Coisotropic Branes and Homological Mirror Symmetry for Tori

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(based on Yingdi Qin's PhD thesis)[1]

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(Simons Collaboration on Homological Mirror Symmetry)

[1] except for the sign mistakes, which are entirely mine

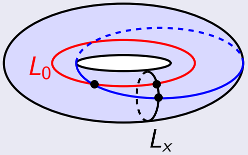
Jacobi theta functions and counting triangles

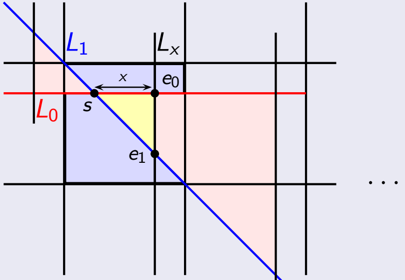
Jacobi theta function on the elliptic curve $E = \mathbb{C} / \mathbb{Z}+\tau\mathbb{Z}$

All doubly periodic holomorphic functions are constant, but we can ask for *quasi-periodic* functions: $s(z+1) = s(z)$, $s(z+\tau) = e^{-\pi i\tau - 2\pi i z}s(z)$ (section of deg. 1 line bundle $\mathcal{L} \to E$)

Only one up to scaling! $s(z) = \vartheta(\tau; z) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau + 2\pi i n z)$. (Jacobi, 1820s)

Counting triangles in $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ (weighted by area)





$$\boxed{?} = \cdots + q^{(x-1)^2/2} + q^{x^2/2} + q^{(x+1)^2/2} + \ldots$$
$$= q^{x^2/2}\sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2+nx} = e^{\pi i\tau x^2}\vartheta(\tau;\tau x) \quad (q = e^{2\pi i\tau})$$

Homological mirror symmetry (Kontsevich 1994)

Algebraic (or analytic) geometry

Coherent sheaves (eg: $\mathcal{O}_V$, vector bundles $\mathcal{E} \to V$, skyscrapers $\mathcal{O}_{p\in V}$, ...)
Morphisms (+ extensions): $H^* hom(\mathcal{E}, \mathcal{F}) = Ext^*(\mathcal{E}, \mathcal{F})$.

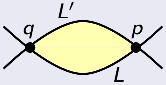
Derived category = complexes $0 \to \cdots \to \mathcal{E}^i \xrightarrow{d^i} \mathcal{E}^{i+1} \to \cdots \to 0 \;/\sim$
Eg: functions, intersections, cohomology...

$\Updownarrow$ **Mirror symmetry:** $D^b Coh(V) \simeq D^\pi \mathcal{F}(X, \omega)$ in general: over Novikov field
here: over $\mathbb{C}$

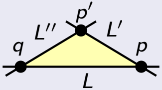
Symplectic geometry: **Fukaya category** $\mathcal{F}(X, \omega)$

(X, ω) loc.$\simeq$ $(\mathbb{R}^{2n}, \sum dx_i \wedge dy_i)$, Lagrangian submanifolds L (*dim. n*, $\omega_{|L} = 0$) + rk 1 loc. system ∇.
Floer cohomology measures intersections (physicists' version: over $\mathbb{C}$ instead of Novikov field)

$CF^*(L, L') = \mathbb{C}^{|L\cap L'|}$
(⊗ local coefficients)

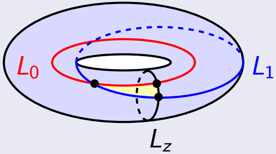


$\partial p = \exp(2\pi i \int (B + i\omega))\, hol_\nabla\, q$

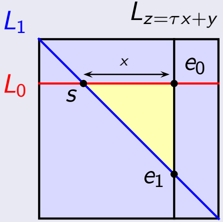


Product $CF(L', L'') \otimes CF(L, L') \to CF(L, L'')$: $p' \cdot p = \exp(2\pi i \int (B + i\omega))\, hol_\nabla\, q$

$T^2_\tau = \mathbb{R}^2/\mathbb{Z}^2,\ B+i\omega = \tau\, dr\wedge d\theta$ vs. $E_\tau = \mathbb{C}\,/\,(\mathbb{Z}+\tau\mathbb{Z})$



$L_{z=\tau x+y} = \{x\}\times S^1_\theta \quad (\nabla = d + 2\pi i y\, d\theta)$



In $\mathcal{F}(T^2_\tau)$, $L_0 \xrightarrow{s} L_1 \xrightarrow{e_1} L_z$ (mirror to: $\mathcal{O}\to\mathcal{L}\to\mathcal{O}_z$ on E_τ)

$e_1\cdot s = \boxed{?}\ e_0$ $\quad s \sim$ section of $\mathcal{L}$, $e_0\sim$ evaluation $\mathcal{O}\to\mathcal{O}_z$

$$\boxed{?} = \sum_{n\in\mathbb{Z}} e^{\pi i\tau(x+n)^2+2\pi i(x+n)y} + \ldots = e^{\pi i\tau x^2+2\pi i x y}\vartheta_\tau(z)$$

$$\vartheta_\tau(z) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau + 2\pi i n z) \in H^0(E_\tau,\mathcal{L}): \begin{cases}\vartheta(z+1)=\vartheta(z),\\ \vartheta(z+\tau) = e^{-\pi i\tau-2\pi i z}\vartheta(z)\end{cases}$$ (Jacobi, 1820s)

Similarly for rest of $\mathcal{F}(T^2_\tau)\simeq \mathrm{Coh}(E_\tau)$. In higher dim., $\tau\in Mat_{n\times n}(\mathbb{C})$ (Fukaya, Kontsevich-Soibelman, Abouzaid-Smith)

What is the mirror of multiplication by i?

Over $\mathbb{C}$, the Fukaya category of $T_i^2 = (\mathbb{R}^2/\mathbb{Z}^2, \omega_0)$ has an autoequivalence mirror to complex multiplication by i on $E_i = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$. This should "exchange position and holonomy":

$$L_{z=ix+y} = (\{x\} \times S^1_\theta, \nabla = d + 2\pi i y\, d\theta) \longleftrightarrow L_{iz=iy-x} = (\{y\} \times S^1_\theta, \nabla = d - 2\pi i x\, d\theta).$$

No Lagrangian correspondence in $(T^2 \times T^2, \omega = -dr_1\, d\theta_1 + dr_2\, d\theta_2)$ induces such a functor.

A coisotropic correspondence ?

Consider the line bundle $(\xi_C \to T^2 \times T^2,\ \nabla_C = d - 2\pi i(r_1\, d\theta_2 + r_2\, d\theta_1))$. Then the "mult. by i" functor maps to $(\xi \to L, \nabla)$ to $(\pi_2)_*(\xi_C \otimes \pi_1^*\xi)$ (check: this has Lagrangian support).

We think of $C = (\xi_C \to T^2 \times T^2, \nabla_C)$ as a coisotropic correspondence.
The curvature $F = dr_1\, d\theta_2 + dr_2\, d\theta_1$ satisfies $(\omega^{-1}F)^2 = -1$.

(Similarly for $\mathcal{F}(T^2_\tau) \simeq \mathcal{F}(T^2_{-1/\tau})$). **Q:** How does C fit into $\mathcal{F}(T^2 \times T^2)$?

Note: none of this occurs over the Novikov field / for non-archimedean abelian varieties!

Coisotropic branes (Kapustin-Orlov 2001)

Kapustin-Orlov observed: for (T^4, ω_0), the image of $ch : \mathcal{F}(T^4) \to H_2(T^4)$, $L \mapsto [L]$ has rank 5, while for $V = E_i \times E_i$, $ch : D^b(V) \twoheadrightarrow \bigoplus_p H^{p,p}(V)$ which has rank 6. So $\mathcal{F}(T^4) \not\simeq D^b(V)$.
Fix: take split-closure $Tw^\pi \mathcal{F}(T^4)$ (Abouzaid-Smith), or add coisotropics (Kapustin-Orlov).

Definition (Coisotropic branes) (without B-field) (Kapustin-Orlov)

A *coisotropic brane* consists of a coisotropic submanifold $C^{n+k} \subset (X^{2n}, \omega)$ with a U(1)-bundle (ξ, ∇) such that $\tilde{F} = \frac{1}{2\pi i} F_\nabla$ satisfies:
(i) $\tilde{F} = 0$ on the isotropic leaves $TC^{n-k}_{iso} = \ker \omega_{|TC}$,
(ii) $(\omega^{-1}\tilde{F})^2 = -1$ complex structure on TC/TC_{iso}. ($\Rightarrow \tilde{F} + i\omega$ holom. symplectic, and k is even).
Proposal: $End(C, \nabla) = H^*(C, \mathcal{O}_C)$ (loc. constant in TC_{iso}, holomorphic in TC/TC_{iso}).

(Note: in X^4, $\tilde{F} \wedge \tilde{F} = \omega \wedge \omega$ and $[\tilde{F}] \in H^2(X, \mathbb{Z}) \Rightarrow$ coisotropic branes only exist at special locus in Kähler moduli space)
In (T^4, ω_0), the "missing" generator is mirror to $\mathcal{O}_\Gamma$, where $\Gamma = \{(z, iz)\} \subset V$ (mult. by i).
Candidate: $C = T^4$, $\nabla = d - 2\pi i(r_1\, d\theta_2 - r_2\, d\theta_1)$.

Question: how to enlarge $\mathcal{F}(T^{2n})$ to include coisotropic branes? (hom(L, C)? compositions?)
Theorem (Yingdi Qin): this can be done by a *doubling* construction.

Reformulating the theta function

Classical fact: $f(z) = e^{\pi z^2/2}\vartheta_i(z) = e^{\pi z^2/2}\sum_{n\in\mathbb{Z}} e^{-\pi n^2} e^{2\pi i n z}$ satisfies $f(iz) = f(z)$!

Slogan: understanding mult. by i in HMS is difficult because this invariance property is not obvious!

Alternative, invariant expression: $c\, f(z) = \sum_{(m,n)\in\mathbb{Z}^2} (-1)^{mn} e^{-\pi(m^2+n^2)/2}\, e^{\pi(m+in)z}$ $\quad (c = \sqrt{2}\,\vartheta(0))$

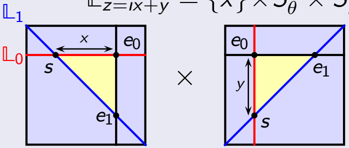
Symplectic interpretation: Floer product in $T^2 \times T^2$

In $\mathbb{T} = T^2 \times T^2$, with

$\omega = \frac{1}{2}(dr\wedge d\theta + d\hat{r}\wedge d\hat{\theta})$,

$B = \frac{1}{2}(d\hat{r}\wedge dr + d\hat{\theta}\wedge d\theta)$

$\mathbb{L}_{z=ix+y} = \{x\}\times S^1_\theta \times S^1_{\hat{r}} \times \{y\}$ $\quad (\nabla = d + 2\pi i y\, d\theta)$



$\Rightarrow e_1 \cdot s \sim f(z)\, e_0$.

In general, $(-2i\tau)^{1/2}\vartheta_\tau(0)\, e^{\pi i z^2/2\tau}\,\vartheta_\tau(z) = \sum_{m,n}(-1)^{mn} e^{\pi i n^2\tau/2} e^{-\pi i m^2/2\tau} \exp(\pi i(\frac{m}{\tau} + n)z)$

Similar interpretation in $T^2 \times T^2$ with mutually inverse symplectic areas!

Lifting branes to the doubled torus (Yingdi Qin)

Definition

Given a symplectic torus $(T = V/\Lambda, \omega)$, the linear dual torus (with inverse symplectic form) $(T^* = V^*/\Lambda^*, \omega^{-1})$, and the standard pairing $\sigma_0 = \sum_i d\hat{x}_i \wedge dx_i$, define $\mathbb{T} = T \times T^*$, with symplectic form $\frac{1}{2}(\omega \oplus \omega^{-1})$ and B-field $\frac{1}{2}\sigma_0$.

So: $\boldsymbol{B} + i\boldsymbol{\omega} = \frac{1}{2}\sigma_0 + \frac{1}{2}(\omega \oplus \omega^{-1})$. (Eventually T should be allowed to carry a B-field!)

Definition

Given a brane (L, ∇, ε) in T, let

$$\mathbb{L} = \{(x, x^*) \in T \times T^* \,|\, x \in L \text{ and } \forall \gamma \in H_1(L) \subset \Lambda,\ \exp(2\pi i \langle x^*, \gamma\rangle) = (-1)^{\varepsilon(\gamma)} hol_\nabla(x + S^1_\gamma)\},$$

with $\boldsymbol{\nabla} = \pi^*\nabla$. This is a Lagrangian brane in $\mathbb{T}$.

$\varepsilon : H_1(L, \mathbb{Z}) \to \mathbb{Z}/2$ such that $\varepsilon(\gamma + \gamma') - \varepsilon(\gamma) - \varepsilon(\gamma') = c_1(\nabla)(\gamma \wedge \gamma') \bmod 2$ ($\Leftrightarrow$ rel. spin structure)

When L is Lagrangian, $\mathbb{L} = L \times (L^\perp + hol_\nabla)$ (translated) conormal.
For $L = T$ space-filling coisotropic, $\mathbb{L} =$ "graph of ∇" (Lagr. for $\boldsymbol{\omega} = \frac{1}{2}(\omega \oplus \omega^{-1})$ using $(\omega^{-1}F)^2 = -1$).

The mirror of the doubled torus (Yingdi Qin)

For $T = (\mathbb{R}/\mathbb{Z})^{2n}$, $\omega = a\,dr \wedge d\theta$ ($\tau = ia \in M_{n\times n}(\mathbb{C})$), with SYZ fibers $F = \{r\} \times T^n_\theta$,
the mirror abelian variety is $E = \mathbb{C}^n/(\mathbb{Z}^n + \tau(\mathbb{Z}^n))$.

For $T^* = (\mathbb{R}/\mathbb{Z})^{2n}$, $\omega^{-1} = a^{-1}\,d\hat{r} \wedge d\hat{\theta}$, with SYZ fibers $F^* = T^n_{\hat{r}} \times \{\hat{\theta}\}$,
the mirror abelian variety is $\hat{E} = \mathbb{C}^n/(\mathbb{Z}^n + \tau^{-1}(\mathbb{Z}^n))$ $\quad (\simeq E)$.

For $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^{4n}$, $\boldsymbol{B} + i\boldsymbol{\omega} = \frac{1}{2}(\sigma_0 + ia\,dr \wedge d\theta + ia^{-1}\,d\hat{r} \wedge d\hat{\theta})$, $\mathbb{F} = \{r\} \times T^n_\theta \times T^n_{\hat{r}} \times \{\hat{\theta}\}$,
the mirror is $\mathbb{E} = (\mathbb{C}^n \times \mathbb{C}^n)/(\mathbb{Z}^{2n} + \boldsymbol{\tau}(\mathbb{Z}^{2n}))$, $\boldsymbol{\tau} = \frac{1}{2}\begin{pmatrix} \tau & \mathbb{1} \\ -\mathbb{1} & -\tau^{-1} \end{pmatrix}$.
Using coordinates $u = z + \tau\hat{z}$, $v = z - \tau\hat{z}$, we have: $\mathbb{E} \simeq E \times E$.

There are simpler mirrors to $E \times E$; this one has the property that, even if a sheaf $\mathcal{E} \in Coh(E)$ corresponds to a coisotropic in T, a closely related sheaf on $E \times E$ corresponds to a *Lagrangian* in $\mathbb{T}$.

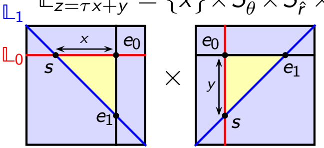
SYZ fibers $F \subset T$ lift to fibers $\mathbb{F} \subset \mathbb{T}$ which correspond to points in $E \times 0$, i.e. $v = 0$.
$\{r\} \times T^n_\theta \times T^n_{\hat{r}} \times \{\hat{\theta}\}$, $\nabla = d + 2\pi i(y\,d\theta + \hat{y}\,d\hat{r}) \leftrightarrow (z = \frac{\tau}{2}r + y - \frac{\hat{\theta}}{2}, \hat{z} = \frac{\tau^{-1}}{2}\hat{\theta} + \hat{y} + \frac{r}{2})$.
Lifts from T have $\hat{\theta} = y$ and $\hat{y} = 0$, so $(z = \frac{1}{2}(\tau r + y), \hat{z} = \frac{1}{2}(\tau^{-1}y + r))$, hence $(u = \tau r + y, v = 0)$.
$L \sim \mathcal{O}_p \Rightarrow \mathbb{L} \sim \mathcal{O}_p \boxtimes \mathcal{O}_0$. Similarly, $L \sim \mathcal{E} \Rightarrow \mathbb{L} \sim \mathcal{E} \boxtimes \mathcal{E}_0$. (if $\mathcal{E} \in Pic_d(E)$, then $\mathcal{E}_0$ = origin of Pic_d)

$HF(\mathbb{L},\mathbb{L}')$: first examples

$\mathbb{L} = \{(x,x^*) \in T\times T^* \,|\, x\in L \text{ and } \forall\gamma\in H_1(L),\ \exp(2\pi i\langle x^*,\gamma\rangle) = (-1)^{\varepsilon(\gamma)} hol_\nabla(x+S^1_\gamma)\}$,
$\mathbb{V} = \pi^*\nabla$. (For L Lagrangian, $\mathbb{L} = L\times(L^\perp + hol_\nabla)$.)
If T is mirror to E then $\mathbb{T}$ is mirror to $E\times E$, and $L\sim\mathcal{E} \Rightarrow \mathbb{L}\sim\mathcal{E}\boxtimes\mathcal{E}_0$.

$HF(\mathbb{L},\mathbb{L}')$ works well for deg. 1 line bundles

$\mathbb{L}_{z=\tau x+y} = \{x\}\times S^1_\theta\times S^1_{\hat r}\times\{y\}$ $(\nabla = d+2\pi i y\, d\theta)$



$\Rightarrow e_1\cdot s \sim \vartheta_\tau(z)\, e_0$.

and for graph of mult. by i in square torus:
$C = T^4$, $\nabla = d - 2\pi i(r_1\, d\theta_2 - r_2\, d\theta_1)$
$\mathbb{C} = \{\hat r_1 = \theta_2,\ \hat\theta_1 = r_2,\ \hat r_2 = -\theta_1,\ \hat\theta_2 = -r_1\}$, ∇
$\mathbb{L}_z$: $r_i = \mathrm{Im}\, z_i$, $\hat\theta_i = \mathrm{Re}\, z_i$, $\nabla = d + 2\pi i\sum(\mathrm{Re}\, z_j)\, d\theta_j$
$\Rightarrow HF(\mathbb{C},\mathbb{L}_z)\neq 0$ iff $z_2 = iz_1$.
(but then… $HF(\mathbb{C},\mathbb{L}_z)\simeq H^*(T^2)$ instead of $H^*(S^1)$)

In general, Hom spaces are too large: $Ext^*(\mathcal{E}\boxtimes\mathcal{E}_0,\mathcal{F}\boxtimes\mathcal{F}_0) = Ext^*(\mathcal{E},\mathcal{F})\otimes Ext^*(\mathcal{E}_0,\mathcal{F}_0)$.
For Lagrangians, $\dim HF(\mathbb{L},\mathbb{L}') = (\dim HF(L,L'))^2$; restrict to "u-part"?

The u-part: $HF(\mathbb{L},\mathbb{L})_u \subset HF(\mathbb{L},\mathbb{L})$ (Yingdi Qin)

$HF(\mathbb{L},\mathbb{L}) \simeq H^*(\mathbb{L};\mathbb{C}) = \bigwedge H^1(\mathbb{L};\mathbb{C})$. Among all deformations of $\mathbb{L}$ in $\mathcal{F}(\mathbb{T})$, only consider those which are lifted from $Def(L \subset T)$. On mirror: $Ext^1(\mathcal{E}\boxtimes\mathcal{E}_0,\mathcal{E}\boxtimes\mathcal{E}_0) \supset Ext^1(\mathcal{E},\mathcal{E})\otimes 1$.

This has a nice geometric characterization:

A complex structure on $\mathbb{T}$

On $\mathbb{T}$, let $\boldsymbol{B} + i\boldsymbol{\omega}^\pm = \frac{1}{2}\sigma_0 + \frac{i}{2}(\omega \oplus \pm\omega^{-1})$. Then $\mathbb{J} = \boldsymbol{B}^{-1}\boldsymbol{\omega}^-$ is a complex structure ($\mathbb{J}^2=-1$), mapping $v \in T(T) = V$ to $\iota_v\omega \in V^* = T(T^*)$.

Fact: $\mathbb{L} \subset \mathbb{T}$ is a complex submanifold! (but not complex Lagrangian w.r.t. holom. sympl. structure)

The deformations of $\mathbb{L}$ which come from lifting correspond to $H^{0,1}_{\mathbb{J}}(\mathbb{L}) \subset H^1(\mathbb{L};\mathbb{C})$.

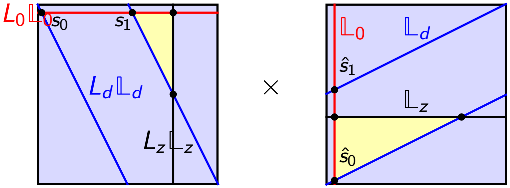
Definition

$HF(\mathbb{L},\mathbb{L})_u := H^{0,*}_{\mathbb{J}}(\mathbb{L}) \subset H^*(\mathbb{L};\mathbb{C}) = HF(\mathbb{L},\mathbb{L})$.

Similarly for the "continuous" part of $\mathbb{L}\cap\mathbb{L}'$ when $\mathbb{L},\mathbb{L}'$ not transverse.

In T^2_τ, consider $L_0 : \{\theta = 0\}$ and $L_d : \{\theta = -d\,r\}$ (mirrors to $\mathcal{O}$, $\mathcal{L}^{\otimes d}$).
The generators $s_k = (\frac{k}{d}, 0) \in L_0 \cap L_d$ of $HF(L_0, L_d)$ correspond to the ϑ-basis of $H^0(E, \mathcal{L}^{\otimes d})$:

$$\vartheta_{k/d}(z) = \sum_{n\in\mathbb{Z}} \exp\Big(\pi i d(n+\tfrac{k}{d})^2\tau + 2\pi i d(n+\tfrac{k}{d})z\Big)$$



In $\mathbb{T}$, the generator $s_j \otimes \hat{s}_k \in CF(\mathbb{L}_0, \mathbb{L}_d)$ corresponds to

$$\sum_{\ell\in\mathbb{Z}/d} e^{2\pi i k\ell/d}\,\vartheta_{(j-\ell)/d}(u)\,\vartheta_{\ell/d}(v) \in H^0(E\times E, \mathcal{L}^{\otimes d}\boxtimes\mathcal{L}^{\otimes d}).$$

Definition

$HF(\mathbb{L}_0,\mathbb{L}_d) \supset HF(\mathbb{L}_0,\mathbb{L}_d)_u$ = span of $s_j \otimes (\sum_k \hat{s}_k)$ $(j\in\mathbb{Z}/d)$ $\quad (\leftrightarrow \vartheta_{j/d}(u)\,\vartheta_{0/d}(v))$

Main result

For L, L' linear (Lagrangian or coisotropic) branes in (T, ω), consider their lifts $\mathbb{L}, \mathbb{L}'$ to Lagrangian branes in $(\mathbb{T}, \boldsymbol{B} + i\boldsymbol{\omega})$, and define

$$HF(\mathbb{L}, \mathbb{L}')_u = T^*\text{-translation-invariant, } (0,1)_{\mathbb{J}} \text{ part of } HF(\mathbb{L}, \mathbb{L}').$$

with product $x \cdot y = \pi_T(i(x) \cdot i(y))$ using inclusion and projection $HF(\mathbb{L}, \mathbb{L}')_u \underset{i}{\overset{\pi_T}{\leftrightarrows}} HF(\mathbb{L}, \mathbb{L}')$.

Theorem (Yingdi Qin)

Let $H^*\mathcal{F}(\mathbb{T})_u$ be the category whose objects are linear (Lagrangian or coisotropic) branes in T, with $\hom(L, L') = HF(\mathbb{L}, \mathbb{L}')_u$*, and composition $x \cdot y = \pi_T(i(x) \cdot i(y))$. The Donaldson-Fukaya category $H^*\mathcal{F}(T)$ of linear Lagrangians in T embeds fully faithfully into $H^*\mathcal{F}(\mathbb{T})_u$.*

This gives a version of the Fukaya category which includes coisotropic branes!

$H^*\mathcal{F}(T,\omega) \simeq H^*\mathcal{F}(T^*,\omega^{-1})$

Given dual tori (T,ω) and (T^*,ω^{-1}), the doubled tori $\mathbb{T} = T \times T^* \simeq T^* \times T = \mathbb{T}^*$ carry the same symplectic form $\frac{1}{2}(\omega \oplus \omega^{-1})$, but opposite B-fields $\pm\frac{1}{2}\sigma_0 = \pm\frac{1}{2}\sum_i d\hat{x}_i \wedge dx_i$.

$[\sigma_0] \in H^2(\mathbb{T},\mathbb{Z})$: the tautological bundle $(\xi_\mathcal{T}, \nabla_\mathcal{T} = d + 2\pi i \sum \hat{x}_i\, dx_i)$ has curvature $2\pi i \sigma_0$.
Hence $\mathcal{F}(\mathbb{T}^*) \simeq \mathcal{F}(\mathbb{T})$ via B-twist $\beta = - \otimes (\xi_\mathcal{T}, \nabla_\mathcal{T})$.

Under β, branes lifted from $T \longleftrightarrow$ branes lifted from T^*.
The subspaces $HF(\mathbb{L},\mathbb{L}')_u$ are different ((u,v)-splitting is the same, but $\vartheta_{\tau^{-1},\frac{0}{d}}(\tau^{-1}v) \neq \vartheta_{\tau,\frac{0}{d}}(v)$) but the projections π_T, π_{T^*} induce isomorphisms. Hence $H^*\mathcal{F}(\mathbb{T})_u \simeq H^*\mathcal{F}(\mathbb{T}^*)_u$.
Restricting to lifts of Lagrangians, $H^*\mathcal{F}(T,\omega) \simeq H^*\mathcal{F}(T^*,\omega^{-1})$. (e.g. T^2's of inverse areas).
Note: $\mathcal{F}(T) \simeq \mathcal{F}(T^*)$ is induced by a coisotropic corresp. in $T \times T^*$, which lifts to the diagonal in $\mathbb{T} \times \mathbb{T}^* \simeq \mathbb{T} \times \mathbb{T}$!)

This also works for partial dualization. E.g., in $\mathbb{T} = T^4 \times T^4$, $\mathbb{C} = \{\hat{r}_1 = \theta_2,\ \hat{\theta}_1 = r_2,\ \hat{r}_2 = -\theta_1,\ \hat{\theta}_2 = -r_1\}$, (with ∇ dependent on B-field twist) is lifted from any of the following branes:

$C = T^4_{r_1,\theta_1,r_2,\theta_2}$, $\nabla = d - 2\pi i(r_1\, d\theta_2 - r_2\, d\theta_1)$
$C = T^4_{\hat{r}_1,\hat{\theta}_1,\hat{r}_2,\hat{\theta}_2}$, $\nabla = d + 2\pi i(\hat{r}_1\, d\hat{\theta}_2 - \hat{r}_2\, d\hat{\theta}_1)$
(mirror to $\mathcal{O}_\Gamma$, $\Gamma = \{z_2 = i z_1\}$)

$L = \{\hat{r}_1 = \theta_2,\ \hat{\theta}_1 = r_2\} \subset T^4_{\hat{r}_1,\hat{\theta}_1,r_2,\theta_2}$
$L = \{\hat{r}_2 = -\theta_1,\ \hat{\theta}_2 = -r_1\} \subset T^4_{r_1,\theta_1,\hat{r}_2,\hat{\theta}_2}$
(on mirror, rotate 2nd factor: $\mathcal{O}_\Delta$, $\Delta = \{z_2 = z_1\}$).