

INFINITELY MANY MONOTONE LAGRANGIAN TORI IN \mathbb{R}^6

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ABSTRACT. We construct infinitely many families of monotone Lagrangian tori in \mathbb{R}^6 , no two of which are related by Hamiltonian isotopies (or symplectomorphisms). These families are distinguished by the (arbitrarily large) numbers of families of Maslov index 2 pseudo-holomorphic discs that they bound.

1. INTRODUCTION

The study and classification of Lagrangian submanifolds in symplectic manifolds is a central topic of modern symplectic topology; in spite of spectacular advances in the last few decades, it remains poorly understood, even in very simple symplectic manifolds such as the standard symplectic vector space $(\mathbb{R}^{2d}, \omega_0)$.

By a celebrated result of Gromov, there are no closed exact Lagrangian submanifolds in \mathbb{R}^{2d} , and in fact any closed Lagrangian in \mathbb{R}^{2d} must bound some pseudo-holomorphic discs of non-zero area [10]. (This is in sharp contrast with the situation for immersed Lagrangians, see e.g. [6].) Thus, the nicest condition that one could impose on a closed Lagrangian submanifold $L \subset \mathbb{R}^{2d}$ is for it to be *monotone*, i.e. that the symplectic area of discs with boundary on L is (positively) proportional to their *Maslov index*.

The simplest examples of monotone Lagrangians in \mathbb{R}^{2d} are the tori obtained as products of d circles of equal radius, $L = S^1(r) \times \cdots \times S^1(r)$. In the early 1990s Chekanov found the first examples of Lagrangian tori in \mathbb{R}^{2d} that cannot be related to product tori by Hamiltonian isotopies (or symplectomorphisms) [3] (see also [7]). Subsequent work of Chekanov and Schlenk has led to more examples, the so-called *monotone twist tori* [4]; the number of tori produced by this construction grows exponentially with the dimension, but remains finite for all d .

More recently, Renato Vianna's thesis [12] shows that $\mathbb{C}\mathbb{P}^2$ contains at least one new kind of monotone Lagrangian torus besides product and Chekanov tori; this result was recently improved to show that $\mathbb{C}\mathbb{P}^2$ contains *infinitely many* non-isotopic monotone Lagrangian tori [9, 13].

In this paper, we construct infinitely many families of monotone Lagrangian tori in \mathbb{R}^6 , no two of which are related by symplectomorphisms. Specifically, the invariants

This work was partially supported by NSF grants DMS-1406274 and DMS-1264662, and by a fellowship from the Simons Foundation.

that we use to distinguish these tori are the algebraic counts of Maslov index 2 pseudo-holomorphic discs whose boundary passes through a given point (see §3.1); these invariants were already used by Eliashberg-Polterovich to distinguish the Chekanov torus in \mathbb{R}^4 [7] and in much of the subsequent work [4, 12, 13].

Theorem 1. *For each integer $n \geq 0$, and for any choice of monotonicity constant, there exists a monotone Lagrangian torus $L \subset (\mathbb{R}^6, \omega_0)$ such that there are $n + 2$ distinct Maslov index 2 classes in $\pi_2(\mathbb{R}^6, L)$ for which the algebraic count of pseudo-holomorphic discs passing through a point of L is non-zero (and the sum of these counts is $2^n + 1$). Therefore, for different n these tori cannot be related by symplectomorphisms.*

Remark.

- (1) Taking the product of these tori with circles of the appropriate radius, we also obtain infinitely many examples in \mathbb{R}^{2d} for all $2d \geq 6$ (similarly distinguished by counts of Maslov index 2 pseudo-holomorphic discs).
- (2) For $n = 1$ our tori are most likely symplectomorphic to standard product tori. For $n = 0$ they can be shown to be symplectomorphic to the product of a circle in \mathbb{R}^2 with the monotone Chekanov torus in \mathbb{R}^4 .
- (3) Vianna's recent result concerning the existence of infinitely many monotone Lagrangian tori in $\mathbb{C}\mathbb{P}^2$ ([13], see also [9]) should also imply a result similar to Theorem 1, by considering the preimages of these tori under the natural projection map from the unit sphere $S^5 \subset \mathbb{R}^6$ to $\mathbb{C}\mathbb{P}^2$. However, the construction we give here is substantially simpler.
- (4) Monotonicity plays a key role in the construction. Indeed, after arbitrarily small Lagrangian isotopies (not preserving monotonicity), our tori become Hamiltonian isotopic to standard product tori.
- (5) The least elementary part of our argument is the discussion of orientations of moduli spaces. The reader unwilling to delve into these should be content to work with mod 2 counts of holomorphic discs; the number of Maslov index 2 classes for which the algebraic count of discs is non-zero mod 2, and the number of integer points in their convex hull inside $\pi_2(\mathbb{R}^6, L) \simeq \mathbb{Z}^3$, are in fact sufficient to distinguish the monotone tori we construct for different n .

Acknowledgements. While the methods of this paper are elementary, some of the key conceptual ideas come from the joint work of the author with Mohammed Abouzaid and Ludmil Katzarkov [1], and from Renato Vianna's thesis [12] (see §5). I thank all three of them for helping shape my thoughts on this subject. I also thank Felix Schlenk and the anonymous referees for their careful comments.

This work was partially supported by NSF grants DMS-1406274 and DMS-1264662, and by a fellowship from the Simons Foundation.

2. KÄHLER REDUCTION AND MOSER FLOW ON THE REDUCED SPACE

Our main object of study is the manifold

$$(1) \quad X = \{(x, y, z, w) \in \mathbb{C}^4 \mid xy = h(z, w)\},$$

where for $n \geq 0$,

$$(2) \quad h(z, w) = cz^n + c^{-1}w - 1,$$

for $c \gg 1$ a constant (e.g., $c = 10$). As a complex manifold X is isomorphic to \mathbb{C}^3 via projection to the coordinates (x, y, z) , as $w = c(xy + 1) - c^2z^n$. We equip X with the Kähler form

$$(3) \quad \omega_X = \frac{i}{2}dz \wedge d\bar{z} + \frac{i}{2}dw \wedge d\bar{w} + \kappa \left(\frac{i}{2}dx \wedge d\bar{x} + \frac{i}{2}dy \wedge d\bar{y} \right),$$

where $\kappa > 0$ is a small positive constant to be determined below. We note that up to a rescaling of the x and y coordinates ω_X is simply the restriction to X of the standard Kähler form of \mathbb{C}^4 .

The action of S^1 on X by

$$(4) \quad e^{i\theta} \cdot (x, y, z, w) = (e^{i\theta}x, e^{-i\theta}y, z, w)$$

is Hamiltonian, with moment map

$$(5) \quad \mu_X = \frac{\kappa}{2}(|x|^2 - |y|^2).$$

We will consider the reduced space

$$(6) \quad X_{red} = \mu_X^{-1}(0)/S^1.$$

As a complex manifold, X_{red} can be naturally identified with \mathbb{C}^2 via projection to the coordinates (z, w) . Indeed, for fixed (z, w) the part of the conic $xy = h(z, w)$ where $|x| = |y|$ consists of a single S^1 -orbit; the reduced space is therefore naturally a smooth complex manifold, even though $\mu_X^{-1}(0)$ is singular at the fixed points of the S^1 -action, i.e. where $h(z, w) = 0$ and $x = y = 0$.

Lemma 1. *The reduced Kähler form on $X_{red} \simeq \mathbb{C}^2$ is given by*

$$(7) \quad \omega_{red} = \frac{i}{2}dz \wedge d\bar{z} + \frac{i}{2}dw \wedge d\bar{w} + \frac{i\kappa}{4} \frac{dh \wedge d\bar{h}}{|h|} = \omega_0 + \frac{\kappa}{2} dd^c(|h|).$$

(As expected this form is singular along the complex curve $h(z, w) = 0$.)

Proof. Given any point of X_{red} where $h(z, w) \neq 0$, we choose a local square root of h , and observe that a local section of the quotient map from $\mu_X^{-1}(0)$ to X_{red} is given by setting $x = y = h(z, w)^{1/2}$. By definition, the reduced Kähler form ω_{red} agrees with the pullback of ω_X under this local section map. Setting $x = y = h^{1/2}$, we find that

$$dx \wedge d\bar{x} + dy \wedge d\bar{y} = 2d(h^{1/2}) \wedge d(\bar{h}^{1/2}) = \frac{1}{2|h|} dh \wedge d\bar{h}.$$

The first part of (7) follows immediately by substitution into (3). The second equality follows from the observation that

$$dd^c(|h|) = 2i\partial\bar{\partial}(h^{1/2} \cdot \bar{h}^{1/2}) = \frac{i}{2|h|} dh \wedge d\bar{h}.$$

□

Next we recall the following explicit form of Moser's lemma in the Kähler case.

Lemma 2. *Let ω_0 and $\omega_1 = \omega_0 + dd^c\varphi$ be two Kähler forms on a complex manifold. Denote by $g_t = (1-t)g_0 + tg_1$ the Kähler metric corresponding to the Kähler form $\omega_t = \omega_0 + tdd^c\varphi$ for $t \in [0, 1]$, by $\xi_t = -\nabla_{g_t}(\varphi)$ the gradient of φ with respect to g_t , and by ψ_t the isotopy generated by ξ_t wherever it is well-defined. Then $\psi_t^*(\omega_t) = \omega_0$. Moreover, when $\omega_0 = d\theta_0$ is exact, setting $\theta_t = \theta_0 + td^c\varphi$, the pullback $\psi_t^*(\theta_t)$ differs from θ_0 by an exact form.*

Proof. The result follows from Moser's trick and the observation that

$$\omega_t(\xi_t, \cdot) = -g_t(\xi_t, J\cdot) = d\varphi(J\cdot).$$

Thus, $\iota_{\xi_t}\omega_t = -d^c\varphi$, and

$$\frac{d}{dt}(\psi_t^*\omega_t) = \psi_t^*\left(\frac{d}{dt}\omega_t + L_{\xi_t}\omega_t\right) = \psi_t^*(dd^c\varphi + d\iota_{\xi_t}\omega_t) = 0.$$

Similarly, in the exact case,

$$\frac{d}{dt}(\psi_t^*\theta_t) = \psi_t^*\left(\frac{d}{dt}\theta_t + L_{\xi_t}\theta_t\right) = \psi_t^*(d^c\varphi + \iota_{\xi_t}(d\theta_t) + d(\iota_{\xi_t}\theta_t)) = \psi_t^*(d\iota_{\xi_t}\theta_t)$$

is exact as claimed. □

Applying this to the case at hand, we obtain:

Lemma 3. *Let U be the complement of an arbitrarily small neighborhood of $h^{-1}(0)$ inside an arbitrarily large ball in \mathbb{C}^2 . Then there exists a constant $\kappa_0 > 0$ (depending on U) and an isotopy $(\psi_\kappa)_{\kappa \in [0, \kappa_0]}$ defined on U , $\psi_0 = \text{id}$, such that for all $\kappa \in (0, \kappa_0)$, ψ_κ gives an exact symplectomorphism between $U \subset (\mathbb{C}^2, \omega_0)$ and $\psi_\kappa(U) \subset (X_{red}, \omega_{red})$.*

Proof. Let Ω be a compact subset of $\mathbb{C}^2 \setminus h^{-1}(0)$ whose interior contains the closure of U . On Ω , the function $|h|$ is smooth and has bounded derivatives, and the Kähler metric g_κ associated to $\omega_{red} = \omega_0 + \frac{\kappa}{2}dd^c(|h|)$ is bounded between fixed multiples of the standard metric g_0 for all $\kappa \in [0, 1]$. Thus, the vector field $\xi_\kappa = -\frac{1}{2}\nabla_{g_\kappa}|h|$ is smooth and has bounded norm on Ω . Applying Lemma 2, the isotopy ψ_κ generated by ξ_κ is well-defined on U for small enough κ and gives the desired symplectomorphisms. □

3. MONOTONE TORI IN X_{red} AND X

3.1. An enumerative invariant of monotone Lagrangians. Before proceeding with our construction, we recall some basic facts about holomorphic discs and the invariant we use to distinguish our tori. (See also [7, 2, 12].)

Let L be a closed oriented spin Lagrangian submanifold in a symplectic manifold (M^{2d}, ω) equipped with a compatible almost-complex structure J . When M is non-compact we always assume that ω is convex at infinity (in our case, this follows from the properness and strict plurisubharmonicity of the Kähler potential).

Given a J -holomorphic map $u : (D^2, \partial D^2) \rightarrow (M, L)$, the Maslov index $\mu([u]) \in 2\mathbb{Z}$ is the homotopy class of the loop of Lagrangian spaces given by TL along the boundary of u (relative to a trivialization of u^*TM). The deformation of u as a J -holomorphic map is governed by a Cauchy-Riemann type operator (in the integrable case, an honest $\bar{\partial}$ operator) on the space of sections of u^*TM taking values in u^*TL along the boundary. The index of this operator is $\text{ind}(\bar{\partial}) = d + \mu([u])$, and when it is surjective (i.e., u is *regular*) the space of pseudo-holomorphic maps is locally a smooth manifold of this dimension.

Assume now that L is monotone, and fix a homotopy class $\beta \in \pi_2(M, L)$ with $\mu(\beta) = 2$. We consider the moduli space of J -holomorphic discs with one boundary marked point $1 \in \partial D^2$, i.e. the quotient

$$(8) \quad \mathcal{M}_1(L, \beta, J) = \{u : (D^2, \partial D^2) \rightarrow (M, L) \mid \bar{\partial}_J u = 0, u_*[D^2] = \beta\} / \text{Aut}(D^2, 1).$$

Since $\mu(\beta) = 2$ takes the smallest possible positive value, and the monotonicity of L guarantees that the symplectic area of discs is positively proportional to their Maslov index, discs in the class β have the smallest possible symplectic area. Therefore, bubbling can be excluded *a priori*. Moreover, all J -holomorphic discs in the class β are somewhere injective, and so a generic choice of J ensures their regularity. $\mathcal{M}_1(L, \beta, J)$ is then a smooth compact manifold of dimension $d + \mu(\beta) - 2 = d$.

Fix an orientation and a spin structure on L . The spin structure determines an orientation of $\mathcal{M}_1(L, \beta, J)$ (cf. [8, 5]), and the degree of the evaluation map

$$\begin{aligned} ev : \mathcal{M}_1(L, \beta, J) &\rightarrow L, \\ [u] &\mapsto u(1) \end{aligned}$$

is then a well-defined integer – essentially, a signed count of J -holomorphic discs in the class β whose boundary passes through a given point of L . Moreover, a generic path between two regular almost-complex structures J_0 and J_1 determines an oriented cobordism between $\mathcal{M}_1(L, \beta, J_0)$ and $\mathcal{M}_1(L, \beta, J_1)$, which shows that the degree of the evaluation map is independent of the chosen regular J . We denote its value by $n(L, \beta) \in \mathbb{Z}$.

Definition. We call $n(L, \beta) \in \mathbb{Z}$ the *algebraic count* of pseudo-holomorphic discs in the class β passing through a point of L .

By the same cobordism argument, the algebraic counts $n(L, \beta)$ are invariant under isotopies of L among monotone Lagrangian submanifolds; and they are also invariant under simultaneous deformations of the symplectic form on M and of the Lagrangian submanifold L , as long as convexity at infinity and monotonicity are preserved. Another invariance property concerns symplectomorphisms of M : if $L' = \phi(L)$ for some symplectomorphism ϕ , then $\mathcal{M}_1(L, \beta, J) \simeq \mathcal{M}_1(L', \phi_*\beta, \phi_*J)$, and so (with compatible choices of orientations and spin structures) we have $n(L, \beta) = n(L', \phi_*\beta)$.

As pointed out in the introduction, the reader unwilling to deal with spin structures and orientations of moduli spaces should be content to work with $n(L, \beta) \bmod 2$.

3.2. A monotone torus in X_{red} . Let $T_{std} = \{(z, w), |z| = |w| = 1\}$ be the standard product torus in (\mathbb{C}^2, ω_0) equipped with the standard Kähler form and the standard complex structure. The following is well-known (see e.g. [5]; we sketch the proof for completeness):

Lemma 4. *T_{std} is a monotone Lagrangian torus in (\mathbb{C}^2, ω_0) . There are two families of holomorphic discs of Maslov index 2 with boundary on T_{std} , which can be parametrized by the maps $u_\alpha : z \mapsto (z, e^{i\alpha})$ and $v_\alpha : z \mapsto (e^{i\alpha}, z)$ for $e^{i\alpha} \in S^1$. These discs are all regular, and for a suitable choice of spin structure on T_{std} the algebraic count of discs passing through a point of T_{std} is +1 for each of the two families.*

Proof. The maps u_α and $v_\alpha : (D^2, \partial D^2) \rightarrow (\mathbb{C}^2, T_{std})$ obviously define holomorphic discs. To calculate their Maslov index, we note that the pullback bundle $u_\alpha^*(T\mathbb{C}^2)$ can be identified with the direct sum of two trivial holomorphic line bundles in such a way that, at a point $e^{i\theta} \in \partial D^2$, the pullback of TT_{std} splits into the direct sum of the real lines $\ell_1 = e^{i\theta}\mathbb{R} \subset \mathbb{C}$ in the first factor and $\ell_0 = \mathbb{R} \subset \mathbb{C}$ in the second factor.

Thus, the Maslov index of u_α is equal to the sum of the Maslov indices of the two families of lines ℓ_1 and ℓ_0 in \mathbb{C} , namely $2 + 0 = 2$. Furthermore, the regularity of u_α follows from the surjectivity of the $\bar{\partial}$ operator for complex-valued functions on the disc with boundary conditions in ℓ_1 (resp. ℓ_0) (as follows e.g. from the reflection principle). Similarly for v_α .

To see that these are the only Maslov index 2 discs, we observe that $\beta_1 = [u_\alpha]$ and $\beta_2 = [v_\alpha]$ generate $\pi_2(\mathbb{C}^2, T_{std}) \simeq \pi_1(T_{std}) = \mathbb{Z}^2$, so by linearity the Maslov index of a disc with boundary on T_{std} is equal to twice its algebraic intersection number with the union of the coordinate axes. For holomorphic discs, positivity of intersection implies that a Maslov index 2 disc in (\mathbb{C}^2, T_{std}) intersects only one of the two coordinate axes $z = 0$ and $w = 0$, transversely, and at a single point.

If for example the holomorphic disc $u : (D^2, \partial D^2) \rightarrow (\mathbb{C}^2, T_{std})$ is disjoint from the line $w = 0$, then applying the maximum principle to the projection to the w coordinate, we find that $w \circ u : (D^2, \partial D^2) \rightarrow (\mathbb{C}^*, S^1)$ must take some constant value $e^{i\alpha}$. Meanwhile, the projection to the z coordinate has a single zero of order 1, which means that $z \circ u : (D^2, \partial D^2) \rightarrow (\mathbb{C}, S^1)$ is a biholomorphism from the unit disc to

itself, i.e. the identity map up to reparametrization. Thus u is equivalent to u_α up to reparametrization. Similarly for the other case where the disc is disjoint from $z = 0$ and intersects $w = 0$ once.

Finally, the moduli space $\mathcal{M}_1(L, \beta_1, J_0)$ consists of reparametrizations of the discs u_α , e.g. the maps $z \mapsto (e^{i\beta}z, e^{i\alpha})$ for $(e^{i\beta}, e^{i\alpha}) \in S^1 \times S^1$. Thus $\mathcal{M}_1(L, \beta_1, J_0) \simeq T^2$, and the evaluation map to T_{std} is a diffeomorphism; choosing the “standard” spin structure ensures that this diffeomorphism is orientation-preserving [5], hence $n(L, \beta_1) = +1$. Similarly for the other class β_2 . \square

Next we observe that T_{std} lies away from the complex curve

$$(9) \quad C = h^{-1}(0) = \{(z, w) \in \mathbb{C}^2 \mid cz^n + c^{-1}w - 1 = 0\},$$

and that the disc u_α intersects C transversely at n distinct points, where the z coordinate takes the values

$$z_k = e^{2\pi ik/n} c^{-1/n} (1 - c^{-1}e^{i\alpha})^{1/n},$$

while v_α is disjoint from C .

The regularity of the discs u_α and v_α implies that they deform smoothly under small isotopies of T_{std} . Thus, for small enough values of the constant κ , denoting by ψ_κ the isotopy constructed in Lemma 3, the Lagrangian torus

$$T_{red} = \psi_\kappa(T_{std})$$

in (X_{red}, ω_{red}) again bounds two families of Maslov index 2 holomorphic discs u'_α and v'_α , representing the homotopy classes $\beta'_1 = (\psi_\kappa)_*(\beta_1)$ and $\beta'_2 = (\psi_\kappa)_*(\beta_2)$. We obtain:

Lemma 5. *For $\kappa > 0$ small enough, (X_{red}, ω_{red}) contains a monotone Lagrangian torus T_{red} , disjoint from $C = h^{-1}(0)$, which bounds exactly two families of Maslov index 2 holomorphic discs, representing classes β'_1, β'_2 that span $\pi_2(X_{red}, T_{red}) \simeq \mathbb{Z}^2$. These discs are all regular, and for a suitable spin structure their algebraic counts are $n(T_{red}, \beta'_1) = n(T_{red}, \beta'_2) = +1$. Moreover, the discs in the class β'_1 intersect C transversely in n distinct points, while those in the class β'_2 are disjoint from C .*

Remark. While ω_{red} is singular along C , it can still be integrated over a disc that intersects C transversely, so the notion of monotonicity still makes sense. In fact, symplectic area can also be defined as the integral of the Liouville form

$$\theta_{red} = d^c\left(\frac{1}{4}|z|^2 + \frac{1}{4}|w|^2 + \frac{\kappa}{2}|h|\right)$$

along the boundary of a disc. Perhaps even better, we can modify ω_{red} in a neighborhood of C (disjoint from T_{red}) by a small exact deformation so as to cure its lack of smoothness; this can be achieved simply by replacing $|h|$ by a smooth function $\rho(|h|)$ in the expression for the Kähler potential (taking $\rho : [0, \infty) \rightarrow [0, \infty)$ to be any smooth, convex function which agrees with identity outside of $[0, \epsilon]$ and has vanishing odd derivatives at the origin). This modification does not affect the properties of the isotopy ψ_κ away from C , nor the symplectic areas of holomorphic discs.

Proof of Lemma 5. The existence and regularity for small κ of the two families of holomorphic discs u'_α and v'_α with boundary on $T_{red} = \psi_\kappa(T_{std})$ representing the classes β'_1 and β'_2 , obtained as smooth deformations of the discs u_α and v_α under the isotopy, is a direct consequence of the regularity of the latter discs.

Since the isotopy is exact ($\psi_\kappa^*(\theta_{red})$ agrees with the standard Liouville form θ_0 up to an exact term), the symplectic areas of the discs are preserved, which proves the monotonicity of T_{red} . Moreover, Gromov compactness implies that T_{red} does not bound any other Maslov index 2 holomorphic discs: if such discs existed for arbitrarily small κ , taking the limit of a subsequence with $\kappa \rightarrow 0$ would yield a contradiction.

Finally, because the discs u_α and v_α deform smoothly under the isotopy of T_{std} to T_{red} , for small κ the discs u'_α and v'_α continue to intersect C transversely, and the algebraic counts remain unchanged (in fact the evaluation maps $ev : \mathcal{M}_1(T_{red}, \beta'_i, J_0) \rightarrow T_{red}$ remain diffeomorphisms). \square

3.3. A monotone torus in X . From now on we fix the value of the constant $\kappa > 0$ so that the conclusion of Lemma 5 holds. We then construct a Lagrangian torus T in (X, ω_X) by lifting T_{red} to $\mu_X^{-1}(0)$:

Definition. We denote by T the preimage of T_{red} under the projection map from $\mu_X^{-1}(0) \subset X$ to X_{red} , i.e.

$$(10) \quad T = \{(x, y, z, w) \in X \mid (z, w) \in T_{red} \text{ and } |x| = |y|\}.$$

We also denote by $\pi : X \rightarrow X_{red}$ the projection to the (z, w) coordinates,

$$(11) \quad \pi(x, y, z, w) = (z, w).$$

Lemma 6. T is a monotone Lagrangian torus in (X, ω_X) .

Conceptually, this follows from the observation that T is the image of T_{red} under the monotone Lagrangian correspondence between X_{red} and X induced by $\mu_X^{-1}(0)$. A more elementary argument is as follows.

Proof. Since the restriction of ω_X to $\mu_X^{-1}(0)$ agrees with the pullback of ω_{red} via the projection map π , $\omega_X|_T$ is the pullback of $\omega_{red}|_{T_{red}}$ under the projection from $T \subset \mu_X^{-1}(0)$ to $T_{red} \subset X_{red}$, i.e. it vanishes, and T is Lagrangian.

Let $u : (D^2, \partial D^2) \rightarrow (X, T)$ be a disc with boundary on T (not necessarily holomorphic), and denote by $\gamma : S^1 \rightarrow T$ its boundary loop. Perturbing u if necessary, we can assume that it avoids the fixed point set $F = \{x = y = 0\}$ (which has real codimension 4). In terms of the Liouville form

$$(12) \quad \theta_X = d^c\left(\frac{1}{4}|z|^2 + \frac{1}{4}|w|^2 + \frac{\kappa}{4}|x|^2 + \frac{\kappa}{4}|y|^2\right),$$

the symplectic area of u is given by the integral of θ_X along the boundary loop γ . However, along $\mu_X^{-1}(0)$ we have $|x|^2 = |y|^2 = |h|$, and $|x|^2 + |y|^2$ achieves its fiber-wise minimum so its derivative vanishes in all directions tangent to the fibers of π .

Therefore, at every point of $\mu_X^{-1}(0)$ the 1-form θ_X coincides with

$$\pi^*\theta_{red} = d^c\left(\frac{1}{4}|z|^2 + \frac{1}{4}|w|^2 + \frac{\kappa}{2}|h|\right).$$

Denoting by $u_{red} = \pi \circ u : (D^2, \partial D^2) \rightarrow (X_{red}, T_{red})$ and $\gamma_{red} = \pi \circ \gamma : S^1 \rightarrow T_{red}$ the projections of u and γ , we conclude that

$$(13) \quad \int_{D^2} u^*\omega_X = \int_{S^1} \gamma^*\theta_X = \int_{S^1} \gamma^*(\pi^*\theta_{red}) = \int_{S^1} \gamma_{red}^*(\theta_{red}) = \int_{D^2} u_{red}^*(\omega_{red}),$$

i.e. the disc u and its projection u_{red} have the same symplectic areas. Meanwhile, away from the fixed point locus F , denote by

$$(14) \quad \mathcal{L}_{\mathbb{R}} = \mathbb{R} \cdot (ix, -iy, 0, 0) \quad \text{and} \quad \mathcal{L} = \mathbb{C} \cdot (ix, -iy, 0, 0)$$

the real and complex spans of the vector field generating the S^1 -action. Then \mathcal{L} is a trivial holomorphic subbundle of TX , and $TX/\mathcal{L} \simeq \pi^*TX_{red}$, i.e. away from F we have a short exact sequence of holomorphic vector bundles

$$(15) \quad 0 \longrightarrow \mathcal{L} \longrightarrow TX \xrightarrow{d\pi} \pi^*TX_{red} \longrightarrow 0.$$

Along T , we have a similar short exact sequence of real subbundles,

$$(16) \quad 0 \longrightarrow \mathcal{L}_{\mathbb{R}} \longrightarrow TT \xrightarrow{d\pi} \pi^*TT_{red} \longrightarrow 0.$$

Since the trivial subbundles $(u^*\mathcal{L}, \gamma^*\mathcal{L}_{\mathbb{R}})$ do not contribute to the Maslov index, $\mu([u])$ can be computed by considering the quotient bundles $(u^*(TX/\mathcal{L}), \gamma^*(TT/\mathcal{L}_{\mathbb{R}})) \simeq (u_{red}^*(TX_{red}), \gamma_{red}^*(TT_{red}))$. In other terms,

$$(17) \quad \mu([u]) = \mu([u_{red}]).$$

Comparing (13) and (17), we find that the proportionality between Maslov index and symplectic area for discs in X_{red} with boundary on T_{red} implies the same proportionality for discs in X with boundary on T . \square

Lemma 7. *The projection $u_{red} = \pi \circ u : (D^2, \partial D^2) \rightarrow (X_{red}, T_{red})$ of a holomorphic disc $u : (D^2, \partial D^2) \rightarrow (X, T)$ is a holomorphic disc, and $\mu([u_{red}]) = \mu([u])$.*

Conversely, let $u_{red} : (D^2, \partial D^2) \rightarrow (X_{red}, T_{red})$ be a holomorphic disc that intersects $C = h^{-1}(0)$ transversely in k points, and fix a point $p_0 \in T$ such that $\pi(p_0) = u_{red}(1)$. Then there are exactly 2^k holomorphic discs $u : (D^2, \partial D^2) \rightarrow (X, T)$ such that $\pi \circ u = u_{red}$ and $u(1) = p_0$. Moreover, if u_{red} is regular then all these discs are regular.

Proof. The first statement follows immediately from the holomorphicity of π and the Maslov index calculation in the proof of Lemma 6 (equation (17)).

For the second part, let u_{red} be a holomorphic disc in X_{red} that intersects C transversely, with $u_{red}^{-1}(C) = \{t_1, \dots, t_k\} \subset D^2$, and let u be a lift of u_{red} to a disc in X with boundary on T . Along the holomorphic disc u , the product $xy = h(z, w)$ has simple zeroes at t_1, \dots, t_k , i.e. u intersects $\pi^{-1}(C) = \{x = 0\} \cup \{y = 0\}$ transversely at the k points $u(t_1), \dots, u(t_k)$. The quotient $q = x/y$ then defines a meromorphic function on the disc, which has either a simple zero or a simple pole at each of t_1, \dots, t_k , and

no other zeroes or poles. Moreover, on the boundary we have $|x| = |y|$, so q maps the unit circle to itself.

Given any function $\varepsilon : \{1, \dots, k\} \rightarrow \{\pm 1\}$, set

$$(18) \quad \vartheta_\varepsilon(z) = \prod_{j=1}^k \left(\frac{z - t_j}{1 - \overline{t_j}z} \right)^{\varepsilon(j)},$$

which is a meromorphic function on the unit disc, mapping the unit circle to itself, and with simple zeroes (resp. poles) at all t_j such that $\varepsilon(j) = +1$ (resp. -1).

Thus, choosing $\varepsilon(j) = \text{ord}_{t_j}(q)$ according to the poles and zeroes of $q = x/y$ along the disc u , we find that ϑ_ε and q have the same zeroes and poles on the unit disc, and their ratio defines a nowhere vanishing holomorphic function on the unit disc, taking values in the unit circle at the boundary. By the maximum principle this function is constant, i.e. there exists $e^{i\theta} \in S^1$ such that $q = e^{i\theta}\vartheta_\varepsilon$.

By construction the holomorphic functions $(h \circ u_{red})\vartheta_\varepsilon^{\pm 1}$ only have double zeroes, and so we can choose square roots

$$\zeta_\pm = ((h \circ u_{red})\vartheta_\varepsilon^{\pm 1})^{1/2},$$

with $\zeta_+/\zeta_- = \vartheta_\varepsilon$ and $\zeta_+\zeta_- = h \circ u_{red}$. We obtain that along the disc u the coordinates x and y are given by

$$x = e^{i\theta/2}\zeta_+ \quad \text{and} \quad y = e^{-i\theta/2}\zeta_-,$$

for some $e^{i\theta/2} \in S^1$. Conversely, these formulas determine holomorphic lifts of u_{red} for all $\varepsilon : \{1, \dots, k\} \rightarrow \{\pm 1\}$ and for all $e^{i\theta/2} \in S^1$, and the condition that $u(1) = p_0$ determines the normalization factor $e^{i\theta/2}$ uniquely for given ε . Hence there are 2^k lifts of u_{red} as claimed, determined by the choice of whether x or y vanishes at each point where u_{red} intersects C .

Finally, we note that none of the lifts u pass through the fixed point locus of the S^1 -action (since x and y do not vanish simultaneously). Thus, pulling back the exact sequences (15) and (16) along u , we find that the holomorphic vector bundle u^*TX admits a trivial holomorphic line subbundle $u^*\mathcal{L}$, with a trivial real subbundle at the boundary $u^*_{|S^1}\mathcal{L}_\mathbb{R}$. Since the $\bar{\partial}$ operator for complex-valued functions on the disc with the trivial real boundary condition $\mathbb{R} \subset \mathbb{C}$ on the unit circle is surjective, the surjectivity of the $\bar{\partial}$ operator on sections of u^*TX with boundary conditions $u^*_{|S^1}(TT)$ is equivalent to that of the $\bar{\partial}$ operator on the quotient bundle $u^*TX/u^*\mathcal{L} \simeq u^*_{red}TX_{red}$ with boundary conditions $u^*_{|S^1}(TT)/u^*_{|S^1}(\mathcal{L}_\mathbb{R}) \simeq u^*_{red|S^1}(TT_{red})$. Thus, the regularity of u is equivalent to that of u_{red} as claimed. \square

Corollary 8. *There are $n + 2$ distinct Maslov index 2 classes in $\pi_2(X, T)$ for which the algebraic count of pseudo-holomorphic discs is non-zero, and for a suitable choice of spin structure the sum of these counts is $2^n + 1$.*

Proof. By Lemma 7, the holomorphic discs of Maslov index 2 bounded by T are lifts of those bounded by T_{red} in X_{red} , which are determined by Lemma 5.

The discs representing the class $\beta'_2 \in \pi_2(X_{red}, T_{red})$ are disjoint from C , hence they admit a unique lift up to the S^1 -action. Denoting by $\hat{\beta}_2 \in \pi_2(X, T)$ the class of these lifts, the moduli space $\mathcal{M}_1(T, \hat{\beta}_2, J_0)$ is an S^1 -bundle over $\mathcal{M}_1(T_{red}, \beta'_2, J_0)$, and the evaluation map to T is equivariant with respect to the S^1 -action; thus the evaluation map $ev : \mathcal{M}_1(T, \hat{\beta}_2, J_0) \rightarrow T$ is again a diffeomorphism, and its degree is ± 1 .

Meanwhile, the discs representing the class $\beta'_1 \in \pi_2(X_{red}, T_{red})$ intersect C transversely in n points (cf. Lemma 5), so by Lemma 7 they can be lifted in 2^n different ways up to the S^1 -action. Observe that elements of $\pi_2(X, T) \simeq \mathbb{Z}^3$ are determined by their intersection numbers with the three hypersurfaces $x = 0$, $z = 0$, and $w = 0$. Thus, the lifts live in $n + 1$ different classes $\hat{\beta}_{1,\ell} \in \pi_2(X, T)$, $\ell = 0, \dots, n$, depending on the intersection number of the lifted disc with the hypersurface $x = 0$; each value of ℓ is achieved by $\binom{n}{\ell}$ of the 2^n lifts. The moduli space $\mathcal{M}_1(T, \hat{\beta}_{1,\ell}, J_0)$ then projects to $\mathcal{M}_1(T_{red}, \beta'_1, J_0)$ with fiber a union of $\binom{n}{\ell}$ circles. The evaluation map $ev : \mathcal{M}_1(T, \hat{\beta}_{1,\ell}, J_0) \rightarrow T$ is thus an unramified $\binom{n}{\ell}$ -sheeted covering.

To determine the orientations, we briefly recall the construction in [8, Chapter 8] (see also [5, Prop. 5.2] for a simpler presentation that suffices for the case at hand). A spin structure on T determines a trivialization of its tangent bundle along the boundary of a holomorphic disc u . Using this trivialization, the $\bar{\partial}$ operator can be deformed to the direct sum of a complex linear operator and a $\bar{\partial}$ operator for sections of a trivialized complex vector bundle with trivial real boundary condition (namely, the tangent bundles to X and T along the boundary of u , with the trivialization determined by the spin structure). Since the kernel of the latter operator can be identified with the tangent space to T at the marked point, an orientation of T then determines an orientation of the tangent space to the moduli space at u .

In our case, we choose the spin structure on T to be standard along the orbits of the S^1 -action and consistent under the splitting (16) with that previously chosen on T_{red} . Thus, the preferred trivialization of TT along the boundary of a holomorphic disc u agrees with that induced via (16) by the trivialization of TT_{red} along the boundary of $u_{red} = \pi \circ u$ and the natural trivialization of the trivial line bundle $\mathcal{L}_{\mathbb{R}}$. The orientation at u of the moduli space of holomorphic discs in (X, T) then agrees with that induced by the orientation at u_{red} of the moduli space of holomorphic discs in (X_{red}, T_{red}) and the chosen orientation of the orbits of the S^1 -action. With this understood, the orientation-preserving nature of the evaluation maps for discs in (X_{red}, T_{red}) implies that the evaluation maps for discs in (X, T) are also orientation-preserving, i.e. the degrees are positive. \square

(For the reader working mod 2, we note that the odd values of $n(T, \beta)$ are achieved for $\hat{\beta}_2$ and those $\hat{\beta}_{1,\ell}$ for which $\binom{n}{\ell}$ is odd, including the extremal cases $\hat{\beta}_{1,0}$ and $\hat{\beta}_{1,n}$.)

4. PROOF OF THEOREM 1

In light of Corollary 8 and the invariance properties of the algebraic counts $n(T, \beta)$, the only thing that remains to be done is to construct an isotopy between the Kähler form ω_X on (a bounded subset of) $X \simeq \mathbb{C}^3$ and the standard Kähler form. We will again rely on Moser's trick (Lemma 2). We denote by

$$(19) \quad \Phi_1 = \frac{\kappa}{4}|x|^2 + \frac{\kappa}{4}|y|^2 + \frac{1}{4}|z|^2$$

the Kähler potential for the standard (up to rescaling) Kähler form on \mathbb{C}^3 ,

$$\omega_1 = dd^c \Phi_1 = \frac{i}{2} dz \wedge d\bar{z} + \kappa \left(\frac{i}{2} dx \wedge d\bar{x} + \frac{i}{2} dy \wedge d\bar{y} \right).$$

The Kähler potential for ω_X is

$$\Phi_X = \Phi_1 + \frac{1}{4}|w|^2,$$

where we recall that w is determined as a function of the coordinates (x, y, z) by

$$(20) \quad w = c(xy + 1) - c^2 z^n.$$

The estimate that ensures the existence of the Moser flow is the following:

Lemma 9. *Given any bounded subset $B \subset \mathbb{C}^3$, there exist positive constants C and M such that the real-valued function $\varphi = C\Phi_1 - \Phi_X$ is bounded above by M on B , and the connected component Ω of $\varphi^{-1}((-\infty, M])$ which contains B is compact.*

Proof. We equip \mathbb{C}^3 with the Euclidean metric for which the positive definite quadratic form Φ_1 is the square of the distance to the origin (i.e., a rescaling of the usual metric).

Let $R > 0$ be such that B is contained within the ball $B(0, R)$ of radius R (for this metric), denote by K the supremum of $\frac{1}{4}|w|^2/\Phi_1$ in $B(0, 2R) \setminus B(0, R)$, and set $C = 2K + 1$. Then in $B(0, 2R) \setminus B(0, R)$ we have

$$K\Phi_1 \leq \varphi = (C - 1)\Phi_1 - \frac{1}{4}|w|^2 \leq 2K\Phi_1,$$

and the upper bound continues to hold inside $B(0, R)$.

Then inside $B(0, R)$ we have $\varphi \leq 2K\Phi_1 \leq 2KR^2$, while in $B(0, 2R) \setminus B(0, \sqrt{3}R)$ we have $3KR^2 \leq K\Phi_1 \leq \varphi$. Thus, setting $M = \frac{5}{2}KR^2$, there is a connected component Ω of $\varphi^{-1}((-\infty, M])$ for which $B(0, R) \subset \Omega \subset B(0, \sqrt{3}R)$. \square

Choosing B to be a polydisc in \mathbb{C}^3 large enough to contain T , and taking C as in Lemma 9, we now apply Lemma 2 to the Kähler forms ω_X and $C\omega_1$, to construct an exact isotopy ψ_t such that $\psi_1^*(C\omega_1) = \omega_X$. Because the isotopy is generated by the negative gradient of $\varphi = C\Phi_1 - \Phi_X$ (with respect to a varying family of Kähler metrics), the values of φ decrease along the flow. Thus, the compact subset $\Omega \supset B$ constructed in Lemma 9 is preserved, so the isotopy is well-defined everywhere in it, and in particular in B .

Since the isotopy is exact, $\psi_t(T)$ is a monotone Lagrangian torus in \mathbb{C}^3 equipped with the Kähler form $\omega_t = Ct\omega_1 + (1 - t)\omega_X$, and the algebraic counts of Maslov

index 2 holomorphic discs remain constant along the isotopy. For $t = 1$ we obtain a monotone Lagrangian torus in $(\mathbb{C}^3, C\omega_1)$ with the desired properties. Rescaling the coordinate axes by suitable constant factors, we obtain a monotone Lagrangian torus in \mathbb{C}^3 equipped with the standard Kähler form, and by further rescaling we obtain tori with arbitrary monotonicity constants and the same algebraic counts of pseudo-holomorphic discs.

5. COMMENTS ON THE CONSTRUCTION

Our construction is inspired by ideas from mirror symmetry, and more precisely the Strominger-Yau-Zaslow (SYZ) conjecture, whereby the mirror of a given Kähler manifold is constructed geometrically from a Lagrangian torus fibration on the complement of a complex hypersurface. The numbers of Maslov index 2 discs bounded by the fibers exhibit discontinuities across a set of *walls* which separate the fibration into *chambers*, each with its own enumerative behavior; each chamber corresponds to a distinguished coordinate chart on the mirror (cf. [1, §2] and [2]).

In a given Lagrangian fibration, the vast majority of fibers are not monotone, and the counts of Maslov index 2 discs are not invariant under Hamiltonian isotopies. However, by deforming the fibration suitably it is often possible to arrange the existence of a monotone fiber in any given chamber. For example, the complement of a smooth cubic in $\mathbb{C}\mathbb{P}^2$ admits a Lagrangian torus fibration with 3 singular fibers and infinitely many chambers; Vianna's constructions in [12, 13] can be understood as modifying the fibration to place the monotone fiber in a prescribed chamber.

The construction of Theorem 1 relies on the fact that \mathbb{C}^3 can be presented as a conic bundle $\{xy = h(z, w)\}$ over \mathbb{C}^2 with a discriminant curve $h^{-1}(0) \subset \mathbb{C}^2$ of arbitrarily large degree. SYZ mirror symmetry for conic bundles over toric varieties has been studied in detail in [1], where it was shown that the chamber structure is governed by the tropical geometry of $h^{-1}(0)$ (or, in more classical terms, by the various manners in which product tori in \mathbb{C}^2 can be linked with $h^{-1}(0)$). Thus, by increasing the degree of h we can exhibit Lagrangian torus fibrations on open dense subsets of \mathbb{C}^3 (namely, those points where z and w are nonzero) with arbitrarily many chambers. Choosing the coefficients of h suitably ensures the existence of monotone fibers in the “most interesting” chamber. (In fact, choosing h to be analytic rather than algebraic one could obtain a single fibration with infinitely many chambers, with monotone representatives corresponding to all the values of n in our main construction at once.)

Another perspective on the construction comes from singularity theory: projecting the conic bundle $X \simeq \mathbb{C}^3$ to the coordinate w presents it as an unfolding of the A_{n-1} singularity $xy = cz^n$. The A_{n-1} Milnor fiber contains non-displaceable monotone Lagrangian tori (cf. [1, Corollary 9.1] and [11]). The examples of Theorem 1 can be obtained by transporting these tori along a circle in the w coordinate; even though

the unfolding makes the ambient manifold contractible and the tori displaceable, the distinctive enumerative features of the tori in the fibers persist.

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