Lagrangian Floer theory: from geometry to algebra and back again

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Roadmap

Arnold conjectures: lower bounds on fixed points of Hamiltonian diffeomorphisms and on intersections of Lagrangian submanifolds

↓

Floer (co)homology $HF^*(M,H)$, $HF^*(L,L')$
∂ counts (perturbed) J-holomorphic curves

↓

Fukaya categories
encode the algebraic structure of HF^*

↓

Geometry of Floer theory
family Floer program, sheaves, local-to-global principles

Homological mirror symmetry (Kontsevich 1994) motivates the study of Fukaya categories and their geometric reinterpretation as (noncomm.?) algebraic spaces.

Arnold conjectures

(M^{2n},ω) symplectic manifold (locally $\simeq (\mathbb{R}^{2n}, \sum dp_i \wedge dq_i)$)
$\mathrm{Ham}(M,\omega) = \{$flows of (time dependent) *Hamiltonian* vect. field X_H: $\omega(\cdot, X_H) = dH\}$.

Theorem 1 (Arnold conjecture for non-degenerate Hamiltonians)

If (M,ω) is compact and $\varphi \in \mathrm{Ham}(M,\omega)$ has non-degenerate fixed points, then $\#\,\mathrm{Fix}(\varphi) \geq \sum_i \dim H^i(M;\mathbb{K})$.

Floer (1989) for M monotone, then Hofer-Salamon, Ono, Fukaya-Ono, Liu-Tian ($\mathbb{K} = \mathbb{Q}$, 1990s)
... ... 2020s: over $\mathbb{Z}$ (Bai-Xu, Rezchikov), $char(\mathbb{K}) = p$ (Abouzaid-Blumberg)

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$L^n \subset (M^{2n}, \omega)$ is Lagrangian if $\omega_{|L} = 0$. (eg. $\mathbb{R}^n_p \subset \mathbb{R}^{2n}_{p,q}$, zero section in T^*N, etc.)

Theorem 2 (Arnold-Givental conjecture on Lagrangian intersections)

$L \subset (M,\omega)$ compact Lagrangian + suitable topological assumption. If $\varphi(L)$ is transverse to L, then $\#(L \cap \varphi(L)) \geq \sum_i \dim H^i(L;\mathbb{K})$.

Floer (1988): $[\omega]\cdot\pi_2(M,L) = 0$ ($\Rightarrow L$ doesn't bound any holomorphic discs), $\mathbb{K} = \mathbb{Z}/2$ ...
Fukaya-Oh-Ohta-Ono (2000s): L rel. spin, $i_* : H_*(L) \to H_*(M)$ injective, $\mathbb{K} = \mathbb{Q}$. (e.g. $L = \Delta_M \subset M \times M \Rightarrow$ Theorem 1)

Hamiltonian and Lagrangian Floer cohomologies

Fixed points / Lagr. intersections = generators of the Floer complex

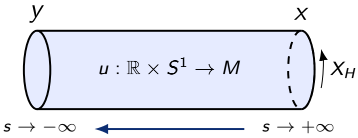
$CF^*(M,H) = \mathbb{K}\langle \mathcal{X}(H)\rangle$ — 1-periodic orbits of X_H

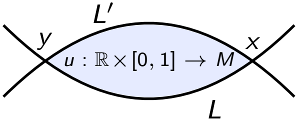
$CF^*(L,L',H) = \mathbb{K}\langle \mathcal{X}(L,L')\rangle$ — time 1 trajectories of X_H from L to L'.

$+\ \partial =$ weighted count of index 1 solutions of Floer's equation $\dfrac{\partial u}{\partial s} + J\left(\dfrac{\partial u}{\partial t} - X_{H_t}\right) = 0,$

on $\mathbb{R}\times S^1$ or $\mathbb{R}\times[0,1]$, asymptotic to given generators at $s\to\pm\infty$.

Codim. 1 boundaries = broken trajectories, hence $\partial^2 = 0$ (if no disc bubbling).





$\rightsquigarrow\ \partial x = T^{E(u)}\, y$

($E(u) \sim$ area)

(Subject to transversality / compactness of moduli spaces of sols.!)

$rank(HF)$ gives a lower bound on $\#\, Fix(\varphi_H)$ resp. $\#(\varphi_H(L)\cap L')$.

Hamiltonian Floer cohomology

The PSS isomorphism (Piunikhin-Salamon-Schwarz)

For compact M, Hamiltonian Floer cohomology is independent of (H, J). The $H \to 0$ limit corresponds to Morse theory $+$ J-holomorphic spheres. Additively, $HF^*(M, H; J) \simeq H^*(M; \mathbb{K})$ ($\Rightarrow$ Arnold conjecture).

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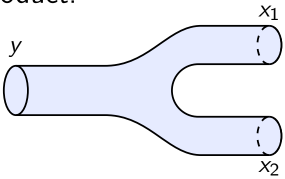
Pair-of-pants product:



$$x_1 \star x_2 = \sum_{y,[u]} \#\mathcal{M}(x_1, x_2, y; [u])\, T^{E(u)}\, y$$

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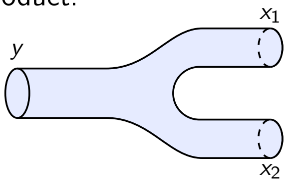

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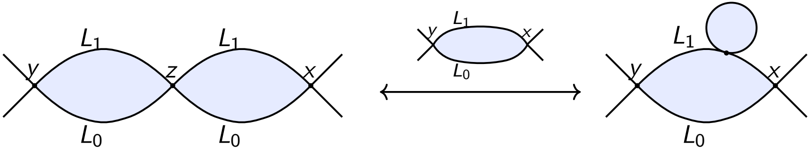
- Noncompact case: HF depends on H; limit as $H \nearrow \infty$ ("wrapping") gives *symplectic cohomology* $SH^*(M)$. (Cieliebak-Floer-Hofer, Viterbo)
- Higher algebraic structures: framed E_2 / homotopy BV algebra (Abouzaid-Groman-Varolgunes 2022)

Lagrangian Floer cohomology

Similarly the $H \to 0$ limit of Lagrangian Floer cohomology $HF^*(L, L; H, J)$ corresponds to Morse theory + J-holomorphic discs.
However, disc bubbling in codim. 1 can cause failure of $\partial^2 = 0$: $HF^*(L, L)$ is not always well defined! and even when defined, not always $\simeq H^*(L; \mathbb{K})$.



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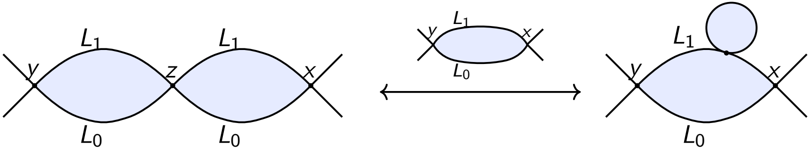


Theorem 2 (Arnold-Givental conjecture)

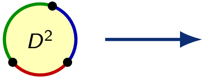
$L \subset (M, \omega)$ compact Lagrangian, s.t. (1) no holom. discs [Floer] or (2) $i_* : H_*(L) \to H_*(M)$ injective [Fukaya-O-O-O]. If $\varphi(L)$ is transverse to L, then $\#(L \cap \varphi(L)) \geq \sum_i \dim H^i(L; \mathbb{K})$.

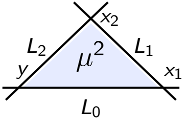
Getting from (1) to (2) requires understanding algebraic structures on Lagrangian Floer cohomology (curved A_∞-algebra, QH^*-module), and deformations (bounding cochains, bulk deformation).

Product

$$\mu^2 : \mathrm{CF}^*(L_1, L_2) \otimes \mathrm{CF}^*(L_0, L_1) \longrightarrow \mathrm{CF}^*(L_0, L_2)$$

defined by weighted counts of (perturbed) J-holomorphic discs with boundary on L_0,L_1, L_2.
(Floer's eq. on disc w. 3 boundary punctures).





Products

$$\mu^d : \mathrm{CF}^*(L_{d-1}, L_d) \otimes \cdots \otimes \mathrm{CF}^*(L_0, L_1) \to \mathrm{CF}^*(L_0, L_d)$$

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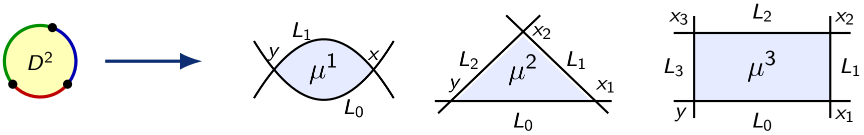

A_∞ **relations**: $\displaystyle\sum_{k+\ell=d+1} \sum_j (-1)^* \mu^k(x_d, \ldots, \mu^\ell(x_{j+\ell}, \ldots, x_{j+1}), \ldots, x_1) = 0.$

$(\mu^1)^2 = 0$, Leibniz rule, μ^2 associative up to homotopy μ^3, ...

Operations on Lagrangian Floer homology

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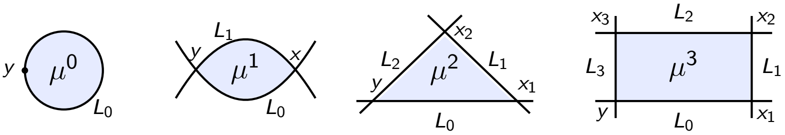

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with curvature μ^0, obstructing $(\mu^1)^2 = 0$. $\quad \mu^1(\mu^1(x)) \pm \mu^2(x, \mu^0) \pm \mu^2(\mu^0, x) = 0.$

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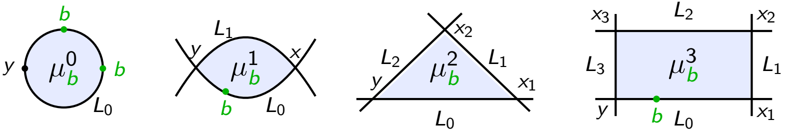

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For $b \in CF^*(L, L)$, deform μ^d_b = allow extra inputs $b, \ldots, b$ along L.
(Weak) **bounding cochain**: $\mu^0_b = \sum_{k\geq 0} \mu^k(b, \ldots, b) = 0$ (or $\mu^0_b = \lambda\, 1_L$).

The Fukaya category

Definition

The Fukaya category $\mathcal{F}(M)$ is an A_∞-category whose objects are unobstructed Lagrangians (+ if needed: spin structure, gradings, local systems, bounding cochains).

$$\text{Morphisms: } \hom_{\mathcal{F}(M)}(L_0, L_1) = \mathrm{CF}^*(L_0, L_1), \qquad \text{Compositions: } \{\mu^d\}_{d\geq 1}.$$

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- Compact, wrapped ($H \nearrow \infty$), partially wrapped Fukaya categories
- Relative and bulk-deformed variants

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This is a lot of data! Organize via algebra: $\mathcal{F}(M)$ is often *(split-)generated* (iterated mapping cones, summands) by finitely many objects $\{G_i\}$; then $\mathcal{F}(M) \hookrightarrow$ mod-$\mathcal{A}$, where $\mathcal{A} = End(\bigoplus G_i)$; or via geometry (see later).

Cotangent bundles

Theorem 3 (Nadler, Fukaya-Seidel-Smith, Abouzaid-Kragh)

Given a compact manifold N, if $L \subset T^*N$ is a compact exact Lagrangian submanifold ($p\,dq_{|L}$ is exact), then L is isomorphic to the zero section in the Fukaya category $\mathcal{F}(T^*N)$. In particular, the projection $L \to N$ has degree 1, induces an isomorphism on cohomology, and is in fact a simple homotopy equivalence.

This is progress towards Arnold's *nearby Lagrangian conjecture*, which asks if L is Hamiltonian isotopic to the zero section N.

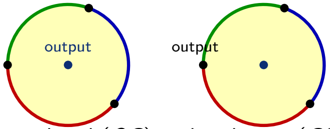
Theorem 4 (Abouzaid)

The wrapped Fukaya category $\mathcal{W}(T^*N)$ is generated by one cotangent fiber T_q^*N, with endomorphism algebra quasi-isomorphic to $C_{-*}(\Omega_q N)$. *(with sign twist if N isn't spin)*

(when $\pi_1(N) = 0$, Koszul duality $\mathcal{F}(T^*N) \leftrightarrow \mathcal{W}(T^*N)$ vs. $C^*(N) \leftrightarrow C_*(\Omega_q N)$.)

Open-closed maps and generation

Floer's eq. on disc with 1 interior and $d+1$ boundary punctures $\Rightarrow$



open-closed (OC) closed-open (CO)

$$OC : \mathrm{HH}_*(\mathcal{F}) \to QH^*(M)$$

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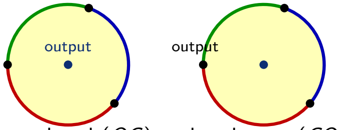

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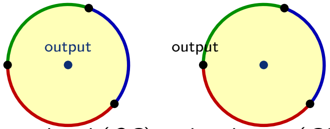

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Theorem 5 (Abouzaid; Ganatra; A-F-O-O-O)

Assume a Hochschild cycle involving morphisms between objects G_i maps to the unit 1 under $OC : HH_*(\mathcal{F}) \to QH^*(M)$ or $HH_*(\mathcal{W}) \to SH^*(M)$. Then
(1) the objects G_i split-generate the Fukaya category.
(2) CO and OC are isomorphisms.
(3) $\mathcal{F}$ resp. $\mathcal{W}$ is a homologically smooth Calabi-Yau category.

Deformations and applications to HMS

Bulk and relative deformations

$CO : QH^* \to HH^*(\mathcal{F})$ (resp. $SH^* \to HH^*(\mathcal{W})$) isom. $\Rightarrow$ every algebraic deformation of the Fukaya category comes from geometry. For $\mathfrak{b} \in QH^*$ or SH^*, bulk-deformed operations $\mu^d_{\mathfrak{b}} = \mu^d + CO^d(\mathfrak{b}) + \cdots$ count discs with interior insertions. For a divisor $D \subset M$, the relative Fukaya category $\mathcal{F}(M, D)$ deforms $\mathcal{F}(M \setminus D)$ by weighting intersections with D.

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Application to homological mirror symmetry (HMS)

Determining the Fukaya category by deformation from "large volume" limit $M \setminus D$ is a key approach to Kontsevich's HMS conjecture, i.e. equivalence between $\mathcal{F}(M, \omega)$ and coherent sheaves on a mirror space $\mathrm{Coh}(Y)$ (or matrix factorizations of a *Landau-Ginzburg model* $\mathrm{MF}(Y, W)$).

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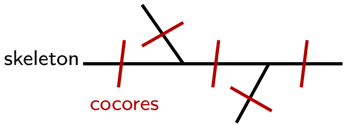
HMS suggests $\mathcal{F}(M)$ is not just an object of noncomm. alg. geometry (a category): often it is an honest (commutative) algebraic space.

Floer theory in families I: skeleta and microlocal sheaves

Families of cotangent fibers

Nadler-Zaslow: $\mathcal{F}(T^*N) \simeq$ local systems / constructible sheaves on N by considering $\{HF^*(F_q, L)\}_{q\in N}$, and dependence on q (eg. locally constant).
This generalizes to: (M,ω) Weinstein manifold (eg. affine alg. var./$\mathbb{C}$) retracts onto an isotropic skeleton B. "Co-cores" ($\sim$ cotangent fibers) generate $\mathcal{W}(M)$.



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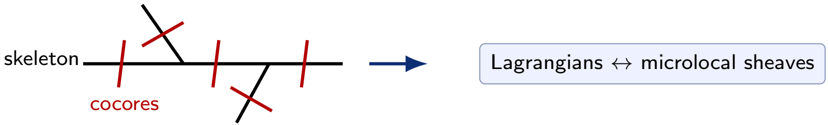

Theorem 6 (Nadler, Ganatra-Pardon-Shende)

The (wrapped) Fukaya category of a stably polarized Weinstein sector M is equivalent to the category of (wrapped) microlocal sheaves on its skeleton.

$\rightsquigarrow$ calculations of Fukaya categories and HMS for affine varieties and Landau-Ginzburg models

(Fang-Liu-Treumann-Zaslow, Nadler, Gammage-Shende ...)

Floer theory in families II: torus fibrations and SYZ mirror symmetry

SYZ parameter space

Given a Lagrangian torus fibration $\pi : M \to B$, consider family of smooth fibers + rank 1 local systems: $Y^0 = \{(F_b, \xi) : b \in B^0,\ \xi \in \mathrm{Hom}(\pi_1(F_b), U(1)_{\mathbb{K}})\}$ ("uncorrected SYZ mirror")

Floer weights $z_\beta = T^{\omega(\beta)}\xi(\partial\beta)$ ($\beta \in H_2(M, F_b)$) are (loc.) analytic functions on Y^0.

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The family Floer functor (Fukaya, Abouzaid)

To $L \in \mathcal{F}(M)$, family Floer theory assigns the family of complexes $\{\mathrm{CF}^*((F_b, \xi), L)\}_{(F_b,\xi)\in Y^0}$, whose differentials and structure maps depend locally analytically on $(F_b, \xi) \in Y^0$.
($\Rightarrow$ complexes of coherent sheaves on Y^0, or matrix factorizations if $\mu^0_{(F_b,\xi)} = W\,\mathrm{id}$)
$(F_b, \xi) \mapsto$ skyscraper sheaves, Lagr. sections $\mapsto$ line bundles

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Theorem 7 (Abouzaid)

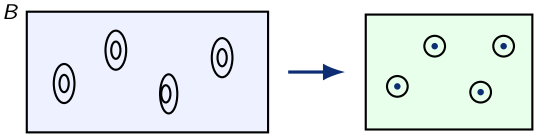
If $\pi : M \to B$ has no singular fibers and admits a Lagrangian section, the family Floer functor gives a fully faithful embedding from $\mathcal{F}(M)$ to $\mathrm{Coh}(Y)$. (i.e., SYZ $\Rightarrow$ HMS)

Wall-crossing and corrections to mirror geometry

Corrected SYZ mirrors

The corrected SYZ mirror Y is constructed as a moduli space of objects of $\mathcal{F}(M)$ supported on the fibers of $\pi : M \to B$. It differs from $Y^0 = \{(F_b, \xi)\}$ by "instanton corrections".

$\pi : M \to B$ is cut into regions of weakly unobstructed fibers delimited by "walls" of fibers which bound holomorphic discs.



$Y^0 = \{(F_b, \xi)\}/\sim$

Wall-crossing and corrections to mirror geometry

Corrected SYZ mirrors

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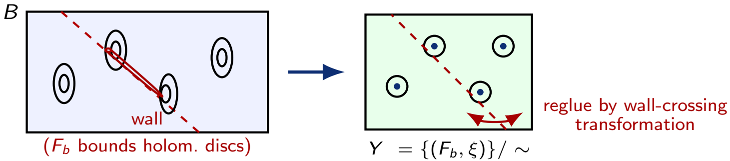

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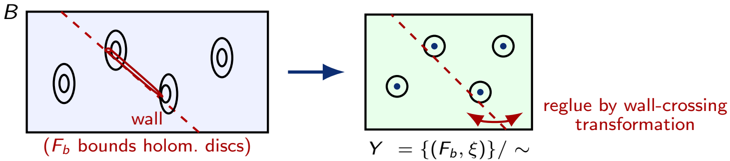

Consistency of wall-crossing transformations (Gross-Siebert, Kontsevich-Soibelman) can fail outside of Calabi-Yau/Fano settings! (A., 2023) Then mirror isn't a genuine analytic space.

Local-to-global I: sectors

Theorem 8 (Ganatra-Pardon-Shende, sectorial descent)

Wrapped Fukaya categories of Liouville sectors are covariantly functorial for proper inclusions. If a sector or Liouville manifold is obtained by gluing

$$M = M_1 \cup_{\mathbb{R}\times F} M_2,$$

then the inclusion functors form a pushout diagram

$$\begin{array}{ccc} \mathcal{W}(F) & \longrightarrow & \mathcal{W}(M_1) \\ \downarrow & & \downarrow \\ \mathcal{W}(M_2) & \longrightarrow & \mathcal{W}(M). \end{array}$$

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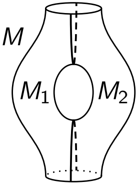
Mirror symmetry interpretation

Sectorial gluing corresponds to gluing categories of coherent sheaves along closed subschemes.

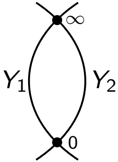
Example: twice-punctured torus

Covariant decomposition

A twice-punctured torus can be cut along two arcs into sectors M_1, M_2, each a cylinder with two stops. Mirror symmetry identifies the pushout with the gluing of two copies $Y_1, Y_2 \simeq \mathbb{P}^1$ at 0 and ∞.



$$\begin{tikzcd} \mathcal{W}(\mathrm{pt} \sqcup \mathrm{pt}) \arrow[r] \arrow[d] & \mathcal{W}(M_1) \arrow[d] & & \mathrm{Coh}(\mathrm{pt} \sqcup \mathrm{pt}) \arrow[r] \arrow[d] & \mathrm{Coh}(Y_1) \arrow[d] \\ \mathcal{W}(M_2) \arrow[r] & \mathcal{W}(M) \arrow[r, phantom, "\overset{\text{mirror}}{\longleftrightarrow}"] & & \mathrm{Coh}(Y_2) \arrow[r] & \mathrm{Coh}(Y) \end{tikzcd}$$



(See also Ganatra's talk tomorrow!)

Local-to-global II: restrictions

Theorem 9 (H. Lee)

Let a Riemann surface Σ be decomposed along nontrivial simple closed curves as $\Sigma = \Sigma_1 \cup_\gamma \Sigma_2$, and let C be a cylindrical neighborhood of γ. The restriction functors form a pullback diagram

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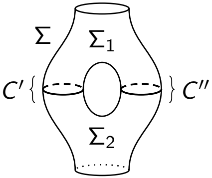
Idea (cf. Abouzaid-Seidel): Set up very steep Hamiltonian "barriers" near the boundary of each subdomain; generators outside a subdomain form an A_∞-ideal, restriction $=$ quotient.

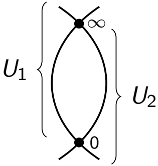
Mirror symmetry interpretation: restriction to pieces of an open cover / Cech model.

Contravariant decomposition

The twice-punctured torus can also be cut along two closed curves into two pairs of pants, glued along cylinders C', C''. Mirror symmetry turns the restriction pullback diagram into the Cech diagram for the affine cover of $Y = \mathbb{P}^1 \cup_{\{0,\infty\}} \mathbb{P}^1$ by two opens $U_i \simeq \mathbb{A}^1 \cup_{\{0\}} \mathbb{A}^1$.



$$\begin{tikzcd}
\mathcal{W}(\Sigma) \arrow[r] \arrow[d] & \mathcal{W}(\Sigma_1) \arrow[d] & & \mathrm{Coh}(Y) \arrow[r] \arrow[d] & \mathrm{Coh}(U_1) \arrow[d] \\
\mathcal{W}(\Sigma_2) \arrow[r] & \mathcal{W}(C' \sqcup C'') \arrow[ur, phantom, "\xleftrightarrow{\text{mirror}}"] & & \mathrm{Coh}(U_2) \arrow[r] & \mathrm{Coh}(U_1 \cap U_2)
\end{tikzcd}$$



When naive locality fails

Deformed local theories

If the domain $V \subset M$ bounds holomorphic discs not contained within it, the restriction picture needs to be modified: Floer theory in V is bulk-deformed by a SH^*-class accounting for contributions of curves outside V. (Borman-Sheridan, Groman-Varolgunes, ...)

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Similarly, *relative wrapped Fukaya categories* should provide sheaf-like local models over covers of the base of an SYZ fibration (Abouzaid-Groman-Varolgunes in progress).

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Conjecture: The *family Floer algebra* of a Lagrangian torus fibration (A. 2023 + in progress) (cochains on M with coefficients in fiberwise loop space, deformed by holomorphic discs) recovers similar structures: Cech model for local Hochschild cohomologies / polyvector fields on the corrected SYZ mirror (even when the mirror isn't a genuine analytic space), as well as a family Floer functor from $\mathcal{F}(M)$ to modules / sheaves.

To be continued...

Thank you!