbb{C}^*) = \text{Ext}^1(\mathbb{Z}, \mathbb{C}^*) = 0$. 

NAH correspondence in rank 1

Note that rank 1 Higgs bundles are always stable since line bundles are always stable. Therefore for $\forall \mathcal{L} \in H^1(C, \mathcal{O}_C^*)$ and $\forall$  
\phi \in H^0(C, K_C) \iff \phi : \mathcal{O}_C \to K_C \iff \phi : \mathcal{L} \to \mathcal{L} \otimes K_C$, then 
$(\phi, \mathcal{L}) \in \mathcal{M}_{Dol}$. Moreover, 

$$\phi \in H^0(C, K_C) \cong H^1(C, \mathcal{O}_C)^* = H^1(C, \text{End}\mathcal{L})^* \cong T^*_L \text{Jac}C.$$  

Therefore 

$$\mathcal{M}_{Dol} = T^* \text{Jac}C = \text{Jac}C \times \mathbb{C}^g.$$  

The $g$ dimensional complex torus 

$$\text{Jac}C = \mathbb{C}^g / \mathbb{Z}^{2g} = (\mathbb{R}/\mathbb{Z})^{2g} = (S^1)^{2g}.$$  

We obtain the diffeomorphism 

$$\mathcal{M}_{Dol} = (S^1)^{2g} \times \mathbb{R}^{2g} \cong (S^1 \times \mathbb{R}_+)^{2g} \cong (\mathbb{C}^*)^{2g} = H^1(C, \mathbb{C}^*) = \mathcal{M}_{deR}.$$
Gaiotto’s correspondence in rank 1 and $GL_1(\mathbb{C}) = \mathbb{C}^*$

We have

$$\mathcal{M}_{Dol} = T^* \text{Jac} C$$

Interpret $H^0(C, K_C)$ as a section of $T^* \text{Jac} C \to H^0(C, K_C)$. Then

$$H^0(C, K_C) \xrightarrow{\gamma} \mathcal{M}_{deR} = H^1(C, \mathbb{C}^*) \xrightarrow{u} H^1(C, \mathcal{O}_C^*)$$

Then an $GL_1(\mathbb{C})$-oper is $\text{Image}(\gamma) = \text{Ker}(u) = u^{-1}(\mathcal{O}_C)$

Then Gaiotto’s correspondence in rank 1 is trivial

$$\gamma(\phi) = (\mathcal{O}_C, \nabla := d - \phi)$$

where $\phi \in H^0(C, K_C)$. 
Hitchin section in rank two

- Fix a spin structure for $C$, a line bundle $\mathcal{L}$ for which $\mathcal{L} \otimes 2 \cong K_C$.
- Let $\zeta_{\alpha\beta}$ be the transition functions for the line bundle $\mathcal{L}$.

Hitchin map is just $(E, \phi) \xrightarrow{H} \text{det}(\phi)$. Let $q \in B = H^0(C, K_C^2)$.

The Hitchin section is $s(q) = (E_0 := K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}}, \phi(q) := \begin{bmatrix} 0 & q \\ 1 & 0 \end{bmatrix})$
Hitchin section (principal $sl_2(\mathbb{C})$)

Let $q := (q_1, \ldots, q_{r-1}) \in B = \bigoplus_{i=1}^{r-1} H^0(C, K_C^{i+1})$. Denote $p_i := i(r - i)$.

\[ X_+ := \begin{bmatrix}
0 & \sqrt{p_1} & 0 & \cdots & 0 \\
0 & 0 & \sqrt{p_2} & \cdots & 0 \\
0 & 0 & 0 & \cdots & \sqrt{p_{r-1}} \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}, \quad X_- := X_+^t, \]

\[ H := \begin{bmatrix}
r - 1 & 0 & \cdots & 0 & 0 \\
0 & r - 3 & \cdots & 0 & 0 \\
0 & 0 & \cdots & -(r - 3) & 0 \\
0 & 0 & \cdots & 0 & -(r - 1)
\end{bmatrix}. \]

Let $E_0 := K_C^{r-1} \oplus K_C^{-r-1} \oplus \cdots \oplus K_C^{-r-1}$ with transition functions $\{\zeta^H_{\alpha\beta} = \exp(H \cdot \log(\zeta_{\alpha\beta}))\}$.

Hitchin section

For any $q \in B$, the Higgs pair $(E_0, \phi(q) := X_- + \sum_{i=1}^{r-1} q_i X_+^i) \in \mathcal{M}_{Dol}$
A stable holomorphic Higgs bundle \((E, \phi)\) corresponds to \((D, \phi, h)\)

- \(h\) is a **hermitian metric**, \(D\) an **\(h\)-unitary connection**

\[
D = D^{(1,0)} + D^{(0,1)}
\]

- \(F_D\) is the curvature, \(2F_D = [D, D]\)
- \(\phi^\dagger h\) is the adjoint of \(\phi\) with respect to \(h\)

satisfying Hitchin’s equations, a nonlinear system of PDE

\[
F_D + [\phi, \phi^\dagger h] = 0 \quad (4)
\]
\[
\bar{\partial}_D \phi = 0 \quad (5)
\]

This is equivalent to the flatness of a family of connections, \(\zeta \in \mathbb{C}^*\)

\[
D(\zeta) := \frac{1}{\zeta} \phi + D + \zeta \phi^\dagger h
\]

\((E, \phi) \xrightarrow{N_{AH}} (V, \nabla := D(1)^{(1,0)}), \ V = (E^{\text{top}}, D(1)^{(0,1)})\)
A stable holomorphic Higgs bundle \((E, \phi)\) corresponds to \((D, \phi, h)\)

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\]

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\]
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\[ F_D + [\phi, \phi^\dagger h] = 0 \]  \hspace{1cm} (4)

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\((E, \phi) \xrightarrow{\text{NAH}} (V, \nabla := D(1)^{(1,0)}), \ V = (E^{\text{top}}, D(1)^{(0,1)})\)
Gaiotto’s conjecture
Take \((E_0, \phi(q))\) on the Hitchin section and twist by \(R \in \mathbb{R}_+\), 
\((E_0, R\phi(q)) \in \mathcal{M}_{Dol}

\[
F_D + R^2 [\phi, \phi^\dagger_h] = 0, \tag{6}
\]
\[
\bar{\partial}_D \phi = 0. \tag{7}
\]

where \(R \in \mathbb{R}_+.\) The solution to Hitchin’s equations corresponds to
family of flat connections on a topologically trivial bundle, \(\zeta \in \mathbb{C}^*\) and 
\(R > 0.\)

\[
D(\zeta, R) := \zeta^{-1} R\phi + D + \zeta R\phi^\dagger.
\]

D. Gaiotto predicted in 2014
For a Higgs bundle on a Hitchin section \(\lim_{R, \zeta \to 0, \frac{\zeta}{R} = \hbar} D(\zeta, R) \) exists
and is an oper.

We recall: in rank 2 oper is \((V, \nabla)\) with \(0 \to K_C^{\frac{1}{2}} \to V \to K_C^{-\frac{1}{2}} \to 0.\)
Gaiotto’s conjecture

Take $(E_0, \phi(q))$ on the Hitchin section and twist by $R \in \mathbb{R}_+$,
$(E_0, R\phi(q)) \in \mathcal{M}_{Dol}$

\begin{align*}
F_D + R^2[\phi, \phi^\dagger h] &= 0, \quad (6) \\
\bar{\partial}_D \phi &= 0. \quad (7)
\end{align*}

where $R \in \mathbb{R}_+$. The solution to Hitchin’s equations corresponds to a family of flat connections on a topologically trivial bundle, $\zeta \in \mathbb{C}^*$ and $R > 0$.

$$D(\zeta, R) := \zeta^{-1} R\phi + D + \zeta R\phi^\dagger.$$ 

D. Gaiotto predicted in 2014

For a Higgs bundle on a Hitchin section $\lim_{R, \zeta \to 0, \frac{\zeta}{R} = \hbar} D(\zeta, R)$ exists and is an oper.
Gaiotto’s conjecture

Take \((E_0, \phi(q))\) on the Hitchin section and twist by \(R \in \mathbb{R}_+\),
\((E_0, R\phi(q)) \in \mathcal{M}_{Dol}\)

\[
F_D + R^2 [\phi, \phi^\dagger] = 0, \tag{6}
\]
\[
\bar{\partial}_D \phi = 0. \tag{7}
\]

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For a Higgs bundle on a Hitchin section \(\lim_{R, \zeta \to 0, \zeta \cdot h = \hbar} D(\zeta, R)\) exists and is an oper.

We recall: in rank 2 oper is \((V, \nabla)\) with \(0 \to K_C^{1/2} \to V \to K_C^{-1/2} \to 0.\)
Definition (Beilinson-Drinfeld, 1993)

Let $V$ be a holomorphic vector bundle of rank $r$. An $SL_r(\mathbb{C})$-oper is a pair $(V, \nabla)$ with a filtration $st$

1. $0 = F_r \hookrightarrow F_{r-1} \hookrightarrow \ldots \hookrightarrow F_0 = V$
2. $\nabla|_{F_{i+1}} : F_{i+1} \to F_i \otimes K_C$ Griffiths transversality.
3. $F_{i+1}/F_{i+2} \cong F_i/F_{i+1} \otimes K_C$ an $\mathcal{O}_C$ linear isomorphism.

Theorem (D-Mulase 2017)

For every $\hbar \in H^1(C, K_C)$,

- There is a unique filtration and vector bundle $V_{\hbar}$ with $F_{r-1} = K_C^{r-1} \otimes 2$.
- $V_{\hbar}$ is given by $\{f^\hbar_{\alpha\beta}\}$ where $f^\hbar_{\alpha\beta} = \exp(H \cdot \log \zeta_{\alpha\beta}) \exp(\hbar \frac{d\log \zeta_{\alpha\beta}}{dz_\beta} \cdot X_+)$
- $V_0 = K_C^{r-1} \otimes 2 \oplus \ldots \oplus K_C^{r-1} \otimes 2$, since $f^\hbar_{\alpha\beta} = 0 = \exp(H \cdot \log \zeta_{\alpha\beta}) = \zeta^H_{\alpha\beta}$.
- For $\hbar \neq 0$ the vector bundles $V_{\hbar}$ and their complex structures are isomorphic. Let $V_1 := V_{\hbar}|_{\hbar=1}$.
- Let $\nabla^\hbar_{\alpha}|_{u_{\alpha}} := d + \frac{1}{\hbar} \phi(q)|_{u_{\alpha}}$. Then $(V_{\hbar}, \nabla^\hbar)$ is a family of opers.
Gaiotto’s conjecture holds

Theorem [D, Fredrickson, Kydonakis, Mazzeo, Mulase, Neitzke]

For an arbitrary simple and simply connected Lie group $G$
\[
\lim_{R, \zeta \to 0, R = h} D(\zeta, R) \text{ exists and is indeed an oper. For } G = \text{Sl}_n(\mathbb{C}) \text{ then it is equal to } (V_h, \nabla^h).
\]

for rank 2

The limit oper is $d + \frac{1}{h} \phi(q)$ i.e. DM quantum curve of Topological Recursion.

- [Hitchin, Simpson] Nonabelian-Hodge correspondence

rank 1 case:

\[ T^* \text{JacC} \cong (\mathbb{C}^*)^{2g}. \]

- [Gaiotto’s conjecture] holomorphic corresp. between Lagrangians

Hitchin section $\longrightarrow$ moduli space of opers
Gaiotto’s conjecture holds

Theorem [D, Fredrickson, Kydonakis, Mazzeo, Mulase, Neitzke]

For an arbitrary simple and simply connected Lie group $G$

$$\lim_{R, \zeta \to 0, \frac{\zeta}{R} = \hbar} D(\zeta, R)$$

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- [Hitchin, Simpson] Nonabelian-Hodge correspondence

$$\mathcal{M}_{Dol} \text{ diffeomorphism } \mathcal{M}_{deR}$$

rank 1 case: $T^* \text{Jac} C \cong (\mathbb{C}^*)^{2g}$.

- [Gaiotto’s conjecture] holomorphic corresp. between Lagrangians

Hitchin section $\text{holomorphic}$ moduli space of opers
sketch of the proof in rank two

- Let \( g = \lambda^2 \cdot dz \cdot d\bar{z} \) a Riemannian metric on \( C \).
- The hermitian metric on the vector bundle of the Hitchin component \( V_0 = K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}} \) is given by \( h = \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{bmatrix} \).
- the Chern connection \( D = d - \partial \log \lambda \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \).
- If \( \phi = \begin{bmatrix} 0 & P \\ 1 & 0 \end{bmatrix} \) then \( \phi^* h = h \cdot \phi^t \cdot h^{-1} = \begin{bmatrix} 0 & \lambda^2 \\ \lambda^{-2} \overline{P} & 0 \end{bmatrix} d\bar{z} \).
- The flatness condition of \( D(\zeta, R) \) gives \( \partial \bar{\partial} \log \lambda - R^2 (\lambda^{-2} \overline{P} \overline{P} - \lambda^2) = 0 \).
- For \( P = 0 \), ie \( \phi = X_- \), solving the harmonicity equation for \( \lambda \) one obtains on \( \mathbb{H} \)
  \( \lambda_0 = \frac{1}{R} \cdot \frac{i}{z - \overline{z}} \).
- In general \( \lambda = \lambda_0 \cdot e^{f(R)} \) analysis proves \( f \) is real analytic with \( f(R) = R^4 + \text{HOT} \).
Non-Abelian Hodge Gaiotto Correspondence = Canonical Biholomorphic Map

Hitchin Section

(E₀, X⁻)

Stable Bundles

Moduli of Higgs Bundles

quantization

semi-classical limit

Oper Moduli

(V₁, d + X⁻)

Narasimhan Seshadri

SUᵣ

SLᵣ(R)

Hitchin Component

Moduli of Holomorphic Connections

Non-Abelian Hodge

Diffeomorphism

(C, K_C¹/²) Chosen

Process of quantization = construction of ℏ deformed \(\mathcal{D}\)-modules through opers.
Thank you for your attention!