Symplectic 4-manifolds, mapping class group factorizations, and fiber sums of Lefschetz fibrations

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Symplectic 4-manifolds

A (compact) symplectic 4-manifold (M^4, ω) is a smooth 4-manifold with a symplectic form $\omega \in \Omega^2(M)$, closed $(d\omega = 0)$ and non-degenerate $(\omega \wedge \omega > 0$ everywhere$)$.

Local model (Darboux): $\mathbb{R}^4$, $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.

E.g.: $(\mathbb{CP}^n, \omega_0 = i\partial\bar{\partial} \log \|z\|^2) \supset$ complex projective surfaces.

The symplectic category is strictly larger

(Thurston 1976, Gompf 1994).

Symplectic manifolds are not always complex, but they are almost-complex, i.e. there exists $J \in \mathrm{End}(TM)$ such that

$$J^2 = -\mathrm{Id}, \quad g(u,v) := \omega(u, Jv) \text{ Riemannian metric.}$$

At any given point (M, ω, J) looks like $(\mathbb{C}^n, \omega_0, i)$, but J is not integrable ($\nabla J \neq 0$; $\bar{\partial}^2 \neq 0$; no holomorphic coordinates).

Hierarchy of compact oriented 4-manifolds:

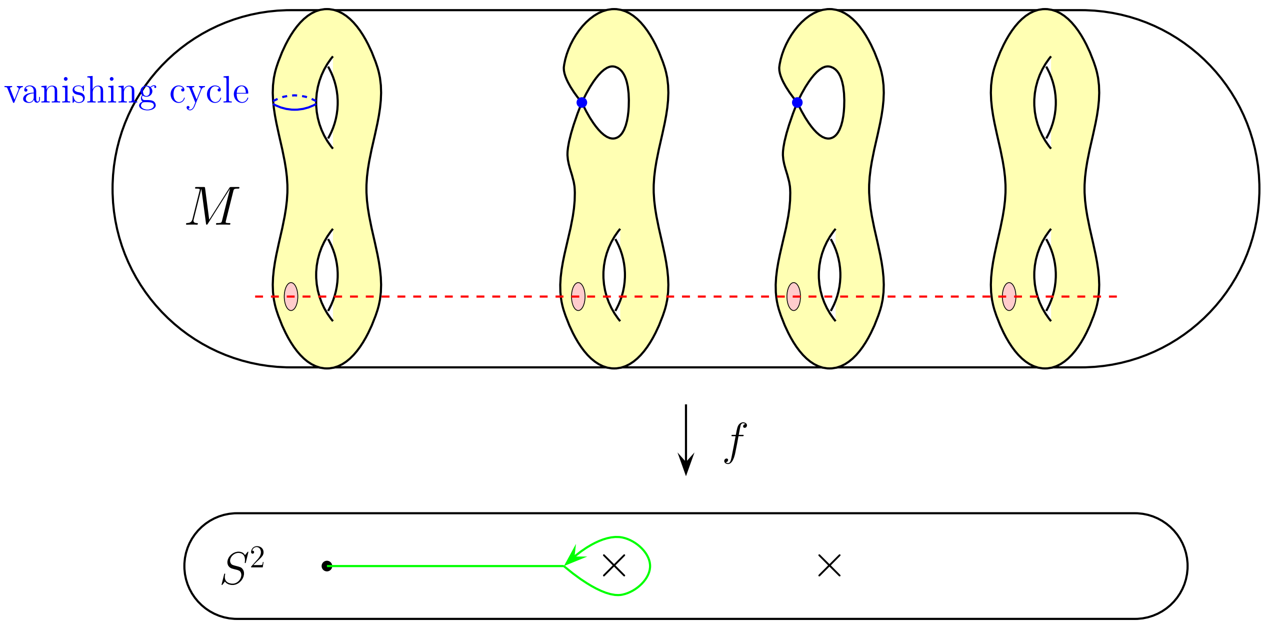
COMPLEX PROJ. $\subsetneq$ SYMPLECTIC $\subsetneq$ SMOOTH

$\Rightarrow$ Classification problems.

Symplectic manifolds retain some (not all!) features of complex proj. manifolds; yet (almost) every smooth 4-manifold admits a "near-symplectic" structure (sympl. outside circles).

Lefschetz fibrations

A **Lefschetz fibration** is a C^∞ map $f : M^4 \longrightarrow S^2$ with isolated non-degenerate crit. pts, where (in oriented coordinates) $f(z_1, z_2) \sim z_1^2 + z_2^2$. (⇒ sing. fibers are nodal)



Monodromy around sing. fiber = **Dehn twist**

Gompf: Assuming [fiber] non-torsion in $H_2(M)$, M carries a symplectic form s.t. $\omega_{|\text{fiber}} > 0$, unique up to deformation.

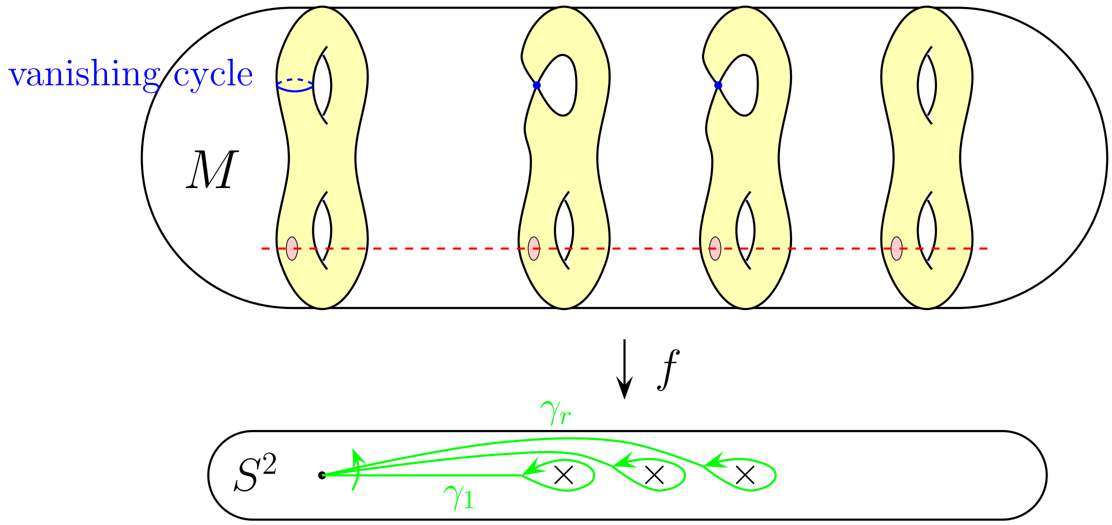
(extends Thurston's result on symplectic fibrations)

Donaldson: Any compact symplectic (X^4, ω) admits a symplectic Lefschetz pencil $f : X \setminus \{\text{base}\} \longrightarrow \mathbb{CP}^1$; blowing up base points, get a sympl. Lefschetz fibration $\hat{f} : \hat{X} \longrightarrow S^2$ with distinguished -1-sections.

(extends classical alg. geometry (Lefschetz); uses "approx. hol. geometry")

($f = s_0/s_1$, $s_i \in C^\infty(X, L^{\otimes k})$, L "ample", $\sup|\bar\partial s_i| \ll \sup|\partial s_i|$)

Monodromy



Monodromy around sing. fiber = **Dehn twist**

Monodromy: $\psi : \pi_1(S^2 \setminus \{p_1, \ldots, p_r\}) \longrightarrow \mathrm{Map}_g$

$\mathrm{Map}_g = \pi_0 \, \mathrm{Diff}^+(\Sigma_g)$ is the genus g mapping class group.

Map_g is generated by Dehn twists.

E.g. for $T^2 = \mathbb{R}^2/\mathbb{Z}^2$: $\mathrm{Map}_1 = SL(2, \mathbb{Z})$; $\tau_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\tau_b = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

Choose an ordered basis $\langle \gamma_1, \ldots, \gamma_r \rangle$ for $\pi_1(S^2 \setminus \{p_i\})$

$\Rightarrow$ factorization of Id as product of positive Dehn twists:

$$(\tau_1, \ldots, \tau_r) \in \mathrm{Map}_g, \quad \tau_i = \psi(\gamma_i), \quad \prod \tau_i = 1.$$

If $g \geq 2$ then the factorization $\tau_1 \cdot \ldots \cdot \tau_r = 1$ determines the fibration f up to isotopy.

- With n distinguished sections: $\hat{\psi} : \pi_1(\mathbb{R}^2 \setminus \{p_i\}) \longrightarrow \mathrm{Map}_{g,n}$
  $\mathrm{Map}_{g,n} = \pi_0 \mathrm{Diff}^+(\Sigma, \partial\Sigma)$ genus g with n boundaries.

$\Rightarrow \tau_1 \cdot \ldots \cdot \tau_r = \delta$ (monodromy at ∞ = boundary twist).

Factorizations

Two natural equivalence relations on factorizations:

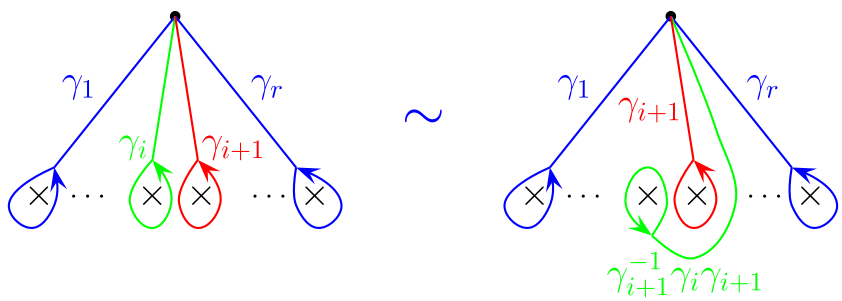
1. Global conjugation (change of trivialization of reference fiber)

$$(\tau_1, \dots, \tau_r) \sim (\phi\tau_1\phi^{-1}, \dots, \phi\tau_r\phi^{-1}) \quad \forall \phi \in \mathrm{Map}_g$$

2. Hurwitz equivalence (change of ordered basis $\langle \gamma_1, \dots, \gamma_r \rangle$)

$$(\tau_1, \dots, \tau_i, \tau_{i+1}, \dots \tau_r) \sim (\tau_1, \dots, \tau_{i+1}, \tau_{i+1}^{-1}\tau_i\tau_{i+1}, \dots, \tau_r)$$
$$\sim (\tau_1, \dots, \tau_i\tau_{i+1}\tau_i^{-1}, \tau_i, \dots, \tau_r)$$

(generates braid group action on r-tuples)



$\{$ genus g Lefschetz fibrations with n sections $\}$ / isotopy

$\updownarrow$ 1-1

$$\left\{ \begin{array}{c} \text{factorizations in } \mathrm{Map}_{g,n} \\ \delta = \prod(\text{pos. Dehn twists}) \end{array} \right\} \Big/ \begin{array}{c} \text{Hurwitz equiv.} \\ + \text{ global conj.} \end{array}$$

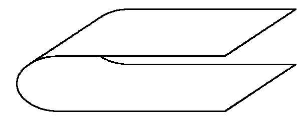
$\Rightarrow$ Classification of $\begin{cases} \text{Lefschetz fibrations?} \\ \mathrm{Map}_{g,n} \text{ factorizations?} \end{cases}$

Branched covers of $\mathbb{CP}^2$

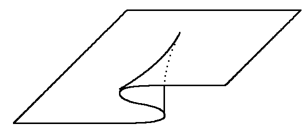
(D.A. '99, D.A.-Katzarkov '00–'02)
(extends work of Zariski, Moishezon-Teicher, ... on alg. surfaces)

Alternative description of symplectic 4-manifolds:
$f : X \to \mathbb{CP}^2$ branched covering, with crit. pts. modelled on

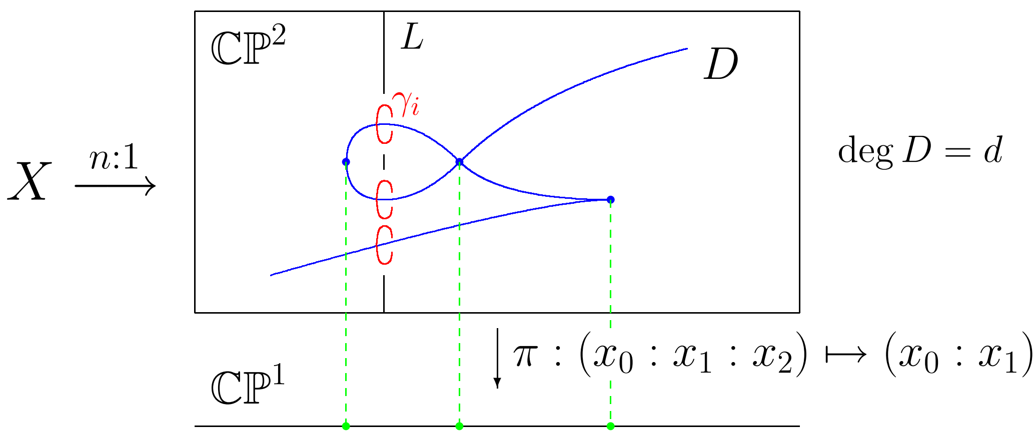
- simple branching: $(x, y) \mapsto (x^2, y)$.



- cusp: $(x, y) \mapsto (x^3 - xy, y)$.



Branch curve: $D = \text{crit}(f) \subset \mathbb{CP}^2$ symplectic curve with (complex) cusp and (+/−) node singularities.



$\Rightarrow$ another combinatorial description of sympl. 4-manifolds:

1) Branch curve: $D \subset \mathbb{CP}^2$

Braid monodromy = $\rho : \pi_1(\mathbb{C} - \{\text{pts}\}) \to B_d$ (braid group)

$\Rightarrow$ D is described by a (liftable) braid group factorization

(involving cusps, nodes, tangencies)

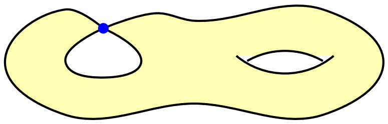
2) Monodromy: $\theta : \pi_1(\mathbb{CP}^2 - D) \to S_n$ $\quad$ ($n = \deg f$)

(surjective, maps γ_i to transpositions)

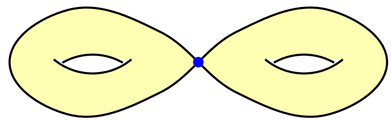
Classification of Lefschetz fibrations

- $g = 0, 1$: classical (genus 1: Moishezon-Livne).
  These are always isotopic to holomorphic fibrations.
  In Map_1: $\quad (\tau_a \cdot \tau_b)^{6k} = 1 \qquad \tau_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tau_b = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

- $g = 2$, assuming no reducible sing. fibers:



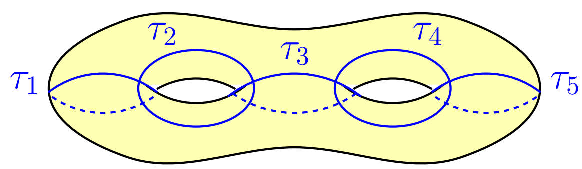
irreducible



reducible

Conj.: always isotopic to holomorphic fibrations, i.e. one of:

$$(\tau_1 \cdot \tau_2 \cdot \tau_3 \cdot \tau_4 \cdot \tau_5 \cdot \tau_5 \cdot \tau_4 \cdot \tau_3 \cdot \tau_2 \cdot \tau_1)^{2k} = 1$$
$$(\tau_1 \cdot \tau_2 \cdot \tau_3 \cdot \tau_4 \cdot \tau_5)^{6k} = 1$$
$$(\tau_1 \cdot \tau_2 \cdot \tau_3 \cdot \tau_4)^{10k} = 1$$



Proved by Siebert-Tian (2003) under a technical assumption.

(Method: pseudo-holomorphic curves)

- $g \geq 3$ (or $g = 2$ with reducible sing. fibers):

  Various infinite families of Lefschetz fibrations not isotopic to any holomorphic fibration!

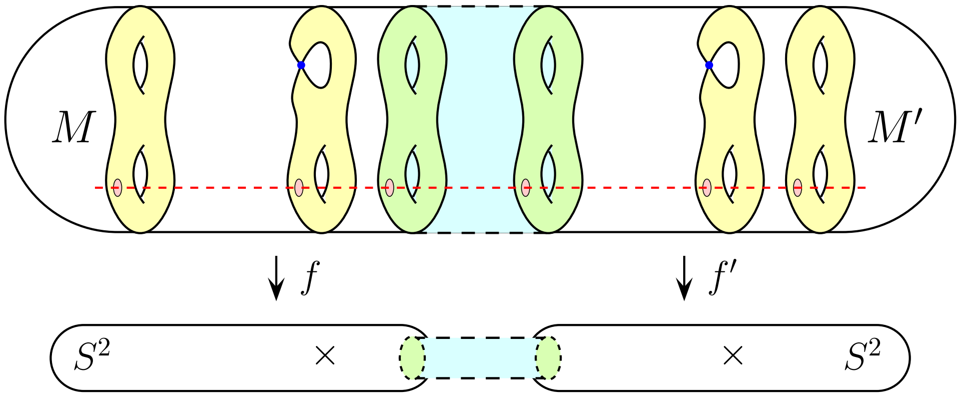
  (Ozbagci-Stipsicz, Smith, Fintushel-Stern, Korkmaz, ...)

  Can we understand anything?

Fiber sums

$f: M \to S^2$, $f': M' \to S^2$ genus g Lefschetz fibrations.
Fix a diffeomorphism between smooth fibers.

$\Rightarrow$ fiber sum $f \# f'$ (fiberwise connected sum)



For factorizations:

$(\tau_1, \dots, \tau_r), (\tau'_1, \dots, \tau'_s) \mapsto (\tau_1, \dots, \tau_r, \tau'_1, \dots, \tau'_s).$

Classification up to fiber sums: (D.A., '04)

$\forall g$ there is a genus g Lefschetz fibration f_g^0 such that:
$\forall\ f_1 : M_1 \to S^2$, $f_2 : M_2 \to S^2$ genus g Lefschetz fibrations,

if $\begin{cases} \chi(M_1) = \chi(M_2),\ \sigma(M_1) = \sigma(M_2) \\ f_1, f_2 \text{ have same \#'s of reducible fibers of each type} \\ f_1, f_2 \text{ have sections of same self-intersection} \end{cases}$

then $\forall n \gg 0$, $f_1 \# n\, f_g^0 \simeq f_2 \# n\, f_g^0$.

Positive factorizations

The proof relies on the following result:

Let $G = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle$ finitely presented group, and $\delta \in G$ a central element.

Assume there exist factorizations $\mathcal{F}_1, \dots, \mathcal{F}_m$ of δ such that:

- all factors in $\mathcal{F}_i$ are in $\{g_1, \dots, g_k\}$;
- every generator g_i appears at least once;
- every relation can be written as an equality of positive words, $w = w'$ where, viewing w, w' as factorizations:
  - either w, w' are Hurwitz equivalent
  - or $w = \mathcal{F}_i$ and $w' = \mathcal{F}_j$ for some i, j.

Then, given $\mathcal{F}', \mathcal{F}''$ factorizations of a same element in G s.t. the factors of $\mathcal{F}'$ are conjugated to those of $\mathcal{F}''$ (up to permutation),

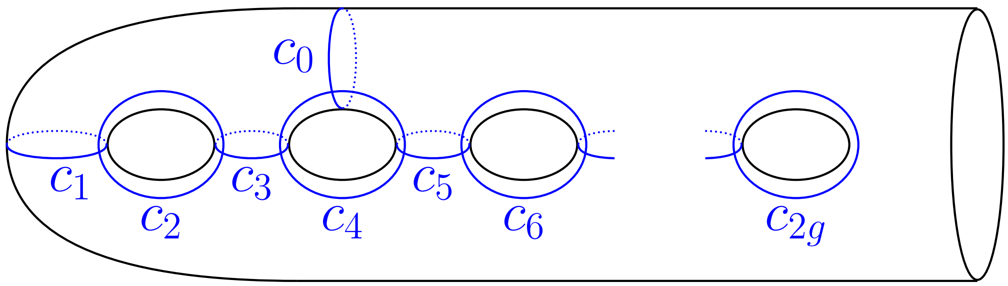
$$\exists\, n'_i, n''_i \in \mathbb{N} \text{ s.t. } \mathcal{F}' \cdot \prod_1^m \mathcal{F}_i^{n'_i} \underset{\text{Hurwitz}}{\sim} \mathcal{F}'' \cdot \prod_1^m \mathcal{F}_i^{n''_i}$$

We apply this result (+ some topology) to $G = \mathrm{Map}_{g,1}$.

(There we have 4 factorizations. Relate $n'_i - n''_i$ to change in $\chi(M)$, $\sigma(M)$

$\Rightarrow$ if preserved then $n'_i = n''_i$. Finally, take $\mathcal{F}^0 = \prod \mathcal{F}_i$)

Factorizations in $\mathrm{Map}_{g,1}$



Generators: $\tau_0, \dots, \tau_{2g}$.
Relations:

- (i) $\tau_i\tau_j = \tau_j\tau_i$ if $c_i \cap c_j = \emptyset$, $\tau_i\tau_j\tau_i = \tau_j\tau_i\tau_j$ if $c_i \cap c_j \neq \emptyset$
- (ii) for $g \geq 2$: $(\tau_0\tau_2\tau_3\tau_4)^{10} = (\tau_0\tau_1\tau_2\tau_3\tau_4)^6$
- (iii) for $g \geq 3$: $(\tau_0\tau_1\tau_2\tau_3\tau_4\tau_5\tau_6)^9 = (\tau_0\tau_2\tau_3\tau_4\tau_5\tau_6)^{12}$

(i): Hurwitz equivalences;
$(ii), (iii)$: both sides can be completed to factorizations of δ.

Corollary: (M_1, ω_1), (M_2, ω_2) compact sympl. 4-manifolds, $[\omega_i] \in H^2(M_i, \mathbb{Z})$, with same $(c_1^2, c_2, c_1 \cdot [\omega], [\omega]^2)$.

$\Rightarrow$ M_1, M_2 become symplectomorphic after (same) blow-ups and fiber sums.

Question: can M_2 be obtained from M_1 by a sequence of surgeries on Lagrangian tori?

Or: given f_1, f_2 as in main theorem, are their factorizations equivalent under Hurwitz moves + partial conjugations?

$$(\tau_1, \dots, \tau_i, \tau_{i+1}, \dots, \tau_r) \sim (\phi\tau_1\phi^{-1}, \dots, \phi\tau_i\phi^{-1}, \tau_{i+1}, \dots, \tau_r) \quad \text{if } [\phi, \tau_1 \dots \tau_i] = 1.$$