Two new constructions of monotone Lagrangian tori

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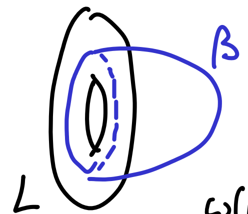
after: D. A., arXiv: 1407.3725
R. Vianna, arXiv: 1409.2850

Question: monotone Lagrangian tori in $\mathbb{R}^{2n}$, $\mathbb{CP}^n$, ...
up to Hamiltonian isotopy?

Monotone:

$L \subset (M, \omega)$
orientable Lagrangian



$\beta \in \pi_2(M, L)$

$\omega(\beta) = \int_\beta \omega \in \mathbb{R}$
symplectic area

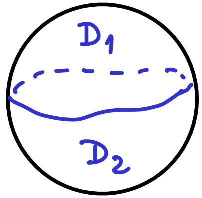
$\mu(\beta) \in 2\mathbb{Z}$
Maslov index

($\mu(\beta) =$ class of $TL_{|\partial\beta}$ (loop of Lagr. subspaces)
in $\pi_1 \, LagGr(\mathbb{R}^{2n}) \simeq \mathbb{Z}$)

Def: L is monotone if $\exists K > 0$ s.t. $\forall \beta \in \pi_2(M, L)$,

$$\omega(\beta) = K \cdot \mu(\beta)$$

- **Example:**



$\gamma \subset S^2$

γ monotone iff $\text{area}(D_1) = \text{area}(D_2)$

$(\mu(D_1) = \mu(D_2) = 2)$.

Unique up to Hamiltonian isotopy.

- **Example:** $S^1(r_1) \times S^1(r_2) \subset (\mathbb{R}^4, \omega_0)$ monotone $\Leftrightarrow r_1 = r_2$.

  ($\mu = 2$ discs $D^2(r_1) \times \{pt\}$, $\{pt\} \times D^2(r_2)$).

- **Thm** (Chekanov, early 90's)

  ‖ $\exists$ monotone $T \subset \mathbb{R}^4$ not Hamiltonian isotopic to a product torus.

Also in $\mathbb{CP}^2$, $S^2 \times S^2$, ... (many constructions)

- Chekanov-Schlenk's monotone twist tori generalize this, giving more examples in $\mathbb{R}^{2n}$ and $\mathbb{CP}^n$ (finitely many for fixed n)

- **Thm 1:** (D.A.) $(\mathbb{R}^6, \omega_0)$ contains infinitely many different monotone Lagrangian tori (not related by Ham. isotopy + scaling)

- **Thm 2:** (R. Vianna) $\mathbb{CP}^2$ contains infinitely many different monotone Lagrangian tori (not related by Ham. isotopy)

(also: Galkin & Mikhalkin, in preparation)

These are all Lagrangian isotopic to product tori (via non-monotone tori)
(however, Evans & Rizell: $\exists$ top. knotted example in $\mathbb{R}^8$)

The invariant (after Eliashberg - Polterovich):

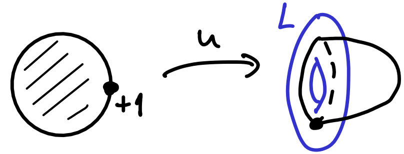
Counts of pseudo-holomorphic Maslov index 2 discs in (M,L).

$$n(L,\beta) \in \mathbb{Z} \qquad \forall \beta \in \pi_2(M,L),\ \mu(\beta)=2.$$

Pick J compatible almost-complex structure.

$$\mathcal{M}_1(L,\beta,J) = \left\{ u : (D^2,\partial) \to (M,L) \;\middle/\; \begin{array}{l} \bar\partial_J u = 0 \\ [u]=\beta \end{array} \right\} \Big/ \mathrm{Aut}(D^2,+1).$$

$$\begin{array}{ccc} \mathcal{M}_1(L,\beta,J) & & u \\ \downarrow \mathrm{ev} & & \downarrow \\ L & & u(+1) \end{array}$$



For generic J, all discs in class β are regular, and $\mathcal{M}_1(L,\beta,J)$ is a smooth manifold of $\dim_{\mathbb{R}} = n-2+\mu(\beta)=n$, compact without boundary

L oriented spin $\Rightarrow \mathcal{M}_1(L,\beta,J)$ oriented.

(L monotone, $\mu(\beta)=2$ $\Rightarrow \omega(\beta)>0$ minimal)

The invariant (after Eliashberg - Polterovich):

Counts of pseudo-holomorphic Maslov index 2 discs in (M,L).

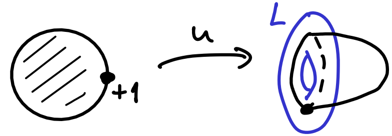
$$n(L,\beta) \in \mathbb{Z} \qquad \forall \beta \in \pi_2(M,L),\ \mu(\beta)=2.$$

L monotone, oriented, spin ; J regular

$$\mathcal{M}_1(L,\beta,J) = \left\{ u : (D^2,\partial) \to (M,L) \;\middle/\; \begin{matrix} \bar\partial_J u = 0 \\ [u]=\beta \end{matrix} \right\} \Big/ \mathrm{Aut}(D^2,+1).$$

compact oriented n-manifolds

$$\begin{array}{ccc} \mathcal{M}_1(L,\beta,J) & & u \\ \big\downarrow \mathrm{ev} & & \downarrow \\ L & & u(+1) \end{array}$$



$\rightsquigarrow \quad n(L,\beta) := \deg(\mathrm{ev}) \in \mathbb{Z}.$

Indep$^{\underline{t}}$ of J & invariant under monotone deformations (by cobordism argument)

M	L	$\#\{\beta \,/\, n(\beta) \neq 0\}$	$\sum_\beta n(\beta)$
$\mathbb{R}^4$	$S^1 \times S^1$	2	2
	Chekanov	1	1

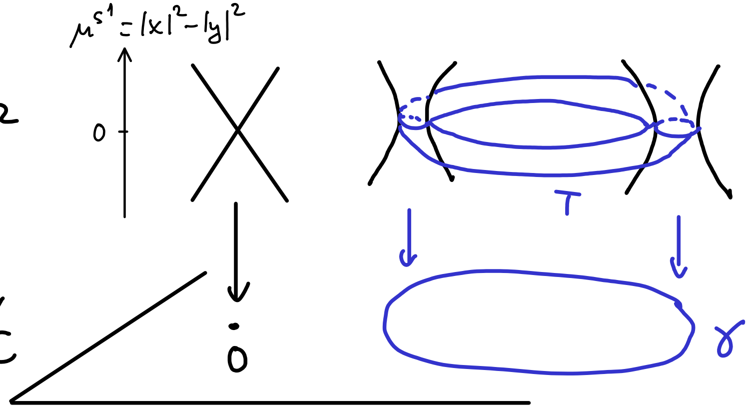
$\beta_1 : D^2 \times pt \qquad n(\beta_1) = 1$

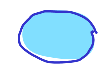
$\beta_2 : pt \times D^2 \qquad n(\beta_2) = 1.$

Chekanov torus: $T = \{ (x,y) \in \mathbb{C}^2 \,/\, |x| = |y| \text{ and } xy \in \gamma \}$

γ: loop in $\mathbb{C}$ not enclosing 0.

$$\begin{array}{c} (x,y) \in \mathbb{C}^2 \\ \downarrow \\ xy \in \mathbb{C} \end{array}$$



$\mu = 2$ holom. discs $=$ sections over  γ

Necess $y/x = $ const.

$\rightsquigarrow$ 1 family, $n(\beta) = 1$.

What if γ encloses 0 ? (Then $\sim$ product torus; 2 classes of sections : x or y has a simple zero.)

M	L	$\#\{\beta / n(\beta) \neq 0\}$	$\sum_\beta n(\beta)$
$\mathbb{R}^4$	$S^1 \times S^1$	2	2
	Chekanov	1	1
$\mathbb{CP}^2$	T_{Clifford}	3	3
	T_{Chekanov}	4	5
$\mathbb{R}^6$	$S^1 \times S^1 \times S^1$	3	3
	$S^1 \times$ Chekanov	2	2
	Chekanov-Schlenk	1	1

M	L	$\#\{\beta / n(\beta) \neq 0\}$	$\sum_\beta n(\beta)$
$\mathbb{R}^4$	$S^1 \times S^1$	2	2
	Chekanov	1	1
$\mathbb{CP}^2$	$T_{\text{Clifford}} = T(1,1,1)$	3	3
	$T_{\text{Chekanov}} = T(1,1,4)$	4	5
Vianna $\{$	$T(1,4,25)$	10	41
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$\mathbb{R}^6$	$n=1$? $S^1 \times S^1 \times S^1$	3	3
	$n=0$ $S^1 \times$ Chekanov	2	2
	Chekanov-Schlenk	1	1
D.A.	T_n $(n \geq 0)$	$n+2$	2^n+1

Question: are these the only ones?

Conj: Vianna's $T(a^2, b^2, c^2)$ are the only ones

Remark: definitely not a complete list, e.g. can lift Vianna's tori to $S^5 \subset \mathbb{R}^6$.

Construction: $(n \geq 0)$

$$X = \{(x,y,z,w) \in \mathbb{C}^4 / xy = \underbrace{10z^n + \frac{w}{10} - 1}_{=: h(z,w)}\}$$

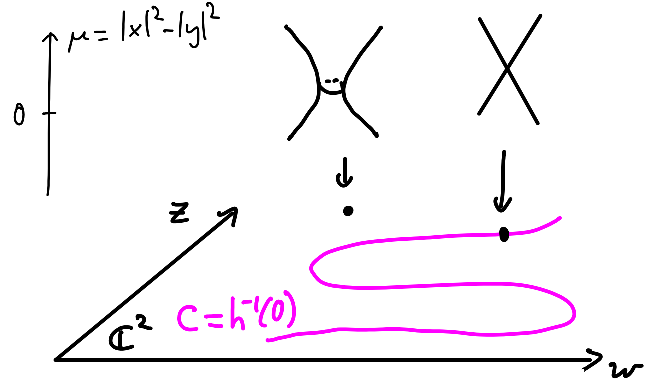
(with restriction of ω_0)

- $X \underset{\text{holom.}}{\simeq} \mathbb{C}^3_{x,y,z}$
- X Stein deform. $\simeq (\mathbb{C}^3, \omega_0)$
  so Lags. in X give Lags. in $\mathbb{C}^3$
- S^1 acts by $(e^{i\theta}x, e^{-i\theta}y, z, w)$,
  reduced spaces

$$X_{red} = \mu^{-1}(\lambda)/S^1 \underset{\text{holom.}}{\simeq} \mathbb{C}^2_{z,w}$$

$(\omega_{red} \neq \omega_0$ but deform. equiv.$)$

$\triangle$ for $\lambda = 0$, ω_{red} singular along $C = h^{-1}(0)$



$\Rightarrow$ build monotone Lagr. tori in X by lifting monotone tori in X_{red} to $\mu^{-1}(0)$.

Construction: $(n \geq 0)$

$$X = \{(x,y,z,w) \in \mathbb{C}^4 \,/\, xy = \underbrace{10z^n + \frac{w}{10} - 1}_{=: h(z,w)}\}$$

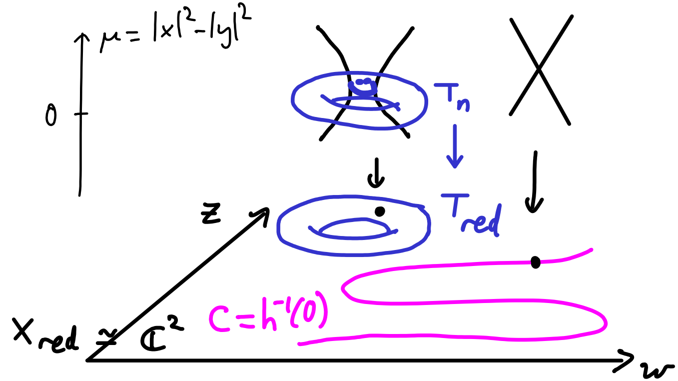
(with restriction of ω_0)

- $T_{red} \subset X_{red} = \mu^{-1}(0)/S^1, \ \omega_{red}$

  $\wr\!\wr \qquad\qquad \wr\!\wr$ away from C

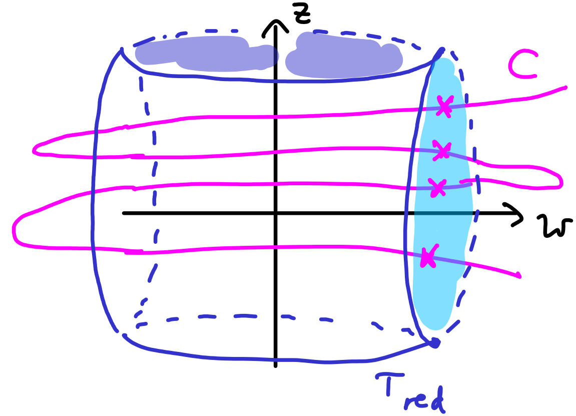
  $S^1 \times S^1 \subset (\mathbb{C}^2_{zw}, \omega_0)$

  monotone Lagr. torus in X_{red} disjoint from C



- $T_n := \{(x,y,z,w) \in X \,/\, |x| = |y|, \ (z,w) \in T_{red}\}$ monotone Lagr. torus in X.

- $T_{red} \subset X_{red}$ ($\sim S^1 \times S^1 \subset \mathbb{C}^2$) is disjoint from C : $10z^n + \frac{1}{10}w - 1 = 0$



- T_{red} bounds 2 families of $\mu = 2$ holom. discs
  - 1st family: all discs are disjoint from C
  - 2nd family: all discs $\cap$ C transversely at n distinct points

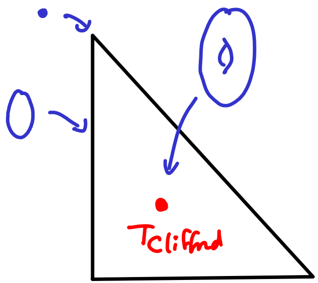
- Holomorphic discs in (X, T_n) project to holom. discs in (X_{red}, T_{red}) of the same Maslov index!

Discs in 1st family: unique lift up to S^1-action (y/x = constant)
2nd family: 2^n lifts in $(n+1)$ different classes
(xy has n simple zeros, choose if x or y vanishes at each point)

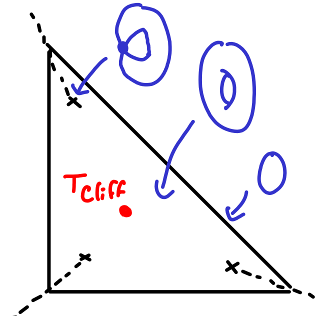
$\rightsquigarrow \sum n(\beta) = 2^n + 1, \ \#\beta = n+2.$

Tori in $\mathbb{CP}^2$: "semi-toric integrable systems" (Symington)

[Vianna]

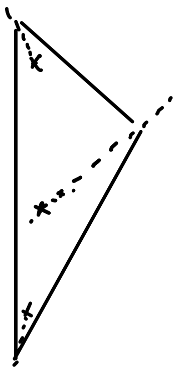


$\longleftrightarrow$ deform toric fibration

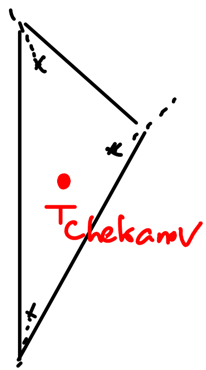


singular Lag. T^2 fibration with 3 nodal fibers (monodromy! $\to$ cuts in affine structure)

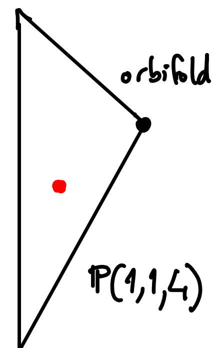
$\longleftrightarrow$ redraw the cut



$\longleftrightarrow$ deform



($\rightsquigarrow$ degenerates to



- Hacking-Prokhorov have classified algebraic toric degenerations of $\mathbb{CP}^2$:

$$\mathbb{P}(a^2,b^2,c^2) = \mathbb{C}^3 - \{0\} \Big/ (\lambda^{a^2}x, \lambda^{b^2}y, \lambda^{c^2}z) \sim (x,y,z)$$

for all Markov triples (a,b,c) st. $a^2+b^2+c^2 = 3abc$.

"Mutation": $(a,b,c) \leftrightsquigarrow (a,b,c' = 3ab-c)$

$\rightsquigarrow (a^2,b^2,c^2) = (1,1,1), (1,1,4), (1,4,25), (1,25,169), \ldots$

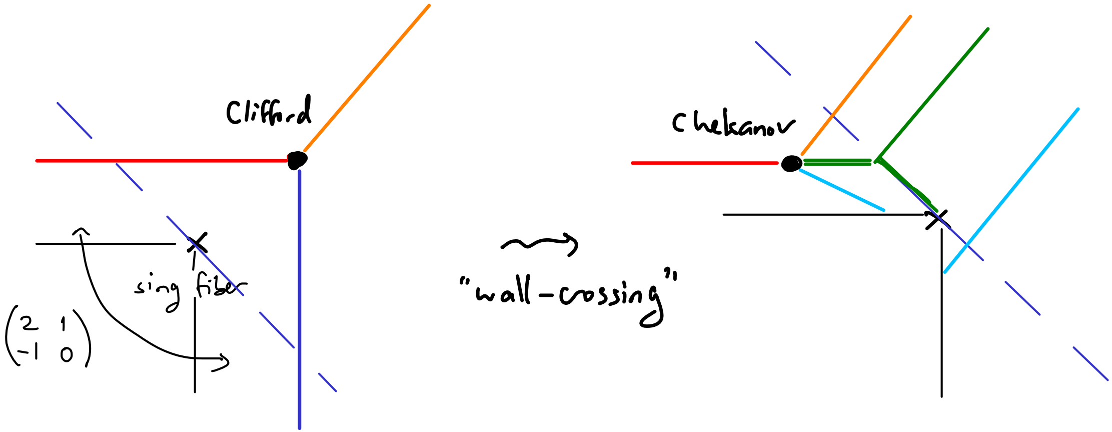
- Semi-toric bases for $\mathbb{CP}^2$ (after moving cuts etc) $\equiv$ "smoothings" of moment polytopes for $\mathbb{P}(a^2,b^2,c^2)$.

- Monotone T^2-orbit in $\mathbb{P}(a^2,b^2,c^2) \overset{\text{smoothing}}{\rightsquigarrow}$ monotone $T(a^2,b^2,c^2) \subset \mathbb{CP}^2$

$T(1,1,1) =$ Clifford, $T(1,1,4) =$ Chekanov, all others are new & distinct.

(Vianna)

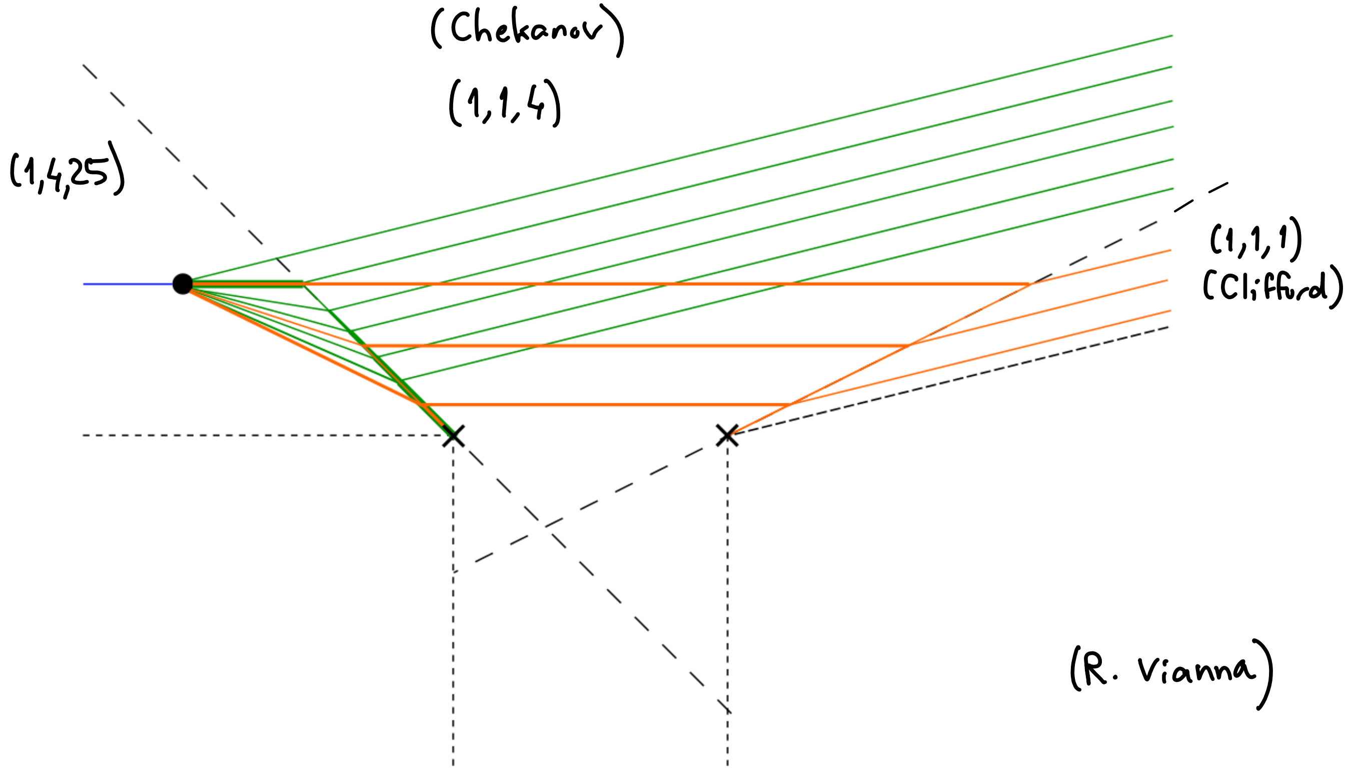
- Tropical geometry can be used to predict the $n(\beta)$'s and their wall-crossing behavior. E.g. for Chekanov vs. Clifford:



3 tropical $\mu=2$ discs

4 tropical $\mu=2$ discs

Typically have arbitrarily many ($X \simeq \mathbb{C}^3$) or infinitely many ($\mathbb{CP}^2$) chambers w/ different $n(\beta)$'s. Deform so the monotone fiber lies in desired chamber.


(Chekanov)
(1,1,4)
(1,4,25)
(1,1,1)
(Clifford)
(R. Vianna)