Lefschetz pencils and the symplectic topology of complex surfaces

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Symplectic 4-manifolds

A (compact) symplectic 4-manifold (M^4, ω) is a smooth 4-manifold with a symplectic form $\omega \in \Omega^2(M)$, closed $(d\omega = 0)$ and non-degenerate $(\omega \wedge \omega > 0)$.

Local model (Darboux): $\mathbb{R}^4$, $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.

E.g.: $(\mathbb{CP}^n, \omega_0 = i\partial\bar{\partial} \log \|z\|^2) \supset$ complex projective surfaces.

The symplectic category is strictly larger (Thurston 1976, Gompf 1994, ...).

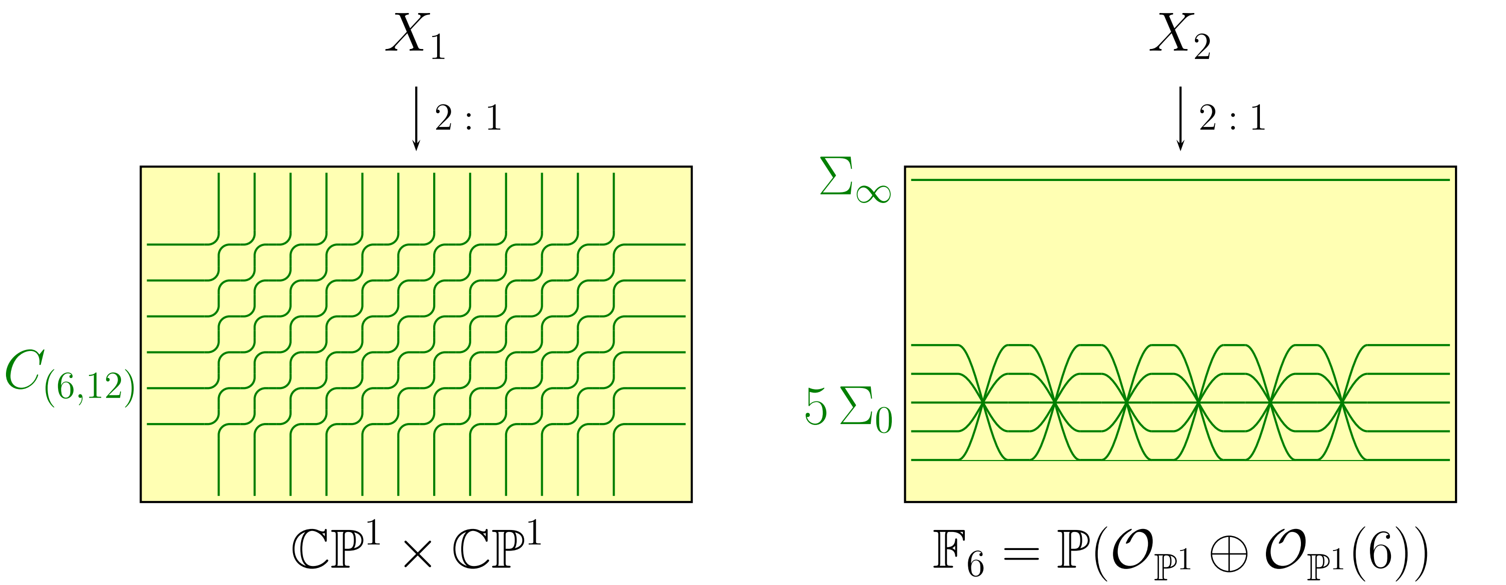
Hierarchy of compact oriented 4-manifolds:

$$\text{COMPLEX PROJ.} \underset{\substack{\text{surgery} \\ \text{Thurston, Gompf...}}}{\subsetneq} \text{SYMPLECTIC} \underset{\substack{\text{SW invariants} \\ \text{Taubes}}}{\subsetneq} \text{SMOOTH}$$

$\Rightarrow$ Classification problems.

Complex surfaces are fairly well understood, but their topology as smooth or symplectic manifolds remains mysterious.

Example: Horikawa surfaces



X_1, X_2 projective surfaces of general type, minimal, $\pi_1 = 1$
X_1, X_2 are not deformation equivalent (Horikawa)
X_1, X_2 are homeomorphic $\quad (b_2^+ = 21,\ b_2^- = 93$, non-spin$)$

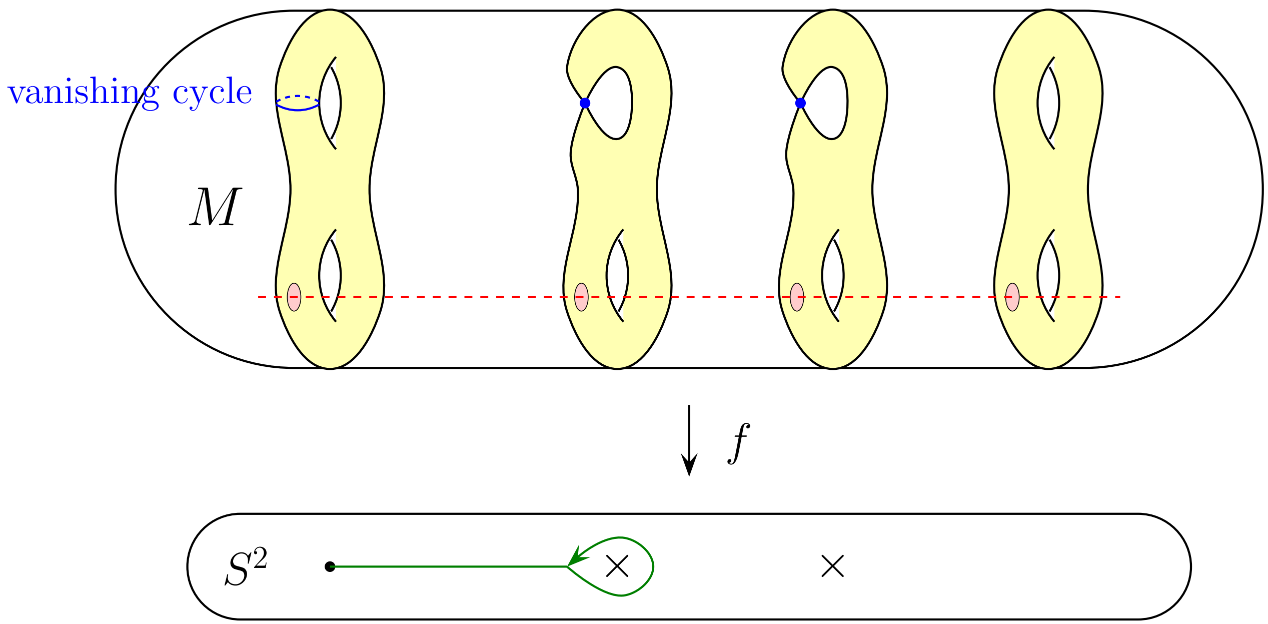
Open problems:

- X_1, X_2 diffeomorphic? $\quad$ (expect: no, even though $SW(X_1) = SW(X_2)$)
- (X_1, ω_1), (X_2, ω_2) (canonical Kähler forms) symplectomorphic?

Remark: projecting to $\mathbb{CP}^1$, Horikawa surfaces carry genus 2 fibrations.

Lefschetz fibrations

A **Lefschetz fibration** is a C^∞ map $f : M^4 \to S^2$ with isolated non-degenerate crit. pts, where (in oriented coords.) $f(z_1, z_2) \sim z_1^2 + z_2^2$. (⇒ sing. fibers are nodal)



Monodromy around sing. fiber = **Dehn twist**

Also consider: Lefschetz fibrations with distinguished sections.

Gompf: Assuming [fiber] non-torsion in $H_2(M)$, M carries a symplectic form s.t. $\omega_{|\text{fiber}} > 0$, unique up to deformation. (extends Thurston's result on symplectic fibrations)

Symplectic manifolds and Lefschetz pencils

Algebraic geometry:

X complex surface + ample line bundle $\Rightarrow$ projective embedding $X \hookrightarrow \mathbb{CP}^N$.
Intersect with a generic pencil of hyperplanes $\Rightarrow$ Lefschetz pencil

(= family of curves, at most nodal, through a finite set of base points).

Blow up base points $\Rightarrow$ Lefschetz fibration with distinguished sections.

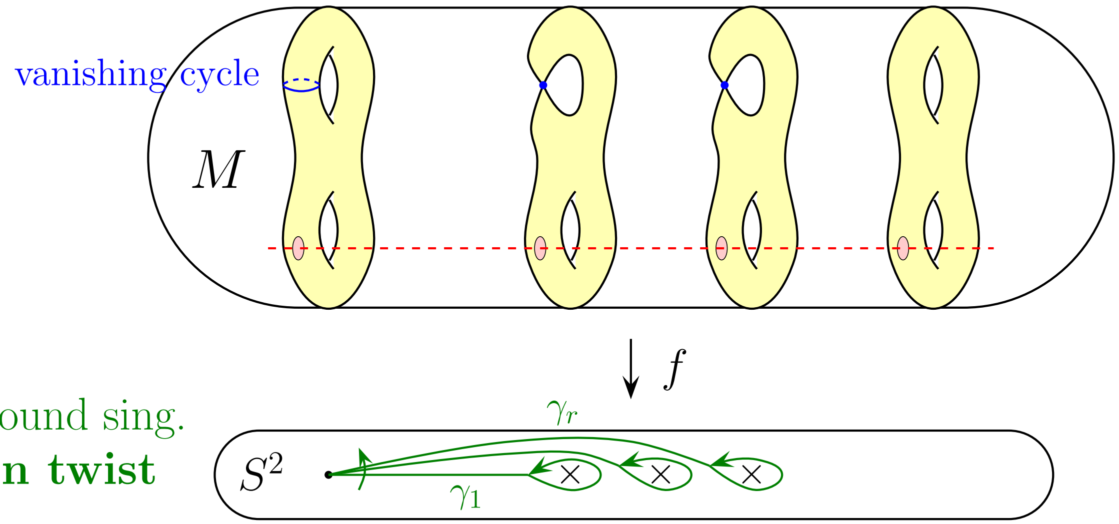
Donaldson: Any compact sympl. (X^4, ω) admits a symplectic Lefschetz pencil $f : X \setminus \{\text{base}\} \to \mathbb{CP}^1$; blowing up base points, get a sympl. Lefschetz fibration $\hat{f} : \hat{X} \to S^2$ with distinguished -1-sections.

(uses "approx. hol. geometry": $f = s_0/s_1$, $s_i \in C^\infty(X, L^{\otimes k})$, L "ample", $\sup|\bar{\partial}s_i| \ll \sup|\partial s_i|$)

In large enough degrees (fibers $\sim m[\omega]$, $m \gg 0$), Donaldson's construction is canonical up to isotopy; combine with Gompf's results $\Rightarrow$

Corollary: the Horikawa surfaces X_1 and X_2 (with Kähler forms $[\omega_i] = K_{X_i}$) are symplectomorphic iff generic pencils of curves in the pluricanonical linear systems $|mK_{X_i}|$ define topologically equivalent Lefschetz fibrations with sections for some m (or for all $m \gg 0$).

Monodromy



Monodromy around sing. fiber = **Dehn twist**

Monodromy: $\psi : \pi_1(S^2 \setminus \{p_1, ..., p_r\}) \longrightarrow \mathrm{Map}_g = \pi_0 \mathrm{Diff}^+(\Sigma_g)$

Mapping class group: e.g. for $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, $\mathrm{Map}_1 = \mathrm{SL}(2, \mathbb{Z})$; $\tau_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\tau_b = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

Choosing an ordered basis $\langle \gamma_1, \ldots, \gamma_r \rangle$ for $\pi_1(S^2 \setminus \{p_i\})$, get

$$(\tau_1, \ldots, \tau_r) \in \mathrm{Map}_g, \quad \tau_i = \psi(\gamma_i), \quad \prod \tau_i = 1.$$

"factorization of Id as product of positive Dehn twists".

- With n distinguished sections: $\hat{\psi} : \pi_1(\mathbb{R}^2 \setminus \{p_i\}) \longrightarrow \mathrm{Map}_{g,n}$

  $\mathrm{Map}_{g,n} = \pi_0 \mathrm{Diff}^+(\Sigma, \partial\Sigma)$ genus g with n boundaries.

  $\Rightarrow \tau_1 \cdot \ldots \cdot \tau_r = \delta$ (monodromy at ∞ = boundary twist).

Factorizations

Two natural equivalence relations on factorizations:

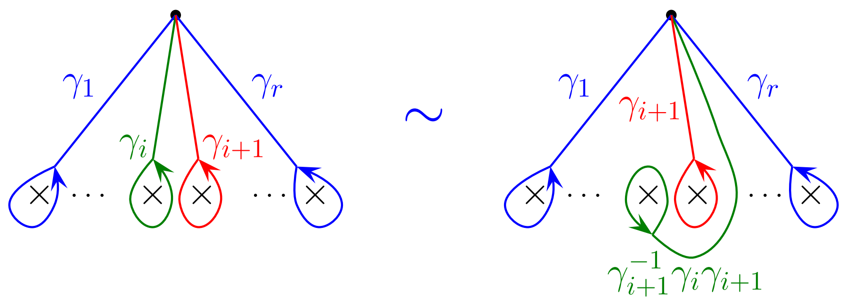
1. Global conjugation (change of trivialization of reference fiber)

$$(\tau_1, \dots, \tau_r) \sim (\phi\tau_1\phi^{-1}, \dots, \phi\tau_r\phi^{-1}) \quad \forall \phi \in \mathrm{Map}_g$$

2. Hurwitz equivalence (change of ordered basis $\langle \gamma_1, \dots, \gamma_r \rangle$)

$$(\tau_1, \dots, \tau_i, \tau_{i+1}, \dots \tau_r) \sim (\tau_1, \dots, \tau_{i+1}, \tau_{i+1}^{-1}\tau_i\tau_{i+1}, \dots, \tau_r)$$
$$\sim (\tau_1, \dots, \tau_i\tau_{i+1}\tau_i^{-1}, \tau_i, \dots, \tau_r)$$

(generates braid group action on r-tuples)



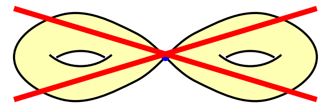
$\{$ genus g Lefschetz fibrations with n sections $\}$ / isomorphism

$\updownarrow$ 1-1 (if $2 - 2g - n < 0$)

$$\left\{ \begin{array}{c} \text{factorizations in } \mathrm{Map}_{g,n} \\ \delta = \prod (\text{pos. Dehn twists}) \end{array} \right\} \Big/ \begin{array}{c} \text{Hurwitz equiv.} \\ + \text{ global conj.} \end{array}$$

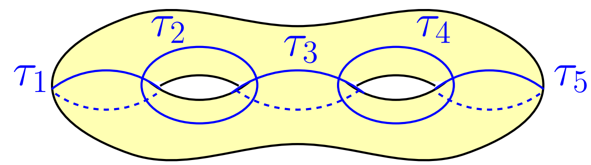
Classification in low genus

- $g = 0, 1$: only holomorphic fibrations ($\Rightarrow$ ruled surfaces, elliptic surfaces).
- $g = 2$, assuming sing. fibers are irreducible:



Siebert-Tian (2003): always isotopic to holomorphic fibrations, i.e. built from:

$$(\tau_1 \cdot \tau_2 \cdot \tau_3 \cdot \tau_4 \cdot \tau_5 \cdot \tau_5 \cdot \tau_4 \cdot \tau_3 \cdot \tau_2 \cdot \tau_1)^2 = 1$$
$$(\tau_1 \cdot \tau_2 \cdot \tau_3 \cdot \tau_4 \cdot \tau_5)^6 = 1$$
$$(\tau_1 \cdot \tau_2 \cdot \tau_3 \cdot \tau_4)^{10} = 1$$



(up to a technical assumption; argument relies on pseudo-holomorphic curves)

- $g \geq 3$: intractable

(families of non-holom. examples by Ozbagci-Stipsicz, Smith, Fintushel-Stern, Korkmaz, ...)

The genus 2 fibrations on X_1, X_2 are different (e.g., different monodromy groups):

X_1: $(\tau_1 \cdot \tau_2 \cdot \tau_3 \cdot \tau_4 \cdot \tau_5 \cdot \tau_5 \cdot \tau_4 \cdot \tau_3 \cdot \tau_2 \cdot \tau_1)^{12} = 1$
X_2: $(\tau_1 \cdot \tau_2 \cdot \tau_3 \cdot \tau_4)^{30} = 1$

... but can't conclude from them!

Canonical pencils on Horikawa surfaces

On X_1 and X_2, generic pencils in the linear systems $|K_{X_i}|$ have fiber genus 17 (with 16 base points), and 196 nodal fibers

$\Rightarrow$ compare 2 sets of 196 Dehn twists in $\mathrm{Map}_{17,16}$?

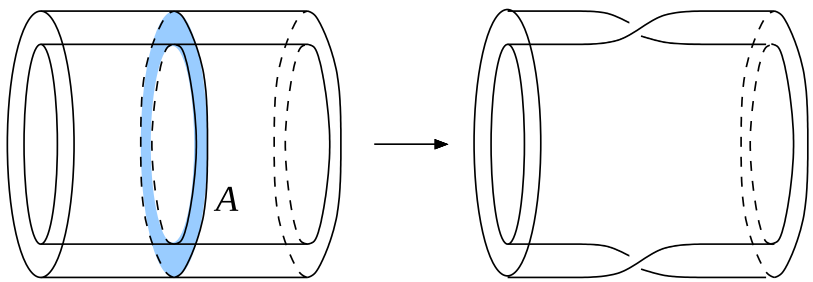
Theorem: The canonical pencils on X_1 and X_2 are related by partial conjugation:

$$(\phi t_1 \phi^{-1}, \dots, \phi t_{64} \phi^{-1}, t_{65}, \dots, t_{196}) \qquad \text{vs.} \qquad (t_1, \dots, t_{196})$$

The monodromy groups $G_1, G_2 \subset \mathrm{Map}_{17,16}$ are isomorphic; unexpectedly, the conjugating element ϕ belongs to the monodromy group.

Key point: $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{F}_6$ are symplectomorphic; the branch curves of $\pi_1 : X_1 \to \mathbb{CP}^1 \times \mathbb{CP}^1$ and $\pi_2 : X_2 \to \mathbb{F}_6$ differ by twisting along a Lagrangian annulus.

disconnected curve
$5\Sigma_0 \cup \Sigma_\infty \subset \mathbb{F}_6$



connected curve
$C_{(6,12)} \subset \mathbb{CP}^1 \times \mathbb{CP}^1$

Perspectives

Theorem: The canonical pencils on X_1 and X_2 are related by partial conjugation; $G_1, G_2 \subset \mathrm{Map}_{17,16}$ are isomorphic; ϕ belongs to the monodromy group.

- The same properties hold for pluricanonical pencils $|mK_{X_i}|$ (in larger $\mathrm{Map}_{g,n}$)
- These pairs of pencils are twisted fiber sums of the same pieces.

- If ϕ were monodromy along an embedded loop (+ more) $\Rightarrow (X_1, \omega_1) \simeq (X_2, \omega_2)$

(but only seems to arise from an immersed loop)

Question: compare these (very similar) mapping class group factorizations??

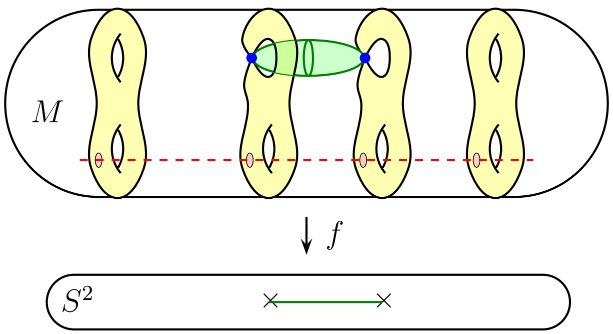
E.g.: "matching paths" (= Lagrangian spheres fibering above an arc). Expect:

H_2-classes represented by Lagrangian spheres

$\Updownarrow$?

"alg. vanishing cycles" (ODP degenerations)

(span $[\pi^* H_2(\mathbb{P}^1 \times \mathbb{P}^1)]^\perp \neq [\pi^* H_2(\mathbb{F}_6)]^\perp$)



(but... $\phi \in G_2$ suggests where to start looking for exotic matching paths?)