Homological Mirror Symmetry for Blowups of $\mathbb{CP}^2$

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(joint work with L. Katzarkov, D. Orlov)
(after ideas of Kontsevich, Seidel, Hori, Vafa, ...)

See: math.AG/0404281, math.AG/0506166

Mirror Symmetry

Complex manifolds: (X, J) locally $\simeq (\mathbb{C}^n, i)$

Look at complex analytic cycles + holom. vector bundles, or better: coherent sheaves
Intersection theory = Morphisms and extensions of sheaves.

Symplectic manifolds: (Y, ω) locally $\simeq (\mathbb{R}^{2n}, \sum dx_i \wedge dy_i)$

Look at Lagrangian submanifolds (+ flat unitary bundles):
$L^n \subset Y^{2n}$ with $\omega_{|L} = 0$ (locally $\simeq \mathbb{R}^n \subset \mathbb{R}^{2n}$; in $\dim_{\mathbb{R}}$ 2, any embedded curve!)

Intersection theory (with quantum corrections) = Floer homology
(discard intersections that cancel by Hamiltonian isotopy)

Mirror symmetry:

D-branes = boundary conditions for open strings.
Homological mirror symmetry (Kontsevich): at the level of derived categories,

A-branes	=	Lagrangian submanifolds,
B-branes	=	coherent sheaves.

HMS Conjecture: Calabi-Yau case

X, Y Calabi-Yau ($c_1 = 0$) mirror pair $\Rightarrow$ $\boxed{\begin{array}{lcl} D^bCoh(X) & \simeq & D\mathcal{F}(Y) \\ D\mathcal{F}(X) & \simeq & D^bCoh(Y) \end{array}}$

$Coh(X)$ = category of coherent sheaves on X complex manifold.

D^b = bounded derived category:

Objects = complexes $0 \to \cdots \to \mathcal{E}^i \xrightarrow{d^i} \mathcal{E}^{i+1} \to \cdots \to 0$.

Morphisms = morphisms of complexes (up to homotopy, + inverses of quasi-isoms)

$\mathcal{F}(Y)$ = Fukaya A_∞-category of (Y, ω). Roughly:

Objects = (some) Lagrangian submanifolds (+ flat unitary bundles)

Morphisms: $\mathrm{Hom}(L, L') = CF^*(L, L') = \mathbb{C}^{|L \cap L'|}$ if $L \pitchfork L'$. (or $\bigoplus \mathrm{Hom}(\mathcal{E}_p, \mathcal{E}'_p)$)

(Floer complex, graded by Maslov index)

with: differential $d = m_1$; product m_2 (composition; only associative up to homotopy); and higher products $(m_k)_{k \geq 3}$ (related by A_∞-equations).

Fukaya categories

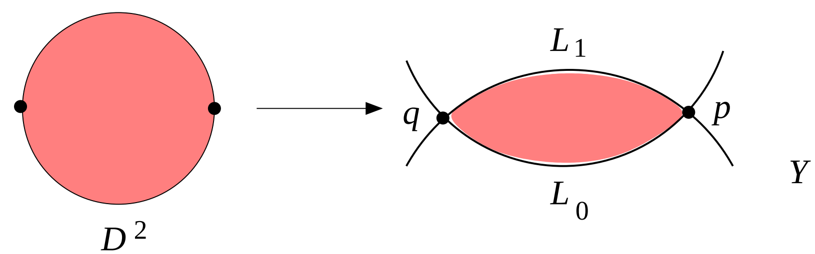
$$\mathrm{Hom}(L,L') = CF^*(L,L') = \mathbb{C}^{|L\cap L'|} \text{ if } L \pitchfork L'. \qquad \Big(\text{or: } \bigoplus_{p\in L\cap L'} \mathrm{Hom}(\mathcal{E}_p,\mathcal{E}'_p)\Big)$$

- Differential $d = m_1 : \mathrm{Hom}(L_0,L_1) \to \mathrm{Hom}(L_0,L_1)[1]$

$$\langle m_1(p),q\rangle = \sum_{u\in\mathcal{M}(p,q)} \pm\exp\Big(-\int_{D^2} u^*\omega\Big)$$

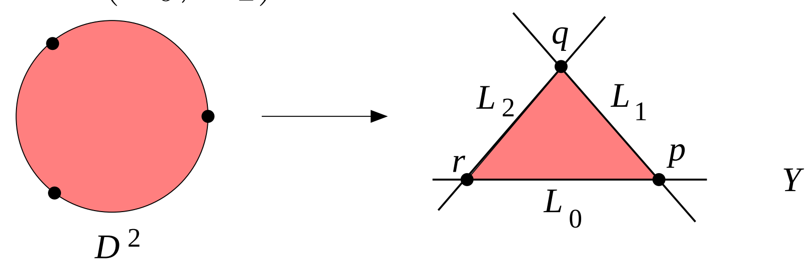
counts pseudo-holomorphic maps

(in $\dim_{\mathbb{R}} 2$: immersed discs with convex corners)



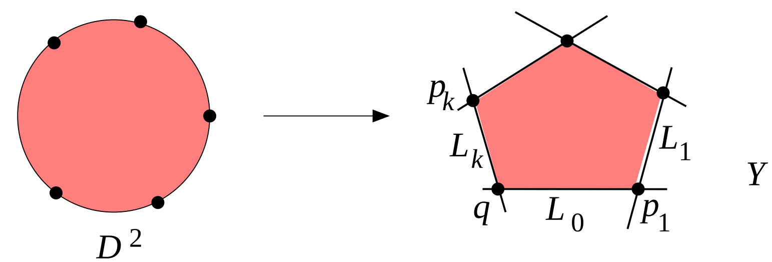
- Product $m_2 : \mathrm{Hom}(L_0,L_1)\otimes\mathrm{Hom}(L_1,L_2) \to \mathrm{Hom}(L_0,L_2)$

$\langle m_2(p,q),r\rangle$ counts pseudo-holomorphic maps



- Higher products $m_k : \mathrm{Hom}(L_0,L_1)\otimes\cdots\otimes\mathrm{Hom}(L_{k-1},L_k) \to \mathrm{Hom}(L_0,L_k)[2-k]$

$\langle m_k(p_1,\dots,p_k),q\rangle$ counts pseudo-holomorphic maps



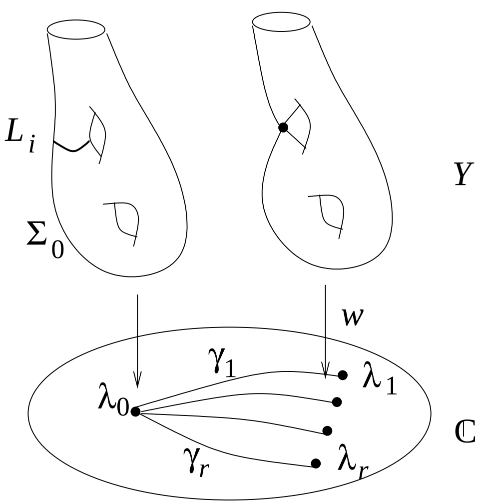
HMS Conjecture: Fano case

X Fano ($c_1(TX) > 0$) $\overset{M.S.}{\longleftrightarrow}$ "Landau-Ginzburg model" $\begin{cases} Y \text{ (non-compact) manifold} \\ W : Y \to \mathbb{C} \text{ "superpotential"} \end{cases}$

$$\begin{array}{lcl} D^b Coh(X) & \simeq & D^b Lag(W) \\ D^\pi \mathcal{F}(X) & \simeq & D^b Sing(W) \end{array}$$

$D^b Lag(W)$ (Lagrangians) and $D^b Sing(W)$ (sheaves) =
symplectic and complex geometries of singularities of W.

If $W : Y \to \mathbb{C}$ is a Morse function (isolated non-degenerate crit. pts):



$L_i \subset \Sigma_0$ Lagrangian sphere = vanishing cycle associated to γ_i
(collapses to crit. pt. by parallel transport)

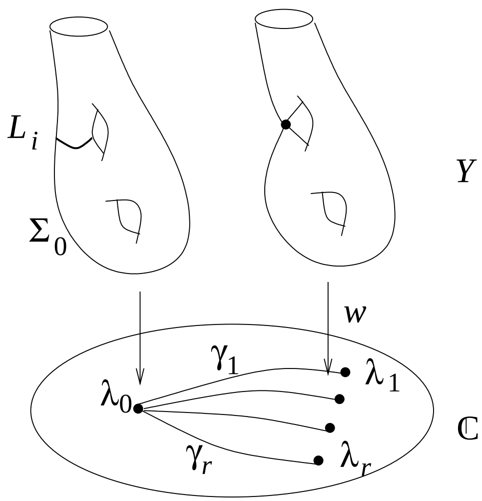
Seidel: $Lag(W, \{\gamma_i\})$ finite, directed A_∞-category.

Objects: $L_1, \ldots, L_r$.

$$\text{Hom}(L_i, L_j) = \begin{cases} CF^*(L_i, L_j) = \mathbb{C}^{|L_i \cap L_j|} & \text{if } i < j \\ \mathbb{C} \cdot \text{Id} & \text{if } i = j \\ 0 & \text{if } i > j \end{cases}$$

Products: $(m_k)_{k \geq 1}$ = Floer theory for Lagrangians $\subset \Sigma_0$.

Categories of Lagrangian vanishing cycles



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Products: $(m_k)_{k \geq 1}$ = Floer theory for Lagrangians $\subset \Sigma_0$.

- $m_k : \mathrm{Hom}(L_{i_0}, L_{i_1}) \otimes \cdots \otimes \mathrm{Hom}(L_{i_{k-1}}, L_{i_k}) \to \mathrm{Hom}(L_{i_0}, L_{i_k})[2-k]$ is trivial unless $i_0 < \cdots < i_k$.
- m_k counts discs in Σ_0 with boundary in $\bigcup L_i$, with coefficients $\pm \exp(-\int_{D^2} u^*\omega)$.
- in our case $\pi_2(\Sigma_0) = 0$, $\pi_2(\Sigma_0, L_i) = 0$, so no bubbling.

Remarks:

- $\langle L_1, \dots, L_r \rangle$ = exceptional collection generating $D^b Lag$.
- objects also represent Lefschetz thimbles (Lagrangian discs bounded by L_i, fibering above γ_i)

Theorem. (Seidel) *Changing $\{\gamma_i\}$ affects $Lag(W, \{\gamma_i\})$ by mutations; $D^b Lag(W)$ depends only on $W : (Y, \omega) \to \mathbb{C}$.*

Example 1: weighted projective planes

(Auroux-Katzarkov-Orlov, math.AG/0404281; cf. work of Seidel on $\mathbb{CP}^2$)

$X = \mathbb{CP}^2(a, b, c) = (\mathbb{C}^3 - \{0\})/(x, y, z) \sim (t^a x, t^b y, t^c z)$ (Fano orbifold).

$D^b Coh(X)$ has an exceptional collection $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(N-1)$ $(N = a + b + c)$

(Homogeneous coords. x, y, z are sections of $\mathcal{O}(a), \mathcal{O}(b), \mathcal{O}(c)$)

$\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j)) \simeq$ deg. $(j - i)$ part of symmetric algebra $\mathbb{C}[x, y, z]$ (degs. a, b, c)

All in degree 0 (no Ext's); composition = obvious.

Mirror: $Y = \{x^a y^b z^c = 1\} \subset (\mathbb{C}^*)^3$, $W = x + y + z$. ($Y \simeq (\mathbb{C}^*)^2$ if $gcd(a, b, c) = 1$)

$\mathbb{Z}/N$ $(N = a + b + c)$ acts by diagonal mult., the N crit. pts. are an orbit; complex conjugation.

We choose ω invariant under $\mathbb{Z}/N$ and complex conj. ($\Rightarrow [\omega] = 0$ exact)

Theorem. *$D^b Lag(W) \simeq D^b Coh(X)$.*

(this should extend to weighted projective spaces in all dimensions; for technical reasons we only have a partial argument when $\dim_{\mathbb{C}} \geq 3$).

Non-commutative deformations

$X = \mathbb{CP}^2(a,b,c)$; $\qquad Y = \{x^a y^b z^c = 1\} \subset (\mathbb{C}^*)^3,\ W = x + y + z,$

Theorem. *If ω is exact, then $D^b Lag(W) \simeq D^b Coh(X)$.*

Can deform $Lag(W)$ by changing $[\omega]$ (and introducing a B-field).

Choose $t \in \mathbb{C}$, and take $\boxed{\int_{S^1\times S^1}[B + i\omega] = t}$ $\quad$ ($S^1 \times S^1$ = generator of $H_2(Y,\mathbb{Z}) \simeq \mathbb{Z}$)

$\longrightarrow$ deformed category $D^b Lag(W)_t$.

This corresponds to a **non-commutative deformation** X_t of X:
deform weighted polynomial algebra $\mathbb{C}[x,y,z]$ to

$$yz = \mu_1\, zy, \quad zx = \mu_2\, xz, \quad xy = \mu_3\, yx, \qquad \text{with } \boxed{\mu_1^a \mu_2^b \mu_3^c = e^{it}}$$

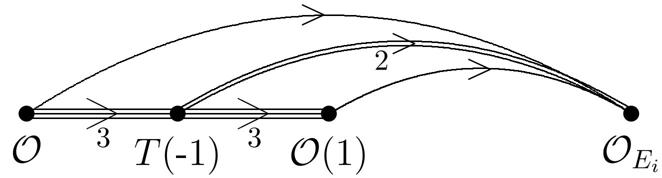
Theorem. $\forall t \in \mathbb{C},\ D^b Lag(W)_t \simeq D^b Coh(X)_t$.

Example 2: Del Pezzo surfaces

(Auroux-Katzarkov-Orlov, math.AG/0506166)

$X = \mathbb{CP}^2$ blown up at $k \leq 9$ points, $-K_X$ ample (or more generally, nef).

$D^bCoh(X)$ has an exceptional collection $\mathcal{O}$, $\pi^* T_{\mathbb{P}^2}(-1)$, $\pi^*\mathcal{O}_{\mathbb{P}^2}(1)$, $\mathcal{O}_{E_1}, \dots, \mathcal{O}_{E_k}$



Compositions encode coordinates of blown up points. For generic blowups, $\mathrm{Hom}(\mathcal{O}_{E_i}, \mathcal{O}_{E_j}) = 0$.
Infinitely close blowups give pairs of morphisms in deg. 0 and 1 (recover $\mathcal{O}_C$ (-2-curve) as a cone).

Mirror: mirror to $\mathbb{CP}^2$ compactifies to $\overline{M}$ = resolution of $\{XYZ = T^3\} \subset \mathbb{CP}^3$, with elliptic fibration $W = T^{-1}(X + Y + Z) : \overline{M} \to \mathbb{C} \cup \{\infty\}$.

W is Morse, with 3 crit. pts. in $\{|W| < \infty\}$; fiber at infinity has 9 components.

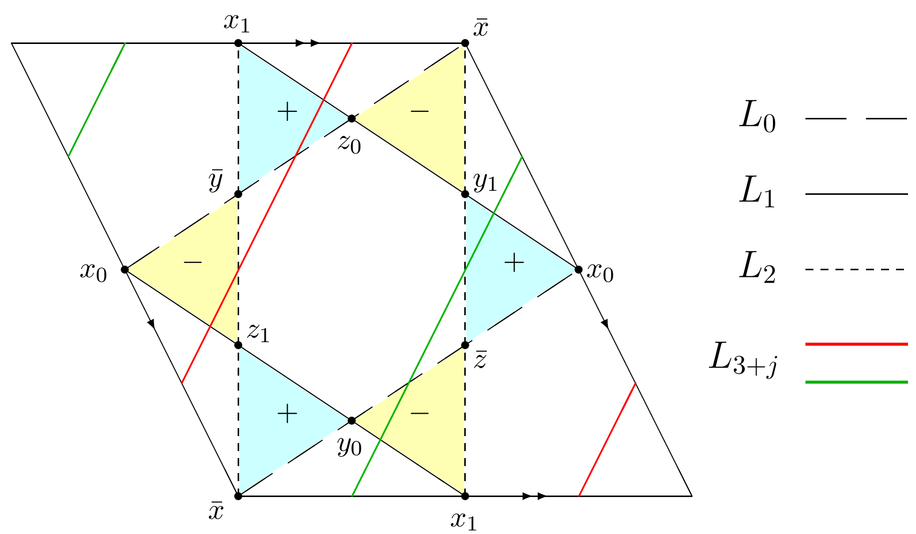
Mirror to X = deform $(\overline{M}, W)$ to bring k of the crit. pts. over ∞ into finite part.

Get an elliptic fibration over $\{|W_k| < \infty\}$: $W_k : M_k \to \mathbb{C}$, with $3 + k$ sing. fibers.

(symplectic form to be specified later)

Theorem. *For suitable choice of* $[B + i\omega]$, $D\,Lag(W_k) \simeq D^bCoh(X_k)$.

The vanishing cycles of W_k



Symplectic deformation parameters: $[B + i\omega] \in H^2(M_k, \mathbb{C})$:

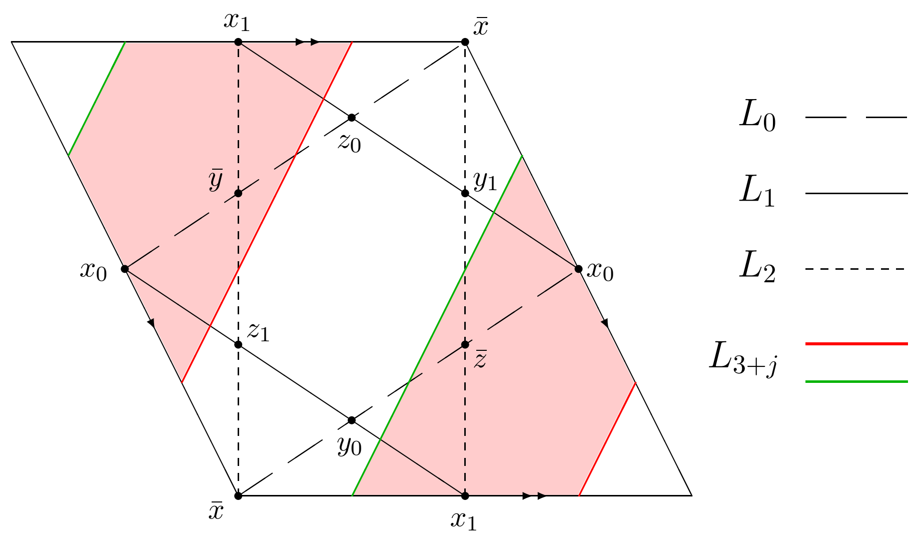
- Area of fiber: $\tau = \frac{1}{2\pi}\int_\Sigma (B + i\omega)$ $\longleftrightarrow$ cubic curve $\mathbb{CP}^2 \supset E \simeq \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$

  (all blowups are at points of E; think of E as zero set of $\beta \in H^0(\Lambda^2 T)$.)

- Area of C ($\partial C = L_0 + L_1 + L_2$): $t = \frac{1}{2\pi}\int_C (B + i\omega)$ $\longleftrightarrow$ $\sigma \in \mathrm{Pic}_0(E)$

  (same parameter as in Example 1; commutative deformations correspond to $t = 0$; takes values in $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$.)

The vanishing cycles of W_k



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- Areas of cycles C_j ($\partial C_j = L_{3+j} + \dots$): $t_j = \frac{1}{2\pi}\int_{C_j}(B + i\omega)$, take values in $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$.

  = positions of blown up points on E.

For $t_i - t_j = 0 \mod (\mathbb{Z}+\tau\mathbb{Z})$, L_{3+i}, L_{3+j} become Ham. isotopic, acquire $HF^*(L_{3+i}, L_{3+j}) \simeq H^*(S^1)$.

This corresponds to infinitely close blowups, where -2-curves appear.