Quiver Representations & Scattering Diagrams

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Application of Mirror Symmetry
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Mirror symmetry

Scattering diagram

Enras
Carlo-Pasquale-Siebert

Theta function (broken line)
Application of Mirror Symmetry

Cluster alg

Quiver rep

Auslander-Reiten quiver

Mirror symmetry

Scattering diagram

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Gross-Hacking-Keel-Kontsevich

Fomin-Zelevinsky
Caldaras- Chapoton

Reineke Bridgeland

Gross-Care-Pampena-Siebert

Theta function (broken line)

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**Motivation (Sketch!)

**Gross-Siebert program** "tropicalize" the SYZ Conjecture

**Idea:** replace special Lagrangian fibration with "good" degeneration.

**Way:** Start with "singular scheme" $X_0$ provide $k$-th order deformation order by order.

**Do:** glue standard thickening "pieces" of $X_0$ modify standard gluing.

E.g. Two charts

\[ \begin{array}{c}
\begin{array}{c}
\text{P} \\
\end{array} \\
\begin{array}{c}
\text{P} \\
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\text{P} \\
\end{array} \\
\begin{array}{c}
\text{P} \\
\end{array}
\end{array} \]

\[ (a, b) \mapsto (a, a+b) \]

have singularities at $p$.

Standard gluing is not well-defined.
modify by

What happens if

Problem: modify giving is not enough.

Solution: scattering.
Scattering diagram

E.g.

\[
\begin{align*}
1 + A_1 & \quad 1 + A_2 \\
1 + A_1^{-1} & \quad 1 + A_1^{-1} A_2 \\
1 + A_2 & \\
\end{align*}
\]

Note $A_i$ is nothing but $X_i$

$N$: lattice of rank $N$, $M = \text{Hom}(N, \mathbb{Z})$, $N \otimes R = N \otimes R$, $M \otimes R = M \otimes R$

for $m = (m_1, \ldots, m_n)$ write $A^m = A_1^{m_1} \cdots A_n^{m_n}$

let $\{ \cdot, \cdot \}: N \times N \to \mathbb{Z}$ be a skew-symmetric bilinear of $N$

Define $p^*: N \to M$

\[
p^*(n) = \{ n, \cdot \}
\]

Assume $p^*$ inj
Wall $(d, fd)$

- $d \leq \text{Mr}$, support of the wall, is a convex rational polyhedral cone of codimension one, $d \leq n^+$ for some $n \in \mathbb{N}^+$ (i.e. $n_i \geq 0$)
- $fd \in C [A_1, \ldots, A_n]$ s.t. $fd = 1 + \sum_{k \geq 1} c_k A^{k \cdot p(n)}$ for some $c_k \in C$

A wall $(d, fd)$ is called outgoing if $-p(n) \in d$

Def: A Scattering diagram $D$ is a collection of walls such that for each $k \geq 0$, the set

$$\{ (d, fd) \mid fd \neq 1 \mod (A_1 \ldots A_n)^k \}$$

is finite.
**Wall Crossing**

Let $\gamma$ be a path on $\mathbb{M}$, and if $\gamma$ passes a wall $(d_i, f_d)$ we define an auto $p_{\gamma} : A^m \rightarrow A^m_{f_d} \langle m, n_0 \rangle$ where $n_0 = \pm n_0$ s.t. $\langle n_0, \gamma'(t) \rangle < 0$.

If $\gamma$ passes thru a sequence of walls $d_1, \ldots, d_s$, $p_{\gamma, \theta} = \prod_{i=1}^{s} p_{\gamma, d_i}$.

**Def.** $D$ is consistent if $p_{\gamma, \theta}$ only depends on the endpt. of $\gamma$ for any path $\gamma$ for which $p_{\gamma, \theta}$ is well-defined.

Thm (Kontsevich-Soibelman, Gross-Siebert)

Given a scattering diagram $D$, there always exists a consistent scattering diagram $D'$ s.t. $D \setminus D'$ consists of only outgoing walls.
**Quiver representation**

\( \Omega \): (acyclic) quiver of rank \( n \) with: \( \Omega_0 \): set of vertices, \( \Omega_1 \): set of arrows

**Def** A \( \mathbb{C} \)-linear representation \( V = (V_a, V_\alpha)_{a \in \Omega_0, \alpha \in \Omega_1} \) of \( \Omega \) is

- each point \( a \) in \( \Omega_0 \) is associated a \( \mathbb{C} \) vector space \( V_a \)
- each arrow \( \alpha : a \to b \) in \( \Omega_1 \) is associated a \( \mathbb{C} \)-linear map \( V_\alpha : V_a \to V_b \)

It is called finite dimensional if each vector space \( V_a \) is f.d.

**Def** Dimension vector of \( V \) as \( \dim V = (\dim V_a)_{a \in \Omega_0} \)

**Def** A representation \( V \) is called indecomposable if \( V \neq 0 \) and if \( V = A \oplus B \) then \( A = 0 \) or \( B = 0 \)

**E.g.** \( i \to 2 \) irreducible rep. are \( C \to 0, 0 \to C, C \to C \)
Let $N = \mathbb{Z} \times \mathbb{Z}$ and $M = \text{Hom}(N, \mathbb{Z})$, i.e. $N = \# \text{vertices in } \mathbb{Q}$.

Now, we define $\{\cdot, \cdot\}$ on $N$ as

$$\{e_i, e_j\} = \{\# \text{arrow from } i \to j\} - \{\# \text{arrow from } j \to i\}$$

Define a bilinear form $\chi(\cdot, \cdot)$, the Euler form, on $N$ as

$$\chi(d, e) = \sum_{i = e_0} \text{d} i \text{e}_i - \sum_{d \neq e_0} \text{d} i \text{e}_j \quad \text{d, e } e N$$

Define $\varepsilon : N \to M$ by $\varepsilon(d) = \chi(\cdot, d)$.
A wall \((d, f_d)\), \(d \leq n^1\)

i.e. \(\langle u, w \rangle = 0 \quad \forall w \neq d\)

"Think \(w\) as a stability condition"

A rep \(E\) is said to be \(w\)-semistable if

- \(w(E) = 0\)
- Every \(w\)-semiobj. \(B \subseteq E\) satisfies \(w(B) \leq 0\)

Thm (Bridgeland) Hall alg. scattering diag.

- Walls consist of maps \(w \in \text{Mor} \, \text{St.}\)

\(~\exists \text{ \(w\)-semistable obj. in rep}(Q)\)~
Auslander-Reiten theory

A way to “line up” irreducible rep. of a quiver.

E.g., \( i \rightarrow 0 \)

\( \begin{array}{c}
0 \rightarrow C \\
C \rightarrow C \\
C \rightarrow 0
\end{array} \)
\[ \gamma : \text{path in } \mathfrak{D} \]
\textbf{Good crossing}

\[ d_1^- \cup d_1^+ \cup \varphi(n) \]

\[ d_1 = d_1^+ \cup \varphi \cup d_1^- \]

\underline{Thm (C)} If \( \gamma \) goes from \( d_1^+ \) to \( d_2 \), and the two crossing are good, then \( D_2 \) is a predecessor of \( D_1 \)

"Reverse the order in Auslander-Reiten quiver"
The original setup from Gross-Hanköing-Keel-Kontsevich is 2014 is more general. They have associated scattering diagram to "cluster algebra." Cluster algebra is defined by Fomin-Zelevinsky in 2000 to understand total positivity in algebraic group and canonical bases in quantum group. It is a subring of a field of rational functions. The (cluster) variables are generated by an iterative process called mutation. Fomin-Zelevinsky proved these "iterative" variables can be expressed as Laurent polynomial of initial variables.

It is conjectured that the coeff. in these Laurent poly are non-negative. GHHK proved it by linking up "theta function" to cluster variables. Can be defined on scattering diag.
Motivation

Broken line: understand Landau-Ginzburg mirror symmetry
Siebert-Carel-Panorka: made use of broken line to construct Landau-
Ginzburg mirrors to varieties with effective
anti-canonical bundle.
Gross-Hacking-Keel-Siebert.

Theta function is classical theta function for the case of abelian varieties.
Theta function

\[ \mathcal{F} : \text{Scattering diag. } m \in M \setminus \{0\} \]
\[ Q \in \{ m \in M \mid <m,e_i> > 0 \} \]

A broken line with initial slope \( m \) and endpoint \( \Theta \) is a piecewise linear continuous proper path \( \gamma : (\infty, 0) \to M_r \setminus \text{Sing}(\Theta) \)

with a finite number of domains of linearity.

A monomial \( e_\lambda a^m e_i \) is attached to each domain of linearity \( L \leq (-\infty, 0) \)

s.t.
Fix $m$ & $Q$. Consider all broken line $\gamma$ with initial slope $m$ & endpoint $Q$.

Take $c_lA_{m_l}$ of the last domain linearly

$\exists_{m,q} = \sum c_lA_{m_l}$
E.g. \( m = (0, -1) \)

Two broken lines: \( \gamma_1, \gamma_2 \)

\[ U_{m,q} = A^{(0, -1)} + A^{(-1, -1)} \]

\[ \text{Thm (GHKK)} \quad \text{If} \quad U_{m,q} \quad \text{is a finite sum, then} \]
\[ U_{m,q} \quad \text{is an elt. of cluster alg.} \]
Many other proposals for bases of cluster alg.

\textbf{Thm. (Caldero-Chapoton)} let $\mathcal{Q}$ be a finite quiver with vertices $1, \ldots, n$ and $D$ a f.d. rep of $\mathcal{Q}$ with dimension vector $\mathbf{d}$. For $e \in \mathbb{N}$,

denote $\text{Gr}(e, D) := \{ E \in \text{mod} \, (\mathcal{Q}) \mid E \leq D, \dim(E) = e \}$

Define

\[
CC(D) = \frac{1}{A_{d_1} \cdots A_{d_n}} \prod_{0 \leq e \leq d} X(\text{Gr}(e, D)) \prod_{i=1}^{n} A_{s_{i}}^{\sum_{j \leq i} d_{j}} + \sum_{i < j} (d_{i} - d_{j})
\]

Then $CC(D) = x_0$ cluster variable obtained from $D$. 
\[ CC(D) = \frac{1}{\text{Ad}} \sum_{0 \leq e \leq d} \chi(G_\text{r}(e, D)) \prod_{i=1}^{n} A_i^j \sum_{j} g_j \cdot (d_j - d_j) \]

\[ = \frac{\prod A_i^j \cdot \sum_{j} g_j}{\prod A_i^j \cdot \sum_{j} g_j} \sum_{0 \leq e \leq d} \chi(G_\text{r}(e, D)) \prod_{i=1}^{n} A_i^j \sum_{j} g_j \cdot (d_j - d_j) \]

\[ = A^{-\chi(d)} \sum_{0 \leq e \leq d} \chi(G_\text{r}(e, D)) \prod_{i=1}^{n} A_i^j \]

\[ \uparrow \]

\[ - \chi(d) + p^*(e) \quad \text{final slope} \]

Note: initial slope
Thm (Bridgeland) (Hall algebra scattering diag.) Denote $H(Q)$ as Hall alg.

- Wall crossing automorphism at a general pt. $w \in d$

\[ \hat{\tau} \text{ is a conjugation by } \]
\[ 1_{ss}(w) = \bigoplus \left[ M(w) \rightarrow M \right] \]

\[ \uparrow \text{obj. are w-semisimple rep.} \]

Let $d$ be a wall in the cluster complex.

\[ 1_{ss}(w) = ( + \sum_{K \in K_{Z1}} \left[ BGL_k(D) \rightarrow M \right] ) \]

think as $D^K$

where $D$ is $\text{sr. w-semisimple rep.}$
Hall algebra broken line.

\( \mathcal{D} : \text{Hall alg. scattering diagram. } m \in \mathcal{M} \setminus \{0\} \text{ and } Q \in \mathcal{M}_{\mathbb{R} \setminus \text{Supp}(\mathcal{D})} \)

A Hall alg. broken line for \( m \) with endpoint \( Q \) is a piecewise linear continuous proper path \( \gamma : (-\infty, 0] \to \mathcal{M}_{\mathbb{R} \setminus \text{Sing}(\mathcal{D})} \) with a finite number of domains of linearity and there is an \( \mathcal{H}(\mathcal{D}) \otimes C[\mathbb{R}] \) attained to each domain of linearity \( L \leq (-\infty, 0) \) of \( \gamma \).

The path \( \gamma \) and the \( [N \to M]A^m \) need to satisfy

1. \( \gamma(0) = Q \)
2. if \( L \) is the first domain of \( \gamma \), then \( [N \to M]A^m = A^m \).

This is saying we have \( [\cdot \to \emptyset] \) zero representation.
In each domain of linearity \( L \subseteq (-\infty, 0] \), the associating Hall alg. monomial \([N \rightarrow M]A^m\)

where \( X(N \rightarrow M)A^m = X(N)A^{p*(\dim W)+m} \)

where \( \gamma'(t) = -1 p*(\dim W) + m \) for \( t \in L \).

- \( \gamma \) bends only when crossing a wall.
  
  If \( \gamma \) bends from the domain of linearity \( L \) to \( L' \) when crossing \( d \),
  then \([N \rightarrow M]A^m\) is a term in \( \Phi_d(d) \cdot ([N \rightarrow M]A^m) \).
Then (C) Hau alg. theta function

\[ \mathcal{U}_m = G_{F(m)}(D) \]

where objects in \( G_{F(m)}(D) \) are reps \( E \) s.t. for any subobj. \( F \leq E, m(F) \leq 0 \).

Furthermore, \( E \) is equipped with an inclusion into \( D \).

Apply Emerynac \( \mathcal{E} \)

\[ \mathcal{U}_m = A^{-3(d)} \sum_{0 \leq \ell \leq d} \chi(G_\ell(D)) A^{p \chi(e)} \]

reprove the CC formula
Setting: \( Y \): broken line with endpoints in positive number

good crossing over outgoing walls

with initial slope \( E(D) \) \( D \) indecompo.

final slope \( E(D) - p(E) \) \( E \in D \) subrep.

Assume \( Y \) bends over line

walls \( \leq f_1^+ \ldots f_s^+ \) with multi. \( \lambda_1 \ldots \lambda_s \)

corresp. to indecompo \( F_1 \ldots F_s \)
First bending

\[ \text{Hom} \quad \text{pr}, f_{1+} (A^{-3}(d)) = \text{Gr}(A^{-3}(d)) \]

where objects in \( G \) are \( [F_i^\lambda \to D] \)

with no kernel of dim vector \( f_i \)

Apply \( X \) to get \( \sum_x \text{Gr}(\lambda, \text{Hom}(F_i, D)) \)

which agrees with usual wall-crossing

Since mult. \( \lambda_i \), define \( V_i := F_i^{\lambda_i} \)
Second bending

\[ \text{Imm}(c) \quad \text{pr}_{f_2} \left( \mathbb{G}_1 A^{-\varepsilon(d)} \right) = \mathbb{G}_2 A^{-\varepsilon(d)} \]

where Poincare poly. of \( \mathbb{G}_2 \) is the same as that of

\[ \text{Gr}(\lambda, \text{Hom}(F_2, D/F_1) - \text{Ext}^1(F_1, F_2)) \times A^\lambda \text{Ext}^1(F_1, F_2) \]

also get a filtration

\[ 0 \leq V_1 \leq V_2 \]

where \( \dim V_2 = \dim F_1^\lambda + \dim F_2^\lambda \)
k-th bending

from (k-1), have \(0 \leq V_1 \leq \ldots \leq V_{k-1}, \ V_i / V_{i-1} = F_i^{\lambda_i}\)

Get \(G_k\) whose Poincaré polyare of the form

\[
\text{Gr} \left( \lambda_k \text{Hom}(F_k, D / V_{k+1}) - \text{Ext}(V_{k-1}, F_k) \right) \times A^{\lambda_k \text{Ext}(V_{k-1}, F_k)}
\]

\& \(0 \leq V_1 \leq \ldots \leq V_{k-1} \leq V_k\)

\[
dim V_k = V_{k-1} + \lambda_k \dim F_k
\]

fill the last wall \(f_5^+\) get \(D = V_5\)

have Harder–Narasimham property in dim 2.
Joint work with Travis Mandel (work in progress)

Fix \( q \in \mathbb{R}^n \).

Now given \( U_{p_1}, \ldots, U_{p_s} \), what is \( U_{p_1} \times \ldots \times U_{p_s} \)?

E.g. \( U_{p_1} \cdot U_{p_2} = \sum_{q} \alpha(p_1, p_2, q) U_q \)

\[ \chi_{(p_1, p_2, q)} = \sum_{(y_1, y_2)} c(y_1)c(y_2) \]

[GHKK 14] \( I(y_1) = \pi, b(y_2) = \pi \) \( F(y_1) + F(y_2) = q \)

Final monomial

\( i.e. \) broken line with initial slope \& final slope satisfying these conditions
Interpret the coefficient of $z^n$ in $U_p, \ldots U_{p^n}$ as Block-Göttsche weighted count of marked tropical curve. 

"&" 

Weighted finite tree into balancing condition 

Limit of classical obj. 

E.g. 

Now: Interpret of the coeff. using the machinery above.
Thank You!