The quantum Johnson homomorphism, and the symplectic mapping class group of 3-folds

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Introduction and context

Statement of the main result
(Naive) construction of the $S^1$-family
The monodromy has infinite order

Idea and proof (sketch)
Motivation from MCG
The technical definition
The factorization (a non-naive construction)
Let $(M, \omega, J)$ be a Kähler manifold.

- **Gromov-Witten invariants** count the number of isolated holomorphic curves in a given homology class, subject to some generic point constraints.

- **Examples:**
  - How many conics ($d = 2$) pass through five generic points? 1.
  - How many lines ($d = 1$) on a cubic surface? 27.

- The 3-point Gromov-Witten invariants serve as structure coefficients in **quantum cohomology algebra** $QH^\bullet(M)$. 
Reminder: Pearly $A_\infty$-structure

Theorem (K. Fukaya ’95, Biran-Cornea ’07, FOOO ’09,...)
There is an $A_\infty$-refinement

$$(A, \mu^1 = 0, \mu^2 = \ast, \mu^3, \ldots)$$

of the quantum cohomology algebra

$$A := (QH^\bullet(M, \omega), \ast).$$

where $\mu^d$ is counting ”pearls” - holomorphic curves connected by Morse flow lines:
Given a MS pair $(f, g)$, the Morse complex is generated by critical points and has a differential $\mu^1$ and product $\mu^2$.

There are higher order multilinear operations defined by solving the perturbed gradient equation on families of metric trees.

Figure: $\mu^3(p_3, p_2, p_1) = \ldots + \lambda_{p_0} \cdot p_0 + \ldots$
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\begin{align*}
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The symplectic mapping class group

Let $(M, \omega)$ be a closed symplectic manifold.

- We denote by

$$\text{Symp}(M, \omega) \subset \text{Diff}^+(M)$$

the group of all $\phi : M \to M$ such that $\phi^*\omega = \omega$.

- The symplectic isotopy problem: understanding the kernel

$$\pi_0\text{Symp}(M, \omega) \to \pi_0\text{Diff}^+(M).$$

- Any symplectomorphism whose isotopy class is in the kernel would be called exotic.
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One of Gromov's original applications for J-curves was to understand $\pi_0\text{Symp}(M, \omega)$ for ruled surfaces.

A symplectic manifold $(M, \omega)$ is **monotone** if:

\[ c_1(M) = \kappa \cdot [\omega], \quad 0 < \kappa \in \mathbb{R}. \]

**Gromov + Taubes (1985):**

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\text{Symp}(S^2 \times S^2, \omega_{\text{mono}}) = (SO(3) \times SO(3)) \ltimes \mathbb{Z}_2, \\
\text{Symp}(Bl_{pt}\mathbb{C}P^2, \omega_{\text{mono}}) = U(2).
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...
Seidel: for every other 4dim complete intersection $M$, there is an exotic symplectomorphism $\tau^2_V$.

1. geodesic flow
2. normalized geodesic flow
3. cut-off away from the zero section

Figure: Model dehn twist in $T^*S^n$ when $n = 1$
J-curves don’t behave as nicely (automatic transversality, positivity of intersections ...) in higher dimensions. As a result, very little is known. Some previous results:

- **Seidel (1998):** Proved the existence of certain $\pi_k$ ($k > 1$ odd) exotic classes for $M = \mathbb{CP}^n \times \mathbb{CP}^m$ in a range.

- **Ivan Smith (2010):** Studied $Q_0 \cap Q_1 \subset \mathbb{CP}^5$, the intersection of two quadrics.

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The formal package: quantum Johnson homomorphism

- Let $X$ be a closed, monotone, symplectic manifold.
- We associate to every symplectomorphism $\phi : X \to X$ an algebraic "mapping torus"

$$\phi \mapsto \mathcal{X}_\phi := QH^\bullet(X_\phi)$$

and to every symplectic isotopy $\phi_t$, a quasi-isomorphism

$$\mathcal{X}_{\phi_0} \to \mathcal{X}_{\phi_1}.$$

- Given a homology class $A \in H^S_2(X)$, for suitable $\phi$, we assign a "characteristic class"

$$q_2 : \phi \mapsto \mathcal{X}_\phi \mapsto o^{3,A}_\phi \in HH^\bullet(...)$$

which is: Natural w.r.t to quasi-isomorphism; vanishes for $\phi = id$, and can "evaluated" on suitable cohomology classes to give quantum Massey products.
Smoothing a nodal curve

Let $C_0$ be a curve of arithmetic genus $g = 4$ with a single node, which separates $C_0$ into two irreducible components of genus 2.

- There is a Lefschetz fibration over the unit disc

$$C \to \Delta$$

such that:

1. The special fiber is the nodal curve $C_0$.
2. The generic fibers are smooth, non-hyperelliptic curves of genus $g = 4$.
3. The monodromy $\psi_C$ is symplectically isotopic to a model Dehn twist around a genus 2 vanishing cycle $\gamma$.

- There is a relatively very ample line bundle (i.e., the relative canonical bundle) over the punctured unit disc

$$\omega_C/\Delta^*.$$
The sections of $\omega_{C/\Delta^*}$ embed $C \hookrightarrow \mathbb{P}(\omega_{C/\Delta^*}) \cong \mathbb{P}^3 \times \Delta^*$.

Each curve is the complete intersection of a smooth quadric and a cubic.

We can perform a (complex-analytic) blowup on the family of smooth embedded curves $C \rightarrow \Delta^*$ and get a new family $\mathcal{X} \rightarrow \Delta^*$, with fibers $X_s = Bl_{C_s} \mathbb{P}^3$.

The further restriction of the base to $S^1 \subset \Delta^*$ is a 7-dim 
Locally Hamiltonian fibration.

Thus, the monodromy obtained by parallel transport around the circle

$$\phi : X \rightarrow X$$

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Family of Fano 3-folds

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is a symplectomorphism.
Pick a base point on $* \in S^1$ and denote $X := X_*$. 

- **Main result (B.)** The symplectic isotopy class $\kappa = [\phi]$ has infinite order in the kernel 

$$
\pi_0\text{Symp}(X, \omega_{\text{mono}}) \to \pi_0\text{Diff}^+(X)
$$

- In fact, there exists Lagrangian spheres $\{V'_1, \cdots, V'_5\}$, and $\{V''_1, \cdots, V''_5\}$ in $X$ such that $\kappa$ has a factorization in the symplectic mapping class group as the product of generalized Dehn twists about them!

- Note: Unlike 4dim, Dehn twist in odd dimensions can have infinite order in $\pi_0\text{Diff}^+(X)$!
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A classical question

Let $C$ be a genus 4 curve. How can we prove that the *seperating* Dehn twist $\psi = \tau_\gamma$ is not isotopic to the identity?

(a) A tubular neighbourhood

$$A \cong N_\gamma \hookrightarrow C$$

(b) Effect of model Dehn twist

$$(s, t) \mapsto (se^{it}, t)$$
How can we prove that the separating Dehn twist $\psi = \tau_\gamma$ is not isotopic to the identity?

- Consider the mapping torus

$$C_\psi := \mathbb{R} \times C / \sim, \ (t, x) \sim (t - 1, \psi(x)).$$

- Fact: on the level of cohomology algebras

$$H^\bullet(C_\psi) \cong H^\bullet(C \times S^1) \cong Q[t]/(t^2),$$

where $Q^\bullet = H^\bullet(C)$ and $t$ is a formal parameter of degree 1.

- D. Johnson (1980): the triple Massey products

$$Q^1 \otimes Q^1 \otimes Q^1 \to t \cdot Q^1 \subset H^2(C_\psi)$$

are non-trivial!
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Non-formality of $C_\psi$ ... or picture proof for $\tau_2(\psi) \neq 0$

The negative gradient flow of a generic point in $C_+$ passes once through every point in $C_-$.

(a) "Rotating cap" has no influence  
(b) New differentials cancel out  
(c) ...A new $\mu^3$ appears

Claim

$$\langle \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b_1 & 0 \\ -b_2 & 0 \end{pmatrix}, \begin{pmatrix} b_1 & b_3 \\ 0 & 0 \end{pmatrix} \rangle = \begin{pmatrix} 0 & tb_3 \\ 0 & 0 \end{pmatrix}$$

is a nontrivial coset (hence $\mathcal{C}$ is non-formal).
Let $Y, X$ be closed symplectic manifolds, and $Y \hookrightarrow X$ a symplectic embedding of codimension $2r$.

- **RTBT theorem (1999).** Assume that the codimension $2r$ is $\geq 8$. Then $Y$ has a nontrivial Massey product $\Rightarrow Bl_Y X$ also has a nontrivial Massey product.

- This was used in the construction of some early examples of non-Kähler symplectic manifolds.

- Contrast with **DGMS (1975):** Let $X$ be a compact Kähler manifold. Then the de-Rham algebra $(\Omega^\bullet(X), d)$ is formal.

- If $X$ is formal then all Massey products in $X$ must vanish.
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Where does “exotic” come from?

What happens after the blowup?

► Unfortunately, all the information stored in the triple Massey product is lost (to classical topology).

► The cohomology of the blowup is (additively)

\[ H^*(Bl_C\mathbb{P}^3) = H^*(\mathbb{P}^3) \oplus H^*(C)[u] \]

where \(|u| = 2\) and \(u^3 = 0\), and

\[ H^3 \otimes H^3 \otimes H^3 \to H^{3+3+3-1} = H^8 = 0. \]

► It’s not a bug - it’s a feature! The isotopy class of the monodromy \(\phi\) is controlled by topological data (rational homotopy + pontryagin classes) up to ”finite ambiguity”.

► Idea. Unlike most cohomology theories, quantum cohomology is not nilpotent, e.g., \(QH^\bullet(\mathbb{C}\mathbb{P}^n) = \mathbb{Z}[h, q]/(h^{n+1} = q)\).
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To define quantum Massey products we need a parametrized version of the quantum $A_\infty$-structure.

- We consider the symplectic fibration $X_\phi \to S^1$.
- Morse trajectories are allowed to move in the total space $X_\phi$.
- $J$-Holomorphic curves live in the fibers $(X_s, \omega_s)$. 

The algebraic mapping torus
The algebraic mapping torus

- A multiple cover type-problem

- Solution: need to let perturbations depend on the simultaneous position of all pearls.
Assume that there is an associative product structure $A = (\mathcal{A}, \cdot)$.

**Definition**

Let $\mathcal{U}(A)$ be the set of $A_\infty$-structures $\mu$ on the underlying vector space of $\mathcal{A}$ with $\mu^1 = 0$ and $\mu^2(x_2, x_1) = (-1)^{|x_1|} x_2 \cdot x_1$. There is a group of *gauge transformations*

$$\mathcal{G}(A) := \left\{ G^1 = id, G^2 : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}[-1], \ldots \right\}$$

which acts on $\mathcal{U}(A)$.

**Idea.** Think of $\mu$ as a *deformation* of the **formal** $A_\infty$-structure $(0, \cdot, 0, 0, \ldots)$. 
The universal Massey product

The pearl complex

Hochschild cohomology groups

The formal structure

Figure: The space of minimal $A_\infty$-structures $\mathcal{U}(A)/\mathcal{G}(A)$
The universal Massey product

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The "angle of arrival" of $\mu^d$ under the $\mathbb{C}^*$-action

$$\mu^d \mapsto \epsilon^{d-2} \mu^d, \ \epsilon \neq 0$$

gives an obstruction class

$$o^3 = [\mu^3] \in HH^2(A)^{-1}.$$
Let \([a] \in A\) be a cohomology class. We want to define a way to "evaluate" \(o^3\) on cohomology classes.

- **Quantum Massey products** are partially defined multilinear operations

\[
\langle [x_3], [x_2], [x_1]\rangle_{[a]} : A^p \otimes A^q \otimes A^r \to A^{p+q+r-1-2c_1([a])}
\]

which depend only on the homology class \([a]\), the inputs \([x_3], [x_2], [x_1]\) and \(o^3 = [\mu^3]\).

- **Vanishing condition.** We must have

\[
[x_2] \ast_{[b]} [x_1] = [x_3] \ast_{[b]} [x_2] = 0 \, , \, \forall [b] \in A^2 \, \text{s.t.} \, c_1([b]) \leq c_1([a]).
\]

- **Ambiguity.** The ideal \(I\) is generated by

\[
A \ast_{[b_2]} [x_1] + [x_3] \ast_{[b_1]} A
\]

for all \(c_1([b_1]), c_1([b_2]) \leq c_1([a]).\)
A*-Massey products

- Choose **bounding cochains**, i.e.,

\[ h = \sum h_B \cdot q^B, \quad g = \sum g_B \cdot q^B \]

such that \( \mu^1(h) = \mu^2(x_3, x_2) \) and \( \mu^1(g) = \mu^2(x_2, x_1) \).

- **Definition.** We set

\[
\langle x_3, x_2, x_1 \rangle_{[a]} = [\mu^3_{[a]}(x_3, x_2, x_1)] \pm \sum_{[b_1] + [b_2] = [a]} \mu^2_{[b_1]}(x_3, h_{[b_2]}) \\
\pm \sum_{[b_1] + [b_2] = [a]} \mu^2_{[b_1]}(h_{[b_1]}, x_1)] / I.
\]

- **Key point:** sort of ”Finite determinicity” makes computations tractable (especially when using an external ray as the homology class).
The effective cone is generated by $F$, and $R$.

- $c_1 = 1$: $F$ (the exceptional fiber), and $R = L - 3F$ (ruling lines).
- $c_1 = 2$: $2F$, $L - 2F$ (lines not in $Q$) and $2L - 6F$ (conics).

Figure: The cone of curves
The "Watchtower correspondence"

The first step is to extend $\mathcal{X} \to \Delta^*$ over the zero fiber and write the whole family as a Kähler degeneration with smooth total space and ADE-singularities in the central fiber.

- **Problem:** the sections of the dualizing sheaf **fails to embed** curves with genus 2 tails like $C_0 \in \delta_2 \subset \overline{M}_4$ in $\mathbb{P}^3$.

- **Solution:** Consider the homogeneous cubic form

  $$f = X_0X_3^2 + X_1^2X_4 - X_0X_2X_4 + X_1X_2X_3.$$  

The zero set in a cubic 3-fold $F_{2A_5} := V(f) \subset \mathbb{P}^4$.

- $F_{2A_5}$ has a ODP at $p = [0 : 0 : 1 : 0 : 0]$ and two $A_5$-singularities.
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- $F_{2A_5}$ has a ODP at $p = [0 : 0 : 1 : 0 : 0]$ and two $A_5$-singularities.
Given any cubic hypersurface \( F = V(f) \subset \mathbb{P}^4 \) with ODP at \( p = [0 : 0 : 1 : 0 : 0] \) as above, we can write

\[
f = X_2 \cdot q + s
\]

with \( q \) and \( s \) homogeneous of degrees 2 and 3 respectively.

- Consider the projection

\[
\pi_p : F \to V(X_2) = \mathbb{P}^3.
\]

- We can resolve \( \pi_p \) by blow up. Denote \( X = Bl_p F \).

- The exceptional locus is the proper transform of lines via \( p \) that lie in \( F \), which is the cone over \( C := V(q, f) \). Thus

\[
Bl_C \mathbb{P}^3 = X = Bl_p F
\]
Theorem (Wall ’98). Let us fix a singular point \( x \in C \), and denote \( \ell = \overline{px} \). \( F \) has exactly two singular points \( p, q \) on \( \ell \). 

*The type of \( q \) is the same as that of \( x \).*

In our case, the complete intersection \( C_{2A_5} := V(q, s) \subset \mathbb{P}^3 \) is the curve

where \( L_1 \) and \( L_2 \) are \((1, 0)\)-ruling lines of the quadric \( Q = V(q) \), and \( L_3 \) is a smooth rational \((1, 3)\)-curve.
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  ![Diagram](attachment:image.png)

  where $L_1$ and $L_2$ are $(1, 0)$-ruling lines of the quadric $Q = V(q)$, and $L_3$ is a smooth rational $(1, 3)$-curve.
We have a new construction of the $S^1$-family – by smoothing $C_{2A_5}$.

(a) The singular fiber

(b) A generic fiber
As evinced by D. Johnson’s work, the rational homotopy theory of mapping tori of $\psi : C \to C$ is very rich, and can detect elements of the mapping class group efficiently.

Degree changing operations (like blowup/branched covers/moduli of bundles...) can kill the Massey products which leads to ’’unknottedness’’ of $\phi : X \to X$.

Symplectic invariants are not nilpotent and can still carry interesting information.

Computing quantum Massey products of low energy allows us to prove the non-triviality of the symplectic isotopy class.
Questions?