Refined Harder–Narasimhan filtrations in modular lattices and iterated logarithms

Fabian Haiden
joint w/ L. Katzarkov, M. Kontsevich, P. Pandit

University of Miami

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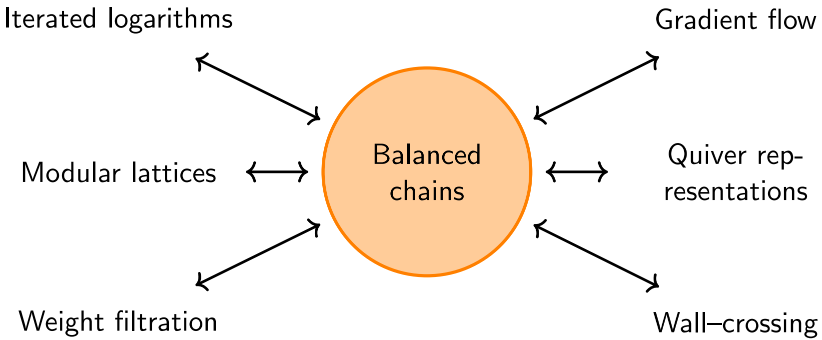
Overview


Iterated logarithms
Gradient flow
Modular lattices
Balanced chains
Quiver representations
Weight filtration
Wall–crossing

Potential applications

1. Asymptotics of solutions to ODEs ✓
2. Asymptotics of geometric PDEs:
    - B–side: Donaldson's heat flow on Hermitian bundles
    - A–side: Lagrangian mean curvature flow
3. Stratification of moduli spaces (c.f. work of F. Kirwan)

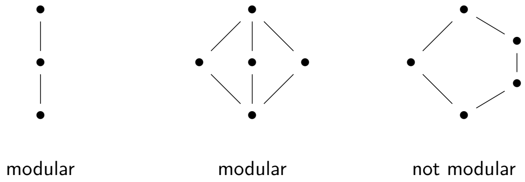
Modular lattices

Lattice: Poset L such that any two elements $a, b \in L$ have a minimum $a \wedge b$ and a maximum $a \vee b$.

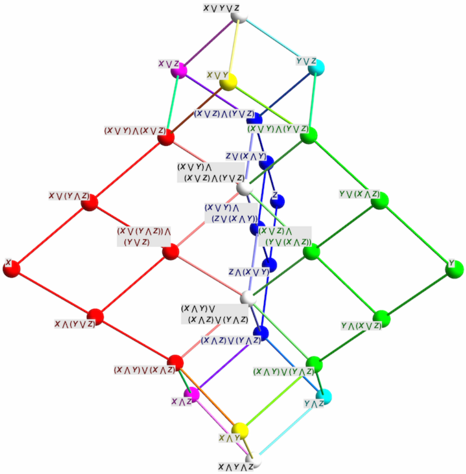
Modular law:

$$a \leq b \implies (x \wedge b) \vee a = (x \vee a) \wedge b$$

Hasse diagrams



Free modular lattice on 3 generators (Dedekind, 1900)



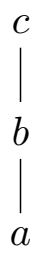
Examples of finite length modular lattices

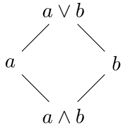
1. $\{0, 1, \ldots, n\}$
2. Normal subgroups of a finite group.
3. Collection of subspaces of a finite dimensional vector space closed under $+$ and $\cap$.
4. Subobjects of a finite-length object in an abelian category.
   - Semistable subobjects, of the same phase, of a semistable object (e.g. in context of Bridgeland stability on a triangulated category).
5. Semistable subbundles, of the same slope, of a semistable Arakelov bundle.

Positive valuations on lattices

L ... finite length modular lattice
$K^+(L)$ ... semigroup generated by intervals $[a,b] \subset L$, $a < b$, with relations

$$[a,b] + [b,c] = [a,c], \qquad [a, a \vee b] = [a \wedge b, b]$$





A **positive valuation** on L is additive homomorphism $K^+(L) \to \mathbb{R}_{>0}$. (canonical choice: *length* of interval)

First main theorem/definition of balanced chain

L ... finite length modular lattice
$X : K^+(L) \to \mathbb{R}_{>0}$ ... positive valuation on L

There exists a unique **balanced chain** $0 = a_0 < a_1 < \ldots a_n = 1$ labeled by $\lambda_1 < \ldots < \lambda_n \in \mathbb{R}$ with

1. $1 \leq k \leq l \leq n, \lambda_l - \lambda_k < 1 \implies [a_{k-1}, a_l]$ complemented
2. $$\sum_{k=1}^{n} \lambda_k X([a_{k-1}, a_k]) = 0$$
3. $b_k \in [a_{k-1}, a_k]$, $k = 1, \ldots, n$, with $1 \leq k < l \leq n$, $\lambda_l - \lambda_k \leq 1 \implies [b_k, b_l]$ complemented, then
$$\sum_{k=1}^{n} \lambda_k X([a_{k-1}, b_k]) \leq 0$$

Strategy of proof

Uniqueness: Consider "double chain" labeled by $\mathbb{R}^2$

Existence:

1. Construct space $\mathcal{B}(L)$ of chains satisfying first condition
2. Local structure at $a \in \mathcal{B}(L)$ controlled by certain modular lattice $\Lambda(a)$
3. Harder–Narasimhan chain in $\Lambda(a)$ gives mass function $m : \mathcal{B}(L) \to \mathbb{R}_{\geq 0}$
4. Show m has minimum
5. Local minimum of m must be balanced chain

Simplest example

$$L = \{0, 1, \dots, n\}$$

$$X([m, n]) = n - m$$

balanced chain in L:

$$a_k = k, \qquad k = 0, \dots, n$$

$$\lambda_k = k - \frac{n+1}{2}, \qquad k = 1, \dots, n$$

(other choice of X: λ_k shifted by constant)

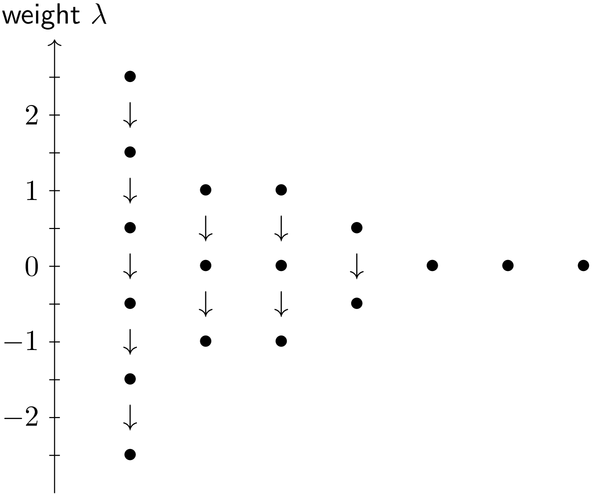
Connection with mixed Hodge structures

V ... finite dimensional vector space
$N : V \to V$ ... nilpotent operator
$L =$ lattice of invariant subspaces (product of lattices of type $\{0, 1, \ldots, n\}$ as in previous example)
$X([A, B]) =$ dimension of subquotient B/A

claim: balanced filtration $=$ "Picard–Lefschetz filtration of a nilpotent operator" as defined by Griffiths

PL filtration provides weight filtration part of MHS on Jacobian ring of isolated hypersurface singularity, N comes from monodromy

Filtration of a nilpotent operator



Iterated balanced chain

Balanced chain a in (L, X)

$\rightsquigarrow$ lattice $\Lambda(a) \supset \Lambda(a)^0 = L'$, comes with positive valuation X'

Balanced chain a' in (L', X') gives refinement of a, labelled by $\mathbb{R}^2$

L finite length $\implies$ process stops after finite number of steps

$\rightsquigarrow$ **iterated balanced chain**: refinement of a labelled by $\mathbb{R}^\infty$

Relation to asymptotics of ODEs suggests interpretation

$$\mathbb{R}^\infty = \mathbb{R}\log t \oplus \mathbb{R}\log\log t \oplus \mathbb{R}\log\log\log t \oplus \dots$$

Quiver representations

Q ... finite quiver (directed multigraph)

representation of Q:

$$\text{vertex } i \longmapsto \text{vector space } V_i$$

$$\text{arrow } \alpha : i \to j \longmapsto \text{linear map } \phi_\alpha : V_i \to V_j$$

(all over $\mathbb{C}$ for now)

metrized representation: V_i equipped with Hermitian mertric h_i

Flow on metrized representations

Potential:

$$S(h) = \sum_{\alpha:i\to j} \mathrm{Tr}\left(h_i^{-1}\phi_\alpha^* h_j \phi_\alpha\right)$$

Metric, depending on parameters $\tau_i > 0$, $i \in Q_0$:

$$\langle v, w\rangle_h = \sum_{i\in Q_0} \tau_i \mathrm{Tr}\left(h_i^{-1} v_i h_i^{-1} w_i\right)$$

Negative gradient flow of S:

$$\tau_i h_i^{-1}\frac{dh_i}{dt} = \sum_{\alpha:i\to j} h_i^{-1}\phi_\alpha^* h_j \phi_\alpha - \sum_{\alpha:j\to i} \phi_\alpha h_j^{-1}\phi_\alpha^* h_i$$

Monotonicity

Crucial property of the flow:

Theorem (Monotonicity): *If g, h are trajectories of the flow with $g(0) \leq h(0)$, then*

$$g(t) \leq h(t), \qquad t \geq 0.$$

Consequence: All solutions have same asymptotics up to bounded factors, i.e.

$$\log(g(t)/h(t)) = O(1), \qquad t \to +\infty$$

Asymptotic behavior

Representation (V_i, ϕ_α) is **semisimple** $\Leftrightarrow$ flow converges to fixed point ("harmonic metric")

$$\sum_{\alpha:i\to j} h_i^{-1}\phi_\alpha^* h_j \phi_\alpha - \sum_{\alpha:j\to i} \phi_\alpha h_j^{-1}\phi_\alpha^* h_i = 0$$

Proof: Apply Kempf–Ness theorem to $\prod Gl(V_i)$-action on ϕ_α's

If representation is **not semisimple**: Metric h grows at different polynomial rates on various subspaces.

$\rightsquigarrow$ Limiting filtration (monotonicity $\implies$ does not depend on initial metric)

Second main theorem

Theorem:

1. limiting filtration = iterated balanced filtration
2. On piece F_λ of the filtration, $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^\infty$ have:

$$\log |h(t)|_{F_\lambda}| = \lambda_1 \log t + \lambda_2 \log \log t + \cdots + \lambda_n \log^{(n)} t + O(1)$$

$\log^{(k)}$ ... k-times iterated logarithm.

Strategy of proof

1. Use framework of $*$-algebras and $*$-bimodules
2. Show solutions up to L^1 are asymptotic solutions (uses monotonicity, homogeneity)
3. Recursively construct solution up to error terms in L^1

$*$-algebra formalism

Finite-dimensional C^*-algebra:

$$B = \bigoplus_{i \in Q_0} \mathrm{End}(V_i)$$

Positive trace:

$$\tau = \sum_{i \in Q_0} \tau_i \mathrm{Tr}(b_i)$$

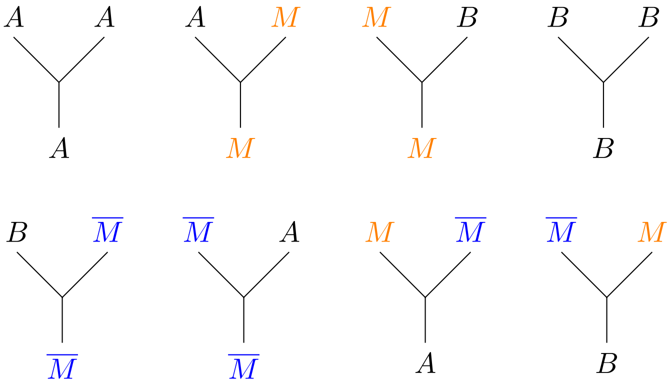
$*$-bimodule:

$$M = \bigoplus_{\alpha : i \to j} \mathrm{Hom}(V_i, V_j)$$

Two B-valued inner products on M:

$$(\phi^* \psi)_i = \frac{1}{\tau_i} \sum_{\alpha : i \to j} \phi^*_\alpha \psi_\alpha, \qquad (\phi \psi^*)_i = \frac{1}{\tau_i} \sum_{\alpha : j \to i} \phi_\alpha \psi^*_\alpha$$

$*$-bimodule operations



Almost everything associative, but $(xy^*)z \neq x(y^*z)$ for $x, y, z \in M$! Instead require $\tau(xy^*) = \tau(y^*x)$.

Solutions up to L^1

Example of Lotka–Volterra type, special case of flow on reps:

$$\frac{d\log x}{dt} = -2x - y$$
$$\frac{d\log y}{dt} = -2x - 2y$$

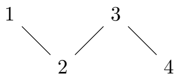
Mathematica gives complicated implicit solution involving Lambert-W function.

But simple explicit solution up to error terms in L^1:

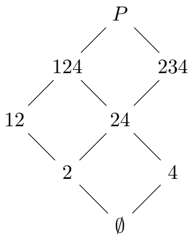
$$x = \frac{1}{2t}\left(1 - \frac{1}{\log t}\right), \qquad y = \frac{1}{t\log t}$$

A_4 zigzag example: lattice

P ... partially ordered set with following Hasse diagram:



L ... lattice of order ideals in P



A_4 zigzag example: dependence on valuation X

$$K^+(L) = \mathbb{Z}^P = \mathbb{Z}^4$$

$\Longrightarrow$ balanced chain depends on 4 parameters

$$X_1, X_2, X_3, X_4 \in \mathbb{R}_{>0}$$

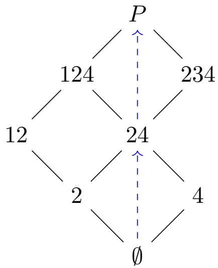
$$D = X_1 X_4 - X_2 X_3$$

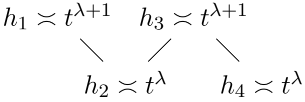
region	length of chain	iteration?
chamber $D < 0$	2	no
wall $D = 0$	2	yes
chamber $D > 0$	4	no

A_4 zigzag example: case $X_1X_4 < X_2X_3$

$$\emptyset < 24 < P$$

$$\lambda = -\frac{X_1 + X_3}{X_1 + X_2 + X_3 + X_4} < \lambda + 1$$

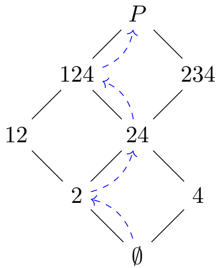


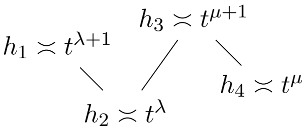


A_4 zigzag example: case $X_1X_4 > X_2X_3$

$$\emptyset < 2 < 24 < 124 < P$$

$$\lambda = -\frac{X_1}{X_1 + X_2} < \mu = -\frac{X_3}{X_3 + X_4} < \lambda + 1 < \mu + 1$$



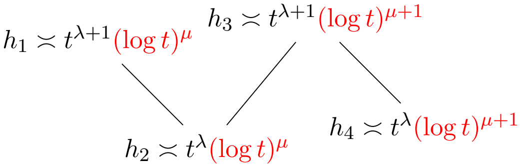


A_4 zigzag example: case $X_1 X_4 = X_2 X_3$

$$\emptyset < 2 < 24 < 124 < P$$

$$(\lambda, \mu) < (\lambda, \mu + 1) < (\lambda + 1, \mu) < (\lambda + 1, \mu + 1)$$

$$\lambda = -\frac{X_1}{X_1 + X_2}, \qquad \mu = -\frac{X_3 + X_4}{X_1 + X_2 + X_3 + X_4}$$



Heat flow on Hermitian bundles

X ... compact Kähler manifold
E ... holomorphic vector bundle over X
h ... Hermitian metric on E
F ... curvature of compatible connection

Flow on space of metrics:

$$\frac{\partial h}{\partial t} h^{-1} = -\Lambda F + \lambda$$

Converges to Hermitian–Einstein metric if bundle is slope–polystable (Donaldson, Uhlenbeck–Yau).

Heat flow on Hermitian bundles (cont.)

E ... direct sum of stable bundles of the same slope
L ... finite length modular lattice of semistable subbundles of the same slope

Conjecture:

1. limiting filtration under flow $=$ iterated balanced filtration in L
2. On piece F_λ of the filtration, $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^\infty$ have:

$$\log |h_x(t)|_{F_\lambda}| = \lambda_1 \log t + \lambda_2 \log\log t + \cdots + \lambda_n \log^{(n)} t + O(1)$$

Flow on Lagrangian submanifolds

X ... compact Calabi-Yau manifold
Ω ... holomorphic volume form
L ... Lagrangian submanifold

Flow:

$$\frac{dL}{dt} = d\mathrm{Arg}(\Omega|_L) \lrcorner \omega^{-1}$$

descreases volume

$$m(L) = \int_L |\Omega|_L|$$

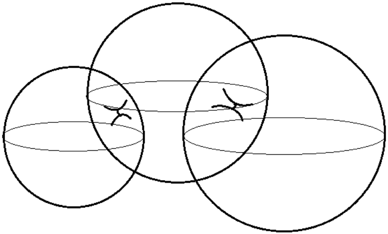
Flow on Lagrangian submanifolds (cont.)

$L_1, \ldots, L_n$ special Lagrangians

- same phase $\operatorname{Arg}(\Omega|_{L_i}) = \text{const.}$
- transverse intersection

Perform surgery introducing small "necks" at intersection points which are degree 1 in morphisms in Fukaya category:

$$\bigcup L_i \rightsquigarrow \text{ Lagrangian } L$$

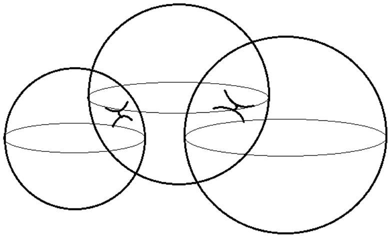


Flow on Lagrangian submanifolds (cont.)

Under flow expect:

$$L_t \to \bigcup L_i$$

Is speed at which necks shrink determined by same asymptotics?



The End

Thank you for your attention!