Geometry and Physics: Mirror symmetry, Hodge theory, and related topics

A conference on the occasion of Ron Donagi' s 60th birthday

University of Miami
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Back in the early '80s, Ron, Maurizio Cornalba, and I met in Utah during the winter. One day, we decided to go skiing. When we got to the bottom of the ski lift, Maurizio asked Ron how long he had been skiing, and Ron said, "I never tried, but it should be fun!"

I have always seen the same fearless spirit in Ron's mathematical work.

Happy Birthday Ron!

Brill-Noether-Petri curves on K3 Surfaces

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Joint work with:

- A. Bruno, E. Sernesi ['15]

- A. Bruno, G. Farkas, G. Saccà ['16]

(Griffiths-Harris ['80], Gieseker ['82])

Theorem. *If $[C] \in M_g$ is a general point, then C is Brill-Noether-Petri (BNP) general.*
i.e. $\forall$ line bundle L on C, the Petri map

$$\mu_0 : H^0(L) \otimes H^0(K_C L^{-1}) \longrightarrow H^0(K_C)$$

is injective.

Remark. *C is BNP general $\iff \quad \forall \quad r, d$*

a) $W_d^r = \{L \in \mathrm{Pic}_d(C) \mid h^0(L) \geq r+1\} \neq \emptyset \qquad \Rightarrow$
$$\Rightarrow \qquad \rho(L) := g - h^0(L) h^0(K_C L^{-1}) = g - (r+1)(g-d+r) \geq 0$$

b) $(W_d^r)_{sing} = W_d^{r+1}$

- *$\rho(L)$ governs the projective realisations of C as deg-d curve in $\mathbb{P}^r$.*

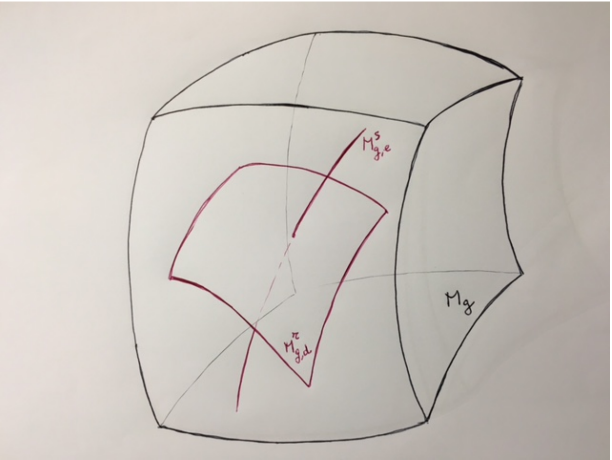
- *e.g.: a d-sheeted cover is BNP general only if $d \geq [(g+2)/2]$.*
- e.g.: a smooth degree d plane curve is BNP general only if $d \leq 4$.

The Brill-Noether subvarieties

When $\rho = g - (r+1)(g-d+r) < 0$

$$M^r_{g,d} = \{[C] \in M_g \mid \exists\ g^r_d \text{ on } C\} \subset M_g$$

is a proper sub-variety of postulated codimension $-\rho$.



Theorem [Lazarsfeld '86]. *Let $S \subset \mathbb{P}^g$ be a K3 surface, $H \subset \mathbb{P}^g$ a general hyperplane, and $C = H \cap S$. Assume that $[C]$ is indecomposable. Then C is Brill-Noether-Petri (BNP) general.*
For $g \geq 12$

$$\mathcal{K}_g = \{[C] \in M_g \mid C = H \cap S, \;\; S \subset \mathbb{P}^g \text{ a K3 surface}\} \subsetneq M_g$$

$$\dim M_g = 3g - 3 \qquad \dim \mathcal{K}_g = g + 19$$

Lazarsfeld: Curves on K3's behave like general curves w.r.t BNP theory for line bundles.

Voisin ['02], Aprodu-Farkas ['12]: Curves on K3's behave like general curves w.r.t Green's conjecture.

Bruno, Sernesi, EA. ['14] : Not so for vector bundles, when $g = 2s + 1, g > 11$.

Wahl ['87], Beauville-Mérindol ['87]

Theorem. *If $C = H \cap S$, $S \subset \mathbb{P}^g$ a K3 surface, then*

$$\nu : \wedge^2 H^0(K_C) \longrightarrow H^0(3K_C)$$
$$s \wedge t \mapsto tds - sdt$$

is not surjective.

- [B-M]'s way to look at this Theorem: For $C = H \cap S$, the map $\nu^\vee$ is not injective since:

$$\nu^\vee : \mathrm{Ext}^1(\omega_C, T_C) \longrightarrow H^1(T_{H|C} \otimes T_C) \qquad H \cong \mathbb{P}^{g-1}$$
$$[0 \to T_C \to T_{S|C} \to \omega_C \to 0] \quad \mapsto \quad 0$$

Question: is the non-surjectivity of ν also a sufficient condition for a canonical curve $C \subset \mathbb{P}^{g-1}$ to be a hyperplane section of a K3 surface $S \subset \mathbb{P}^g$?

Wahl ['96]:

Theorem. *Let $C \subset \mathbb{P}^{g-1}$ be a canonical curve, with defining ideal sheaf $\mathcal{I}_{C/\mathbb{P}}$. Look at*

$$\nu : \wedge^2 H^0(K_C) \longrightarrow H^0(3K_C)$$

$$\left.\begin{array}{ll} a) & \nu \text{ not surjective} \\ b) & H^1(\mathcal{I}^2_{C/\mathbb{P}}(k)) = 0, k \geq 3 \\ c) & \mathrm{Cliff}(C) \geq 3 \end{array}\right\} \Rightarrow \begin{array}{l} \exists \quad \text{surface} \quad \overline{S} \subset \mathbb{P}^g \quad (\text{not a cone}) \\ \text{with} \quad C = \overline{S} \cap H \quad H \cong \mathbb{P}^{g-1} \\ \qquad (g \geq 8) \end{array}$$

$\overline{S}$ is a proj. norm. surface with canonical sections: a "fake K3".

"Proof". $A = \mathbb{C}[x_0, \dots, x_{g-1}]/\mathrm{I}_C$, Spec$(A) \subset \mathbb{C}^g$ is the cone over C. Then:

$(T^1_A)_{-1} = \mathrm{Coker}\,\nu$,

$(T^2_A)_{1-k} = H^1(\mathcal{I}^2_{C/\mathbb{P}}(k))$.

$\mathrm{Cliff}(C) = \min\{d - 2r \mid \exists\ g^r_d = |L|,\ r \geq 1, h^0(K_C L^{-1}) \geq 2\} \geq 3$

$\Rightarrow$ only linear syzygies among quadrics $\supset C$.

Conclusion: a non trivial element in $(T^1_A)_{-1}$ can be integrated to obtain $\overline{S}$. □

$$\left.\begin{array}{ll} a) & \nu \text{ not surjective} \\ b) & H^1(\mathcal{I}^2_{C/\mathbb{P}}(k)) = 0, k \geq 3 \\ c) & \mathrm{Cliff}(C) \geq 3 \quad (\Leftarrow \text{ BNP general}) \end{array}\right\} \Rightarrow \begin{array}{l} \exists \quad \text{surface} \quad \overline{S} \subset \mathbb{P}^g \\ \text{with} \quad C = \overline{S} \cap H, \end{array}$$

Wahl ['96]:

- If $[C] \in M_g$ is general point, then b) holds. Indeed it holds when $C \xrightarrow{5-1} \mathbb{P}^1$. However it does not if $C \xrightarrow{4-1} \mathbb{P}^1$.
- If C is (isom. to) a smooth plane curve of degree $d \geq 7$, a), b), and c) hold, but $\overline{S}$ is non-smoothable.

Conjecture 1. $C \subset \mathbb{P}^{g-1}$ a canonical curve, then c)$\Rightarrow$ b).

Conjecture 2. If C is a BNP general curve of genus $g \geq 8$ then: ν not surj. $\Leftrightarrow$ $C = S \cap H$ with S a K3 surface (or a limit of such).

A. Bruno, E. Sernesi, EA ['15]:

Theorem 1. Conjecture 1 holds for $g \geq 11$.

Theorem 2. Conjecture 2 holds for $g \geq 12$.

“ Proof ”of Th. 1, i.e.

$$\boxed{g \geq 11, \mathrm{Cliff}(C) \geq 3 \Rightarrow H^1(\mathcal{I}^2_{C/\mathbb{P}^{g-1}}(k)) = 0\,,\ \ k \geq 3}.$$

Voisin ['96]:

Theorem. $\mathrm{Cliff}(C) \geq 3 \Leftrightarrow \exists$ *on* C *a base-point-free pencil* $|L| = g^1_{g-2}$ *such that* $g^2_g = |K_C L|^{-1} : C \to \Gamma_g \subset \mathbb{P}^2$ *where* Γ_g *has only double points as singularities.*

$$|L| = g^1_{g-2} \quad \rightsquigarrow \quad C \quad \subset \quad X = \bigcup_{D \in g^1_{g-2}} \langle D \rangle \quad \subset \mathbb{P}^{g-1}, \qquad \langle D \rangle \cong \mathbb{P}^{g-4}$$

X is a degree 3 cone over $\mathbb{P}^1 \times \mathbb{P}^2$ with vertex a $\mathbb{P}^{g-7}$. A desingularization of $\widetilde{X}$ is rank-$(g-4)$ projective bundle over $\mathbb{P}^1$

$$\mathbb{P}^{g-1} \supset X \longleftarrow \widetilde{X} = \mathbb{P}E \longrightarrow \mathbb{P}^1\,, \quad \dim X = g-3$$

Moreover:

$$H^1(\mathcal{I}^2_{C/\mathbb{P}^{g-1}}(k)) = 0 \quad \Leftrightarrow \quad H^1(\mathcal{I}^2_{C/X}(k)) = 0$$

(One step back: Wahl's proof of the vanishing of for a pentagonal curve.

Suppose $C \xrightarrow{5-1} \mathbb{P}^1$. Then $H^1(\mathcal{I}^2_{C/\mathbb{P}^{g-1}}(k)) = 0$, for $k \geq 3$.

$$|L| = g^1_5 \quad \rightsquigarrow \quad C \quad \subset \quad X = \bigcup_{D \in g^1_5} \langle D \rangle \quad \subset \mathbb{P}^{g-1}, \qquad \langle D \rangle \cong \mathbb{P}^3$$

$$X \quad \text{smooth}, \quad \dim X = 4$$

$$H^1(\mathcal{I}^2_{C/\mathbb{P}^{g-1}}(k)) = 0 \quad \Leftrightarrow \quad H^1(\mathcal{I}^2_{C/X}(k)) = 0$$

and $H^1(\mathcal{I}^2_{C/X}(k))$ is computable.)

Go back to an arbitrary curve C with $\mathrm{Cliff}(C) \geq 3$

In our case, instead of a g^1_5, we try to work with a g^1_{g-2}:

$$|L| = g^1_{g-2} \quad \rightsquigarrow \quad C \quad \subset \quad X = \bigcup_{D \in g^1_{g-2}} \langle D \rangle \quad \subset \mathbb{P}^{g-1}, \qquad \langle D \rangle \cong \mathbb{P}^{g-4}$$

X is a degree 3 cone over $\mathbb{P}^1 \times \mathbb{P}^2$ with vertex a $\mathbb{P}^{g-7}$.

$$\mathbb{P}^{g-1} \supset X \longleftarrow \widetilde{X} = \mathbb{P}E \longrightarrow \mathbb{P}^1, \quad \dim X = g - 3$$

$$H^1(\mathcal{I}^2_{C/\mathbb{P}^{g-1}}(k)) = 0 \quad \Leftrightarrow \quad H^1(\mathcal{I}^2_{C/X}(k)) = 0$$

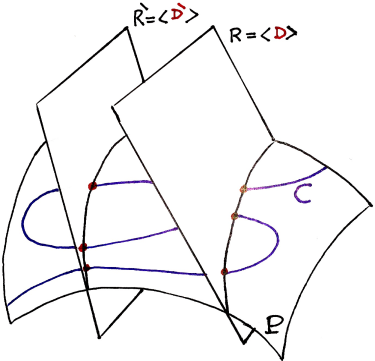
By Voisin's Theorem:

$$|K_C L^{-1}| = g_g^2 \quad \rightsquigarrow \quad C \to \Gamma_g \quad \subset \quad \mathbb{P}^2$$

Γ_g has δ double points $p_1, \dots, p_\delta$. Via the adjoint system:

$$\phi: \quad \mathbb{P}^2 \dashrightarrow P \quad \subset \quad X \quad \subset \quad \mathbb{P}^{g-1}, \qquad \phi(\Gamma_g) = C$$

$$\boxed{C \quad \subset \quad P \quad \subset \quad X = \bigcup_{D \in |L|} < D > \quad \subset \quad \mathbb{P}^{g-1}}$$



(*) $H^1(\mathcal{I}^2_{C/X}(3)) \overset{?}{=} 0$ (case $k=3$)

$$0 \longrightarrow \mathcal{I}^2_{C/X}(3) \longrightarrow \mathcal{I}_{C/X}(3) \longrightarrow N^\vee_{C/X}(3) \longrightarrow 0$$

$$0 \longrightarrow N^\vee_{P/X}(3))_{|C} \longrightarrow N^\vee_{C/X}(3)) \longrightarrow N^\vee_{C/P}(3)) = \mathcal{O}_C(3H-C) \longrightarrow 0$$

The heart of the proof of $(*)$ is this unexpected fact:

Lemma. The rank-$(g-5)$ bundle $N^\vee_{P/X}(3))_{|C}$ is decomposable: $(N^\vee_{P/X}(3))_{|C} = (LK_C)^{\oplus(g-5)}$.

$(*)$ follows from this. $\square$

Theorem 2. *Let C be a BNP general curve of genus $g \geq 12$, then C is the hyperplane section of a K3 surface (or a limit of such) $\Leftrightarrow$ the Gauss-Wahl map ν is non-surjective.*

"Proof". Wahl + Theorem 1. $\Rightarrow$ $\exists$ $\overline{S} \subset \mathbb{P}^g$ s.t.

$$C = \overline{S} \cap H \quad \subset \quad \overline{S} \quad \subset \quad \mathbb{P}^g$$

$\overline{S} \subset \mathbb{P}^g$ a surface with canonical sections: Du Val ['31], Umezu ['81], Epema ['84], Ciliberto-Lopez ['02].

$$\begin{array}{ccc} S & \xrightarrow{\pi} & \overline{S} \subset \mathbb{P}^g \\ {\scriptstyle p}\downarrow & & \\ S_0 & & \end{array}$$

where: S = mimimal resolution

S_0 = mimimal model

If $\overline{S}$ is not a cone over C, then:

$$S_0 = \begin{cases} K3 \quad \text{surface} \\ \mathbb{P}E \to \Gamma, \quad \Gamma \quad \text{a smooth curve} \\ \mathbb{P}^2 \end{cases}$$

$$\begin{array}{ccc} S & \xrightarrow{\pi} & \overline{S} \subset \mathbb{P}^g \\ {\scriptstyle p}\downarrow & & \\ S_0 & & \end{array}$$

The aim:

$C = \overline{S} \cap H$ is BNP general

$\Rightarrow (\overline{S}, H)$ is smoothable

Case: $S_0 = \mathbb{P}^2$. Then

$$|-K_S| = \{J\}, \quad J \cdot C = 0, \quad \pi(J) = \overline{s} \in \overline{S},$$

where $\overline{s}$ is an elliptic singularity. Set $p(J) = J_0\,, p(C) = C_0$. Then J_0 is a plane cubic passing through the singular points of C_0, and S is the blow-up of $\mathbb{P}^2$ at $J_0 \cap C_0$.

$$C = d\ell - \sum_{i=1}^{h} \nu_i E_i\,, \quad J = 3\ell - \sum_{i=1}^{h} E_i\,, \quad 3d = \sum \nu_i\,, \quad J^2 = 9 - h$$

Deformation theory arguments give that:

$$\mathrm{Def}_{(\overline{S},H)} \longrightarrow \mathrm{Def}_{(\overline{S},\overline{s})}$$

is smooth. Moreover : (Pinkham, Laufer ['70], Umezu ['80]) the elliptic singularity $(\overline{S},\overline{s})$ is smoothable $\quad\Leftrightarrow\quad -9 \leq J^2 \leq -1$.

Conclusion: $(\overline{S}, H)$ is smoothable if $10 \leq h \leq 18$.

($h =$ number of singular points of C_0, and $J^2 = 9 - h$).

Must show: C is BNP general $\quad\Rightarrow\quad C_0$ has no more than 18 singular points.

The instrument to test whether C is BNP general, or not, is the linear series cut out on C by the cubics through 8 of the points of $C_0 \cap J_0$.

The analysis shows that, for every genus $g \geq 12$ there are only a finite number of possibilities for the degree d of C_0 and for the values of $\nu_1, \dots, \nu_h$ of the multiplicities of the singular points of C_0, if one insists that the pencil above be a BNP general pencil.

For each $g \geq 12$ one is left with some fourteen cases, and in each of those one gets: $h \leq 18$. $\square$

Among all the cases above, a very interesting one is the following: C_0 is an irreducible plane curve of degree $3g$ having as singularities only nine general points $p_1, \ldots, p_9$ lying on a smooth cubic J_0, with

$$\mu_{p_i}(C_0) = g\,, \quad \text{for} \quad i = 1, \ldots, 8\,, \quad \mu_{p_i}(C_0) = g - 1$$

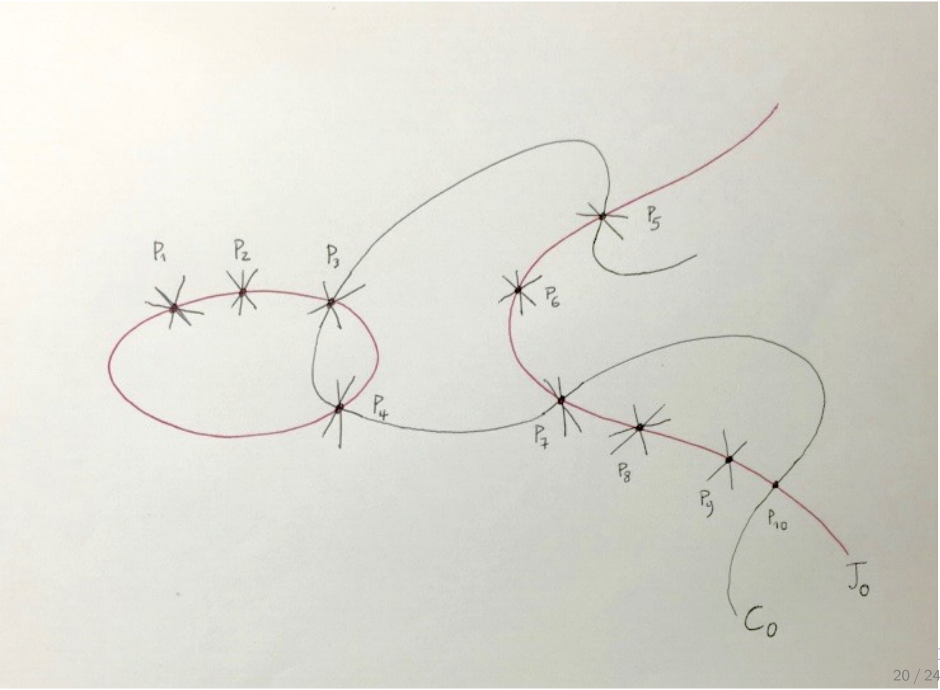
In this case:

$$\deg(C_0) = 3g\,, \quad g(C) = g\,, \quad J_0 \cap C_0 = \{p_1, \ldots, p_9, p_{10}\}\,,$$

In this situation, the minimal resolution S of the $\overline{S} \subset \mathbb{P}^g$ is just $\mathbb{P}^2$ blown up at ten points.

These curves first appear in a paper by Du Val ['31].

We call these plane curves: the Du Val curves.


P1
P2
P3
P4
P5
P6
P7
P8
P9
P10
J0
C0

A. Bruno, G. Farkas, G. Saccà, EA.

Remark. For every value of $g \geq 3$, Du Val curves exist and, for fixed $p_1, \dots, p_9, p_{10}$, they form a g-dimensional system of plane curves.

Theorem 3. *A general Du Val curve is a Brill-Noether-Petri general curve.*

"Proof". Don't go to the minimal resolution S of $\overline{S}$. Just blow-up $\mathbb{P}^2$ at $p_1, \dots, p_9$. Call this surface S'.

Given L on $C' \cong C$, then: L is BNP general $\quad \Leftrightarrow \quad \mathrm{End}(F) \cong \mathbb{C}$ where F is the rank-$h^0(L)$ vector bundle on S' defined by:

$$0 \longrightarrow F \longrightarrow H^0(L) \otimes \mathcal{O}_{S'} \longrightarrow L(p_{10}) \longrightarrow 0$$

Proceed by contradiction: suppose $\phi \in \mathrm{End}(F)$, non-trivial:

$$A = c_1(\mathrm{Im}(\phi)\,, \quad B = c_1(\mathrm{Coker}(\phi)/\,\mathrm{Tors.})\,, \quad T = c_1(\mathrm{Tors.})$$

Then $[C] = A + B + T$.

On S':

$$C'\cdot J' = 1\,,\quad J'^2 = 0\,,\quad C' = A+B+T\,,\quad h^0(A)\neq 0\,,\qquad h^0(B)\neq 0$$

By Nagata: D effective, $D\cdot J' = 0 \Rightarrow D = nJ'$. Thus either $A = nJ'$ or $B = nJ'$. Thus $h^0(J', \mathcal{O}_{J'}(nJ'))\neq 0$, i.e $n(3\ell - p_1 - \cdots - p_9)\sim 0$ on J'. So the p_i's are not general. Absurd. □

Remark 1. The points $p_1, \ldots, p_9$ should be general in a specific way. They should be $3g$-Cremona and $3g$-Halphen general, i.e. $\exists!$ smooth cubic through them; $\nexists$ effective (-2)-curve of degree $k\leq 3g$ on S'; $\exists$ a curve of degree $3d\leq 3g$ with points of multipl. d at $p_1, \ldots, p_9$ and no further singularities, for each $d\leq g$.
Remark 2. Brill-Noether (not necessarily Petri) curves where also described by Treibich.

Finally, a question raised by Harris and Morrison related to Lang's conjecture: Does there exist a Brill-Noether-Petri general curve defined over $\mathbb{Q}$? (Especially when $g\geq 24$, in which case M_g is of general type)

Corollary. For every value of $g \geq 3$ there exist Brill-Noether-Petri general curves defined over $\mathbb{Q}$.

$$DuVal_g \quad \subset \quad \overline{\mathcal{K}}_g \quad \subset \quad M_g$$

unirational subvariety of dimension $g + 10$.

Lefschetz pencils of Du Val curves:

$$\begin{array}{c} C \subset S \\ \downarrow \\ C_0 \subset \mathbb{P}^2 \end{array}$$

Take a pencil in $|C|$. Blow up the $2g - 2$ base points and get a Lefschetz pencil $\mathcal{C} \to \mathbb{P}^1$ and a moduli map $j : \mathbb{P}^1 \to M_g$. Look at the BN loci $M^r_{g,d} \subset M_g$.

Theorem. Either $j(\mathbb{P}^1) \subset M^r_{g,d}$, or

$$j(\mathbb{P}^1) \cap M^r_{g,d} = \emptyset.$$

Consequence: BN general Du Val curves can be written explicitly over $\mathbb{Q}$.

F.-O. Schreyer: computation in some cases with $g \geq 22$, (need 14/15 h).

Happy Birthday Ron!