SPECIAL LAGRANGIAN FIBRATIONS, MIRROR SYMMETRY AND CALABI-YAU DOUBLE COVERS

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Abstract. The first part of this paper is a review of the Strominger-Yau-Zaslow conjecture in various settings. In particular, we summarize how, given a pair (X, D) consisting of a Kähler manifold and an anticanonical divisor, families of special Lagrangian tori in $X \setminus D$ and weighted counts of holomorphic discs in X can be used to build a Landau-Ginzburg model mirror to X. In the second part we turn to more speculative considerations about Calabi-Yau manifolds with holomorphic involutions and their quotients. Namely, given a hypersurface H representing twice the anticanonical class in a Kähler manifold X, we attempt to relate special Lagrangian fibrations on $X \setminus H$ and on the (Calabi-Yau) double cover of X branched along H; unfortunately, the implications for mirror symmetry are far from clear.

To Jean-Pierre Bourguignon on his 60th birthday, with my most sincere gratitude for the time he spent guiding me through the process of becoming a mathematician.

1. Introduction

The phenomenon of mirror symmetry was first evidenced for Calabi-Yau manifolds, i.e. Kähler manifolds with holomorphically trivial canonical bundle. Subsequently it became apparent that mirror symmetry also holds in a more general setting, if one enlarges the class of objects under consideration (see e.g. [14]); namely, one should allow the mirror to be a *Landau-Ginzburg model*, i.e. a pair consisting of a non-compact Kähler manifold and a holomorphic function on it (called *superpotential*).

Our motivation here is to understand how to *construct* the mirror manifold, starting from examples where the answer is known and extrapolating to less familiar situations; generally speaking, the verification of the mirror symmetry conjectures for the manifolds obtained by these constructions falls outside the scope of this paper.

The geometric understanding of mirror symmetry in the Calabi-Yau case relies on the Strominger-Yau-Zaslow (SYZ) conjecture [28], which roughly speaking postulates that mirror pairs of Calabi-Yau manifolds carry dual fibrations by special Lagrangian tori, and its subsequent refinements (see e.g. [10, 21]). This program can be extended to the non Calabi-Yau case, as suggested by Hori [12] and further investigated in [3]. In that case, the input consists of a pair (X, D) where X is a compact Kähler manifold

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and D is a complex hypersurface representing the anticanonical class. Observing that the complement of D carries a holomorphic n-form with poles along D, we can think of $X \setminus D$ as an open Calabi-Yau manifold, to which one can apply the SYZ program. Hence, one can attempt to construct the mirror of X as a (complexified) moduli space of special Lagrangian tori in $X \setminus D$, equipped with a Landau-Ginzburg superpotential defined by a weighted count of holomorphic discs in X. However, exceptional discs and wall-crossing phenomena require the incorporation of "instanton corrections" into the geometry of the mirror (see [3]).

One notable feature of the construction is that it provides a bridge between mirror symmetry for the Kähler manifold X and for the Calabi-Yau hypersurface $D \subset X$. Namely, the fiber of the Landau-Ginzburg superpotential is expected to be the SYZ mirror to D, and the two pictures of homological mirror symmetry (for X and for D) should be related via restriction functors (see Section 7 of [3] for a sketch).

In this paper, we would like to consider a slightly different situation, which should provide another relation with mirror symmetry for Calabi-Yau manifolds. The union of two copies of X glued together along D can be thought of as a singular Calabi-Yau manifold, which can be smoothed to a double cover of X branched along a hypersurface H representing twice the anticanonical class and contained in a neighborhood of D. This suggests that one might be able to think of mirror symmetry for X as a $\mathbb{Z}/2$-invariant version of mirror symmetry for the Calabi-Yau manifold Y. Unfortunately, this proposal comes with several caveats which make it difficult to implement.

Let (X, ω, J) be a compact Kähler manifold, and let H be a complex hypersurface in X representing twice the anticanonical class. Then the complement of H carries a nonvanishing section Θ of $K_X^{\otimes 2}$ with poles along H. We can think of Θ as the square of a holomorphic volume form defined up to sign. In this context, we can look for special Lagrangian submanifolds of $X \setminus H$, i.e. Lagrangian submanifolds on which the restriction of Θ is real. The philosophy of the SYZ conjecture suggests that, in favorable cases, one might be able to construct a foliation of $X \setminus H$ in which the generic leaves are special Lagrangian tori. Indeed, denote by Y the double cover of X branched along H: then Y is a Calabi-Yau manifold with a holomorphic involution. If Y carries a special Lagrangian fibration that is invariant under the involution, then by quotienting we could hope to obtain the desired foliation on $X \setminus H$; unfortunately the situation is complicated by technicalities involving the symplectic form.

Conjecture 1.1. *For a suitable choice of H, $X \setminus H$ carries a special Lagrangian foliation whose lift to the Calabi-Yau double cover Y can be perturbed to a $\mathbb{Z}/2$-invariant special Lagrangian torus fibration.*

If $-K_X$ is effective, we can consider a situation where H degenerates to a hypersurface D representing the anticanonical class in X, with multiplicity 2. As explained above, this corresponds to the situation where Y degenerates to the union of two copies of X glued together along D. One could hope that under such a degeneration the foliation on $X \setminus H$ converges to a special Lagrangian torus fibration on $X \setminus D$. Using the mirror construction described in [3], one can then try to relate a Landau-Ginzburg mirror $(X^\vee, W)$ of X to a Calabi-Yau mirror $Y^\vee$ of Y. The simplest case should be when $K_{X|D}$ is holomorphically trivial (which in particular requires $c_1(X)^2 = 0$). Then $W : X^\vee \to \mathbb{C}$ is expected to have trivial monodromy around infinity (see Remark 2.11), so that $\partial X^\vee \approx S^1 \times D^\vee$ where $D^\vee$ is mirror to D. It is then tempting to conjecture that, considering only the complex structure of the mirror (and ignoring its symplectic geometry), $Y^\vee$ can be obtained by gluing together two copies of the mirror $X^\vee$ to X along their boundary $S^1 \times D^\vee$. Unfortunately, as we will see in §3.5 this is not compatible with instanton corrections.

The rest of this paper is organized as follows. In Section 2 we review the geometry of mirror symmetry from the perspective of the SYZ conjecture, both in the Calabi-Yau case and in the more general case (relatively to an anticanonical divisor). We then turn to more speculative considerations in Section 3, where we discuss the geometry of Calabi-Yau double covers, clarify the statement of Conjecture 1.1, and consider various examples.

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2. The SYZ conjecture and mirror symmetry

2.1. **Motivation.** One of the most spectacular mathematical predictions of string theory is the phenomenon of mirror symmetry, i.e. the existence of a broad dictionary under which the symplectic geometry of a given manifold X can be understood in terms of the complex geometry of a mirror manifold $X^\vee$, and vice-versa. This dictionary works at several levels, among which perhaps the most exciting is Kontsevich's homological mirror conjecture, which states that the derived Fukaya category of X should be equivalent to the derived category of coherent sheaves of its mirror $X^\vee$ [19]; in the non Calabi-Yau case the categories under consideration need to be modified appropriately [20] (see also [1, 13, 18, 26, 27]).

The main goal of the Strominger-Yau-Zaslow conjecture [28] is to provide a geometric interpretation of mirror symmetry. Roughly speaking it says that mirror manifolds carry dual fibrations by special Lagrangian tori. In the Calabi-Yau case, one way to motivate the conjecture is to observe that, given any point p of the mirror $X^\vee$, mirror symmetry should put the skyscraper sheaf $\mathcal{O}_p$ in correspondence with some object $\mathcal{L}_p$ of the Fukaya category of X. As a graded vector space $\mathrm{Ext}^*(\mathcal{O}_p, \mathcal{O}_p)$ is isomorphic to the cohomology of T^n; therefore the most likely candidate for $\mathcal{L}_p$ is a (special)

Lagrangian torus in X, equipped with a rank 1 unitary local system (a flat $U(1)$ bundle). This suggests that one should try to construct $X^\vee$ as a moduli space of pairs (L, ∇) where L is a special Lagrangian torus in X and ∇ is a flat unitary connection on the trivial line bundle over L. Since for each torus L the moduli space of flat connections can be thought of as a dual torus, we arrive at the familiar picture.

When X is not Calabi-Yau but the anticanonical class $-K_X$ is effective, we can still equip the complement of a hypersurface $D \in |-K_X|$ with a holomorphic volume form, and thus consider special Lagrangian tori in $X \setminus D$. However, in this case, holomorphic discs in X with boundary in L cause Floer homology to be *obstructed* in the sense of Fukaya-Oh-Ohta-Ono [6]: to each object $\mathcal{L} = (L, \nabla)$ we can associate an obstruction $\mathfrak{m}_0(\mathcal{L})$, given by a weighted count of holomorphic discs in (X, L), and the Floer differential on $CF^*(\mathcal{L}, \mathcal{L}')$ squares to $\mathfrak{m}_0(\mathcal{L}') - \mathfrak{m}_0(\mathcal{L})$. Moreover, even when the Floer homology groups $HF^*(\mathcal{L}, \mathcal{L})$ can still be defined, they are often zero, so that $\mathcal{L}$ is a trivial object of the Fukaya category. On the mirror side, these features of the theory can be replicated by the introduction of a *Landau-Ginzburg superpotential*, i.e. a holomorphic function $W : X^\vee \to \mathbb{C}$. Without getting into details, W can be thought of as an obstruction term for the B-model on $X^\vee$, playing the same role as $\mathfrak{m}_0$ for the A-model on X. In particular, a point of $X^\vee$ defines a nontrivial object of the category of B-branes $D^b_{sing}(X^\vee, W)$ only if it is a critical point of W [18, 24].

2.2. **Special Lagrangian fibrations and T-duality.** Let (X, ω, J) be a smooth compact Kähler manifold of complex dimension n. If X is Calabi-Yau, i.e. the canonical bundle K_X is holomorphically trivial, then X carries a globally defined holomorphic volume form $\Omega \in \Omega^{n,0}(X)$: this is the classical setting for mirror symmetry. Otherwise, assume that K_X^{-1} admits a nontrivial holomorphic section σ, vanishing along a hypersurface D. Typically we will assume that D is smooth, or with normal crossing singularities. Then $\Omega = \sigma^{-1}$ is a nonvanishing holomorphic $(n, 0)$-form over $X \setminus D$, with poles along D.

The restriction of Ω to a Lagrangian submanifold $L \subset X \setminus D$ does not vanish, and can be expressed in the form $\Omega_{|L} = \psi \operatorname{vol}_g$, where $\psi \in C^\infty(L, \mathbb{C}^*)$ and vol_g is the volume form induced on L by the Kähler metric $g = \omega(\cdot, J\cdot)$.

Definition 2.1. *A Lagrangian submanifold $L \subset X \setminus D$ is* special Lagrangian *if the argument of ψ is constant.*

The value of the constant depends only on the homology class $[L] \in H_n(X \setminus D, \mathbb{Z})$, and we will usually arrange for it to be a multiple of $\pi/2$. For simplicity, in the rest of this paragraph we will assume that $\Omega_{|L}$ is a real multiple of vol_g.

The following classical result is due to McLean [23] (at least when $|\psi| \equiv 1$, which is the case in the Calabi-Yau setting; see §9 of [16] or Proposition 2.5 of [3] for the case where $|\psi| \neq 1$):

Proposition 2.2 (McLean). *Infinitesimal special Lagrangian deformations of L are in one to one correspondence with cohomology classes in $H^1(L,\mathbb{R})$. Moreover, the deformations are unobstructed.*

More precisely, a section of the normal bundle $v \in C^\infty(NL)$ determines a 1-form $\alpha = -\iota_v\omega \in \Omega^1(L,\mathbb{R})$ and an $(n-1)$-form $\beta = \iota_v \mathrm{Im}\,\Omega \in \Omega^{n-1}(L,\mathbb{R})$. These satisfy $\beta = \psi *_g \alpha$, and the deformation is special Lagrangian if and only if α and β are both closed. Thus special Lagrangian deformations correspond to "ψ-harmonic" 1-forms $-\iota_v\omega \in \mathcal{H}^1_\psi(L) = \{\alpha \in \Omega^1(L,\mathbb{R}) |\, d\alpha = 0,\ d^*(\psi\alpha) = 0\}$ (recall $\psi \in C^\infty(L,\mathbb{R}_+)$ is the ratio between the volume elements determined by Ω and g).

In particular, special Lagrangian tori occur in n-dimensional families, giving a local fibration structure provided that nontrivial ψ-harmonic 1-forms have no zeroes.

The base B of a special Lagrangian torus fibration carries two natural affine structures, which we call "symplectic" and "complex". The first one, which encodes the symplectic geometry of X, is given by locally identifying B with a domain in $H^1(L,\mathbb{R})$ (where $L \approx T^n$). At the level of tangent spaces, the cohomology class of $-\iota_v\omega$ provides an identification of TB with $H^1(L,\mathbb{R})$; integrating, the local affine coordinates on B are the symplectic areas swept by loops forming a basis of $H_1(L)$. The other affine structure encodes the complex geometry of X, and locally identifies B with a domain in $H^{n-1}(L,\mathbb{R})$. Namely, one uses the cohomology class of $\iota_v\mathrm{Im}\,\Omega$ to identify TB with $H^{n-1}(L,\mathbb{R})$, and the affine coordinates are obtained by integrating $\mathrm{Im}\,\Omega$ over the n-chains swept by cycles forming a basis of $H_{n-1}(L)$.

In practice, B can usually be compactified to a larger space $\bar{B}$ (with non-empty boundary in the non Calabi-Yau case), by also considering singular special Lagrangian submanifolds that arise as limits of degenerating families of special Lagrangian tori; however the affine structures are only defined on the open subset $B \subset \bar{B}$.

Ignoring singular fibers and instanton corrections, the first candidate for the mirror of X is therefore a moduli space M of pairs (L,∇), where L is a special Lagrangian torus in X (or $X \setminus D$) and ∇ is a flat $U(1)$ connection on the trivial line bundle over L (up to gauge). The local geometry of M is well-understood [11, 22, 8, 3], and in particular we have the following result (see e.g. §2 of [3]):

Proposition 2.3. *M carries a natural integrable complex structure $J^\vee$ arising from the identification*

$$T_{(L,\nabla)}M = \{(v,\alpha) \in C^\infty(NL) \oplus \Omega^1(L,\mathbb{R}) \,|\, -\iota_v\omega + i\alpha \in \mathcal{H}^1_\psi(L) \otimes \mathbb{C}\},$$

a holomorphic n-form

$$\Omega^\vee((v_1,\alpha_1),\dots,(v_n,\alpha_n)) = \int_L (-\iota_{v_1}\omega + i\alpha_1) \wedge \dots \wedge (-\iota_{v_n}\omega + i\alpha_n),$$

and a compatible Kähler form

$$\omega^\vee((v_1,\alpha_1),(v_2,\alpha_2)) = \int_L \alpha_2 \wedge \iota_{v_1}\mathrm{Im}\,\Omega - \alpha_1 \wedge \iota_{v_2}\mathrm{Im}\,\Omega$$

(this formula for $\omega^\vee$ assumes that $\int_L \mathrm{Re}\,\Omega$ has been suitably normalized).

The moduli space of pairs M can be viewed as a complexification of the moduli space of special Lagrangian submanifolds; forgetting the connection gives a projection map $f^\vee$ from M to the real moduli space B. The fibers of this projection are easily checked to be special Lagrangian tori in $(M,\omega^\vee,\Omega^\vee)$.

The special Lagrangian fibrations $f : X \to \bar{B}$ (or rather, its restriction to the open subset $f^{-1}(B)$) and $f^\vee : M \to B$ can be viewed as fiberwise dual to each other. In particular, it is easily checked that the affine structure induced on B by the symplectic geometry of $f^\vee$ coincides with that induced by the complex geometry of f, and vice-versa. Giving priority to the symplectic affine structure, we will often implicitly equip B with the affine structure induced by the symplectic geometry of X, and denote by $B^\vee$ the same manifold equipped with the other affine structure (induced by the complex geometry of X, or the symplectic geometry of M).

Thus, the philosophy of the Strominger-Yau-Zaslow conjecture is that, in first approximation (ignoring instanton corrections), mirror symmetry amounts simply to exchanging the two affine structures on B. However, in general it is not at all obvious how to extend the picture to the compactification $\bar{B}$. The reader is referred to [28], [8], [22] for more details in the Calabi-Yau case, and to [12] and [3] for the non Calabi-Yau case.

2.3. **Mirror symmetry for Calabi-Yau manifolds.** Constructing a special Lagrangian fibration on a Calabi-Yau manifold is in general a challenging task, but there are a few situations where it can be done explicitly, for instance in the case of flat tori, or for hyperkähler manifolds. We give two well-known examples.

Example 2.4 (Elliptic curves)**.** Consider an elliptic curve $E = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$, where $\tau = i\gamma \in i\mathbb{R}_+$, equipped with the holomorphic volume form $\Omega = dz$ and a Kähler form ω such that $\int_E \omega = \lambda \in \mathbb{R}_+$. (The reason why we assume τ to be pure imaginary is that for simplicity we are suppressing any discussion of B-fields). Then the family of circles parallel to the real axis $\{\mathrm{Im}\,(z) = c\}$ defines a special Lagrangian fibration on E, with base $B \simeq S^1$. One easily checks that the length of B with respect to the affine metric is equal to λ for the symplectic affine structure, and γ for the complex affine structure. The mirror elliptic curve $E^\vee$ is obtained by exchanging the two affine structures on B; accordingly, it has modular parameter $\tau^\vee = i\lambda$ and symplectic area $\int_{E^\vee} \omega^\vee = \gamma$. (The reader is referred to [25] for a verification of homological mirror symmetry for the mirror pair $E, E^\vee$.)

Example 2.5 (K3 surfaces)**.** In the case of K3 surfaces, special Lagrangian fibrations can be built using hyperkähler geometry. Let (X, J) be an elliptically fibered K3 surface, for example obtained as the double cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along a suitably chosen algebraic curve of bidegree $(4,4)$: composing the covering map with projection to the first $\mathbb{CP}^1$ factor, we obtain an elliptic fibration $f : X \to \mathbb{CP}^1$ with 24 nodal singular fibers. Equip X with a Calabi-Yau metric g, and denote the corresponding Kähler form by ω_J. Denote by Ω_J a holomorphic $(2,0)$-form on X, suitably normalized, and let $\omega_K = \mathrm{Re}(\Omega_J)$ and $\omega_I = \mathrm{Im}(\Omega_J)$: then $(\omega_I, \omega_J, \omega_K)$ is a hyperkähler triple for the metric g. Now switch to the complex structure $I = g^{-1}\omega_I$ determined by the Kähler form ω_I, and with respect to which $\Omega_I = \omega_J + i\omega_K$ is a holomorphic volume form. Since the fibers of $f : X \to \mathbb{CP}^1$ are calibrated by ω_J, the map f is a special Lagrangian fibration on (X, ω_I, Ω_I).

The affine structures on the base of f are only defined away from the singularities of the fibration. Thus the geometry of (X, ω_I, Ω_I) is characterized by a pair of affine structures on the open subset $B \simeq S^2 \setminus \{24 \text{ points}\}$ of $\bar{B} \simeq S^2$. The monodromies of the two affine structures around each singular point are the transpose of each other, and each individual monodromy is conjugate to the standard matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

The mirror of (X, ω_I, Ω_I) is another K3 surface, carrying a special Lagrangian fibration whose base differs from B by an exchange of the two affine structures. In fact, under certain assumptions (e.g., existence of a section) and for a specific choice of $[\omega_J]$, the mirror may be obtained simply by performing another hyperkähler rotation to get $(X, -\omega_K, \Omega_{-K} = \omega_J + i\omega_I)$; see e.g. §7 of [15]. The reader is also referred to §7 of [8] for more details on the SYZ picture for K3 surfaces.

In the above examples, one can avoid confronting heads-on the delicate issues that arise when trying to reconstruct the mirror from the affine geometry of B. In general, however, the compactification of the mirror fibration over the singularities of the affine structure and the incorporation of instanton corrections are two extremely challenging aspects of this approach. The reader is referred to [21] and [10] for two attempts at tackling this problem.

Another even more important issue is constructing a special Lagrangian torus fibration on X in the first place. When there is no direct geometric construction as in the above examples, the most promising approach seems to be Gross and Siebert's program to understand mirror symmetry via toric degenerations [9, 10]. The main idea is to degenerate X to a union X_0 of toric varieties glued together along toric strata; toric geometry then provides a special Lagrangian fibration on X_0, whose base is a polyhedral complex formed by the union of the moment polytopes for the components of X_0. Gross and Siebert then analyze carefully the behavior of this special Lagrangian fibration upon deforming X_0 back to a smooth manifold, showing how to insert singularities into the affine structure to compensate for the nontriviality of the normal bundles to the singular strata along which the smoothing takes place. Moreover, they also show that, in the toric degeneration limit, exchanging the affine structures on the base of the special Lagrangian fibration can be understood as a combinatorial process called *discrete Legendre transform* [9].

Remark 2.6. The affine geometry of B is a remarkably powerful tool to understand the symplectic and complex geometry of X (and, by exchanging the affine structures, of its mirror). Namely, away from the singularities, the two affine structures on $B = B^\vee$ each determine an integral lattice in the tangent bundle TB; denoting these lattices by Λ for the symplectic affine structure and $\Lambda^\vee$ for the complex affine structure, locally X can be identified with either one of the torus bundles T^*B/Λ^* (with its standard symplectic form) and $TB^\vee/\Lambda^\vee$ (with its standard complex structure). Thus, locally, an integral affine submanifold of B (i.e., a submanifold described by linear equations with integer coefficients in local affine coordinates with respect to the symplectic affine structure) determines a Lagrangian submanifold of X by the conormal construction. Similarly, an integral affine submanifold with respect to the complex affine structure $B^\vee$ locally determines a complex submanifold of X (by considering its tangent bundle). More generally, *tropical subvarieties* of B or $B^\vee$ determine piecewise smooth Lagrangian or complex subvarieties in X; whether these can be smoothed is a difficult problem whose answer is known only in simple cases.

To give a concrete example, let us return to K3 surfaces (Example 2.5) and the corresponding affine structures on $B \simeq S^2 \setminus \{24 \text{ points}\}$. Each singular fiber of the special Lagrangian torus fibration $f : X \to \mathbb{CP}^1$ has a nodal singularity obtained by collapsing a circle in the smooth fiber. The homology class of this vanishing cycle determines a pair of rays in B (straight half-lines emanating from the singular point), with the property that the conormal bundles to these rays compactify nicely to Lagrangian discs in X (possibly after a suitable translation within the fibers). Similarly, the nodal singularity determines a pair of rays in $B^\vee$ (different from the previous ones), whose tangent bundles (again after a suitable translation) compactify to holomorphic discs in X. When two singularities of the affine structure lie in a position such that the corresponding rays in B (resp. in $B^\vee$) align with each other (and assuming the translations in the fibers also match), the line segment joining them in B (resp. $B^\vee$) determines a Lagrangian sphere (resp. a rational curve with normal bundle $\mathcal{O}(-2)$) in X. In the mirror $X^\vee$ the same alignment produces a rational -2-curve (resp. a Lagrangian sphere). In fact, using the hyperkähler structure on X and remembering that the elliptic fibration f is J-holomorphic, these spheres correspond to (special) Lagrangian spheres in (X, ω_J) which arise from the matching path construction and additionally are calibrated by ω_K (resp. ω_I).

2.4. **Mirror symmetry in the complement of an anticanonical divisor.** We now consider special Lagrangian torus fibrations in the complement $X \setminus D$ of an anticanonical divisor D in a Kähler manifold X. We start with a very easy example to make the following discussion more concrete:

Example 2.7. Let $X = \mathbb{CP}^1$, equipped with any Kähler form invariant under the standard S^1-action. Equip the complement of the anticanonical divisor $D = \{0, \infty\}$, namely $\mathbb{CP}^1 \setminus \{0, \infty\} = \mathbb{C}^*$, with the standard holomorphic volume form $\Omega = dz/z$. It is easy to check that the circles $|z| = r$ are special Lagrangian (with phase $\pi/2$). Thus we have a special Lagrangian fibration $f : \mathbb{CP}^1 \setminus D \to B$, whose base B is homeomorphic to an interval. As seen above, B carries two affine structures. With respect to the symplectic affine structure, the special Lagrangian fibration is simply the moment map for the S^1-action on $\mathbb{CP}^1$ (up to a factor of 2π). Thus B is an open interval of length equal to the symplectic area of $\mathbb{CP}^1$, and can be compactified by adding the end points of the interval, which correspond to the S^1 fixed points, i.e. the points of D. On the other hand, with respect to the complex affine structure, B is an infinite line: from this point of view, the special Lagrangian fibration is given by the map $z \mapsto \log |z|$.

We can start building a mirror to X by considering the dual special Lagrangian torus fibration M as in §2.2. M is a non-compact Kähler manifold and, after taking instanton corrections into account, it is in fact the mirror to the open Calabi-Yau manifold $X \setminus D$. Thus, some information is missing from this description. As explained at the end of §2.1, adding in the divisor D very much affects the special Lagrangian tori $X \setminus D$ from a Floer-theoretic point of view, and the natural way to account for the resulting obstructions is to make the mirror a Landau-Ginzburg model by introducing a superpotential $W : M \to \mathbb{C}$.

Recall that a point of M is a pair (L, ∇), where $L \subset X \setminus D$ is a special Lagrangian torus, and ∇ is a flat connection on the trivial line bundle over L. Given a homotopy class $\beta \in \pi_2(X, L)$, we can consider the moduli space of holomorphic discs in X with boundary on L, representing the class β. The virtual dimension (over $\mathbb{R}$) of this moduli space is $n - 3 + \mu(\beta)$, where $\mu(\beta) \in \mathbb{Z}$ is the Maslov index; in our case, the Maslov index is twice the algebraic intersection number $\beta \cdot [D]$ (see e.g. Lemma 3.1 of [3]). When $\mu(\beta) = 2$, in favorable cases we can define a (virtual) count $n_\beta(L)$ of holomorphic discs in the class β whose boundary passes through a generic point $p \in L$, and define

$$W(L, \nabla) = \sum_{\substack{\beta \in \pi_2(X,L) \\ \mu(\beta)=2}} n_\beta(L)\, z_\beta(L, \nabla), \quad \text{where} \quad z_\beta(L, \nabla) = \exp(-\textstyle\int_\beta \omega)\, \mathrm{hol}_\nabla(\partial\beta). \tag{2.1}$$

Thus, W is a weighted count of holomorphic discs of Maslov index 2 with boundary in L, with weights determined by the symplectic area of the disc and the holonomy of the connection ∇ along its boundary.

For example, in the case of $\mathbb{CP}^1$ (Example 2.7), each special Lagrangian fiber separates $\mathbb{CP}^1$ into two discs, one containing 0 and the other one containing ∞. The classes β_1 and β_2 represented by these discs satisfy $\beta_1 + \beta_2 = [\mathbb{CP}^1]$, and hence the corresponding weights satisfy $z_{\beta_1} z_{\beta_2} = \exp(-\int_{\mathbb{CP}^1} \omega)$. One can check that $n_{\beta_1} = n_{\beta_2} = 1$, so that using $z = z_{\beta_1}$ as coordinate on M we obtain the well-known formula for the superpotential, $W = z + e^{-\Lambda}\, z^{-1}$, where Λ is the symplectic area of $\mathbb{CP}^1$.

While the example of $\mathbb{CP}^1$ is straightforward, several warnings are in order. First, unless X is Fano the sum (2.1) is not known to converge. More importantly, if L bounds non-constant holomorphic discs of Maslov index 0 (i.e., discs contained in $X \setminus D$), then the counts $n_\beta(L)$ depend on auxiliary data, such as the point $p \in L$ through which the discs are required to pass, or an auxiliary Morse function on L.

An easy calculation shows that the weights z_β are local holomorphic functions on M (with respect to the complex structure defined in Proposition 2.3), and once all ambiguities are lifted the disc counts $n_\beta(L)$ are locally constant, so that W is locally a holomorphic function on M. However, Maslov index 0 discs determine "walls" in M, across which the counts $n_\beta(L)$ jump and hence the quantity (2.1) presents discontinuities. In terms of the affine geometry of the base of the special Lagrangian fibration, an important mechanism for the generation of walls comes from the rays in $B^\vee$ (the base with its complex affine structure) that emanate from the vanishing cycles at the singular fibers of the special Lagrangian fibration: indeed, by definition any special Lagrangian fiber that lies on such a ray bounds a holomorphic disc in $X \setminus D$ (see Remark 2.6). Intersections between these "primary" walls then generate further walls (which can be visualized as rigid tropical configurations in $B^\vee$).

Fukaya-Oh-Ohta-Ono's results [6] imply that the formulas for W in adjacent chambers of M differ by a holomorphic substitution of variables (see also Proposition 3.9 in [3]). The guiding principle that governs *instanton corrections* is that the various chambers of M should be glued to each other not in the naive manner suggested by the geometry of B, but rather via the holomorphic gluing maps that arise in the wall-crossing formulas. Thus, the instanton-corrected mirror is precisely the analytic space on which the weighted count (2.1) of holomorphic discs in (X, L), and more generally the "open Gromov-Witten invariants" of (X, L) (yet to be defined in the most general setting), become single-valued quantities. The reader is referred to [21] and [10] for more details on instanton corrections (in the Calabi-Yau case, but the general case is similar).

One final issue is that, according to Hori and Vafa [14], the mirror obtained by T-duality needs to be enlarged. The holomorphic volume form Ω has poles along D, which causes B equipped with the complex affine structure to have infinite diameter (after adding in the singular fibers, $B^\vee$ is complete). On the other hand, the fact that ω extends smoothly across D means that, with respect to the symplectic affine structure, B has finite diameter, and compactifies to a singular affine manifold with boundary. The consequence is that, after exchanging the affine structures, the Kähler metric on the mirror is complete but its complex structure is "incomplete": for instance, in Example 2.7 the mirror of $\mathbb{CP}^1$ is naturally a bounded annulus (of modulus equal to the symplectic area of $\mathbb{CP}^1$), rather than the expected $\mathbb{C}^*$. Hori and Vafa's suggestion (assuming that X is Fano) is to symplectically "enlarge" $X \setminus D$ by considering a family of Kähler forms $(\omega_k)_{k\to\infty}$ obtained by symplectic inflation along D, with the property that $[\omega_k] = [\omega] + k\, c_1(X)$, and simultaneously rescaling the superpotential by a factor of e^k (see also §4.2 of [3]). However, this "renormalization" procedure is definitely not desirable in the geometric setting considered in Section 3, so we do not consider it further.

We end here our discussion of the various delicate points that come up in the construction of the mirror and its superpotential, and simply refer the reader to [3] for more details. Instead, we return to examples.

Example 2.8 (Toric varieties)**.** Let (X, ω, J) be a toric variety of complex dimension n, and consider the toric anticanonical divisor D (i.e., the divisor of points where the T^n-action is not free). Recall that $X \setminus D$ is biholomorphic to $(\mathbb{C}^*)^n$, and equip it with the holomorphic $(n,0)$-form $\Omega = d\log z_1 \wedge \cdots \wedge d\log z_n$, which has poles along D. Then the orbits of the standard T^n-action define a special Lagrangian fibration on $X \setminus D \simeq (\mathbb{C}^*)^n$. With respect to the symplectic affine structure, the base B of this fibration is the moment polytope for (X, ω), or rather its interior, and the special Lagrangian fibration is simply given by the moment map. On the other hand, the complex affine structure on B naturally identifies it with $\mathbb{R}^n$; from this point of view the special Lagrangian fibration is the Log map $(z_1, \dots, z_n) \mapsto (\log|z_1|, \dots, \log|z_n|)$.

Exchanging the two affine structures, the mirror of X is naturally a bounded domain in $(\mathbb{C}^*)^n$ (the subset of points whose image under the Log map lies in the moment polytope of X), equipped with a complete Kähler metric and a superpotential W defined by a Laurent polynomial consisting of one term for each component of D. Details can be found in [5] and [7] (see also §4 of [3] for a brief overview, and [1] for a partial verification of homological mirror symmetry).

Example 2.9 ($\mathbb{CP}^2$)**.** Consider $\mathbb{CP}^2$ equipped with the Fubini-Study Kähler form ω_0. Let $D \subset \mathbb{CP}^2$ be a smooth elliptic curve defined by a homogeneous polynomial of degree 3, and let Ω be a holomorphic volume form on $\mathbb{CP}^2$ with poles along D.

Conjecture 2.9. $\mathbb{CP}^2 \setminus D$ *carries a special Lagrangian torus fibration over the disc with (generically) three nodal singular fibers.*

Tentatively, the construction of this special Lagrangian fibration proceeds as follows. Start with the toric setting, i.e. equip $\mathbb{CP}^2$ with a holomorphic volume form with poles along the toric anticanonical divisor D_0 consisting of the three coordinate lines ($\Omega_0 = dx \wedge dy/xy$ in an affine chart). As mentioned above, the orbits of the standard T^2-action define a special Lagrangian fibration on $(\mathbb{C}^*)^2 = \mathbb{CP}^2 \setminus D_0$; with respect to the symplectic affine structure, the base B_0 of this fibration is the moment polytope for $\mathbb{CP}^2$, i.e. a triangle. Deforming this situation to the case of a holomorphic volume form Ω' with poles along a smooth cubic curve D' obtained by smoothing out the three nodal points of D_0 modifies the structure of the special Lagrangian fibration near the three toric fixed points. A local model for what happens near each of these points is described in §5 of [3]. Namely, if we replace Ω_0 by $\Omega_\varepsilon = dx \wedge dy/(xy - \varepsilon)$, then the complement of the anticanonical divisor D_ε formed by the conic $xy = \varepsilon$ and the line at infinity carries a special Lagrangian torus fibration with one nodal singular fiber: the fibers are formed by intersecting the level sets of the moment map for the S^1-action $e^{i\theta} \cdot (x, y) = (e^{i\theta}x, e^{-i\theta}y)$ with the level sets of the function $|xy - \varepsilon|^2$, and the singularity is at the origin [3]. If ε is small then this family is close to the toric family away from the origin. Therefore, general considerations about deformations of families of special Lagrangians suggest that, if the smooth elliptic curve D' lies in a sufficiently small neighborhood of D_0, then $(\mathbb{CP}^2 \setminus D', \omega_0, \Omega')$ carries a special Lagrangian fibration with three nodal singular fibers. From the point of view of the affine geometry of the base B' of this fibration, the smoothing of each node of D_0 amounts to replacing a corner of the triangle B_0 by a singular point in the interior of B' (so that B' is a singular affine manifold with boundary but without corners).

The special Lagrangian fibers over points close to the boundary of B' lie in a tubular neighborhood of D', and collapse to closed loops in D' as one approaches the boundary. Thus their first homology group is generated by a meridian m (the boundary of a small disc that intersects D' transversely once) and by a longitude ℓ (a curve that runs parallel to a closed loop on D'). The monodromy of the special Lagrangian fibration along $\partial B'$ fixes m, but because the normal bundle to D' has degree 9 it maps ℓ to $\ell + 9m$. Thus, in a suitable basis the monodromy along the boundary of B' can be expressed by the matrix $\left(\begin{smallmatrix}1 & 9\\ 0 & 1\end{smallmatrix}\right)$ (see equation (7.2) in [3]).

The general case, where the cubic curve D is not necessarily close to the singular toric configuration $D_0 \subset \mathbb{CP}^2$, should follow from a suitable result on deformations of two-dimensional special Lagrangian torus fibrations with nodal singularities. (To our knowledge such a result hasn't been proved yet; however it should follow from an explicit analysis of the deformations of the nodal singularities and the implicit function theorem applied to the smooth part of the fibration. In our case one also needs to control the behavior of the fibration near the boundary of B.)

When constructing the mirror, the singular fibers create walls, which require instanton corrections. In the case of a cubic D' obtained by a small deformation of the toric configuration D_0, the local model for a single smoothing suggests that the walls run parallel to the boundary of the base B'. In fact, the special Lagrangian fibers which lie sufficiently far from D' are Floer-theoretically equivalent to standard product tori. Thus, in the "main" chamber the superpotential is given by the same formula as in the toric case, $W = x + y + e^{-\Lambda}/xy$ in suitable coordinates (where $\Lambda = \int_{\mathbb{CP}^1} \omega$); in the other chambers it is given by some analytic continuation of this expression (see §5 of [3] for an explicit formula in the case of smoothing a single node of D_0). In fact, ignoring completeness issues (e.g., looking only at $|W| \ll 1$), the overall effect of deforming D_0 to a smooth cubic curve on the complex geometry of the Landau-Ginzburg mirror is expected to be a fiberwise compactification. Simultaneously, the symplectic area of the fiber of the Landau-Ginzburg model, which is infinite in the toric case, is expected to become finite and equal to the imaginary part of the modular parameter of the elliptic curve D' (see also [4]).

Example 2.10 (Rational elliptic surface)**.** Let X be a rational elliptic surface obtained by blowing up $\mathbb{CP}^2$ at the nine base points of a pencil of cubics, equipped with a Kähler form $\hat{\omega}$. Let $\hat{D} \subset X$ be a smooth elliptic fiber (the proper transform of a cubic of the pencil), and let $\hat{\Omega}$ be a holomorphic $(2,0)$-form on X with poles along $\hat{D}$. We expect:

Conjecture 2.10. *$X \backslash \hat{D}$ carries a special Lagrangian torus fibration over the disc with (generically) 12 nodal singular fibers. The monodromy of the affine structure around each singularity is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and the monodromy along $\partial \hat{B}$ is trivial.*

The construction starts with $(\mathbb{CP}^2, D, \omega_0, \Omega)$, where $D \subset \mathbb{CP}^2$ is an elliptic curve and Ω is a holomorphic (2,0)-form with poles along D, as in Example 2.9 above. By Conjecture 2.9, we expect $\mathbb{CP}^2 \setminus D$ to carry a special Lagrangian torus fibration with three nodal singular fibers. Now we blow up $\mathbb{CP}^2$ at nine points on the cubic D, to obtain the rational elliptic surface X. Pulling back Ω under the blowup map yields a holomorphic (2,0)-form $\hat{\Omega}$ on X, with poles along an elliptic curve $\hat{D} \subset X$ (the proper transform of D). On the other hand, the Kähler form $\hat{\omega}$ on X is not canonical, and depends in particular on the choice of the symplectic areas of the exceptional divisors. We claim that, provided these areas are sufficiently small, the blowup should carry a special Lagrangian torus fibration with 12 nodal singular fibers.

The local model for each blowup operation is as follows [2]. Consider a neighborhood of the origin in $\mathbb{C}^2$ equipped with the standard symplectic form, the holomorphic volume form $dx \wedge dy/y$ with poles along $\mathbb{C} \times \{0\}$, and the family of special Lagrangian cylinders $\{\mathrm{Re}(x) = t_1,\ \frac{1}{2}|y|^2 = t_2\} \subset \mathbb{C} \times \mathbb{C}^*$. Equip the blowup $\hat{\mathbb{C}}^2$ with a toric Kähler form $\hat{\omega}_0$ (invariant under the standard T^2-action) for which the area of the exceptional divisor is $\epsilon > 0$, and the holomorphic volume form $\hat{\Omega}_0$ obtained by pulling back $dx \wedge dy/y$ under the blowup map $\pi : \hat{\mathbb{C}}^2 \to \mathbb{C}^2$. The lift to $\hat{\mathbb{C}}^2$ of the S^1-action $e^{i\theta} \cdot (x, y) = (x, e^{i\theta} y)$ preserves $\hat{\omega}_0$ and $\hat{\Omega}_0$; its fixed point set consists of on one hand the proper transform $\hat{D}_0$ of $\mathbb{C} \times \{0\}$, and on the other hand the point where the proper transform of $\{0\} \times \mathbb{C}$ hits the exceptional divisor. Denote by $\mu : \hat{\mathbb{C}}^2 \to \mathbb{R}$ the moment map for this S^1-action, normalized to equal 0 on $\hat{D}_0$ and ϵ at the isolated fixed point. Then it is easy to check that the submanifolds $\{\mathrm{Re}(\pi^* x) = t_1,\ \mu = t_2\} \subset \hat{\mathbb{C}}^2 \setminus \hat{D}_0$ are special Lagrangian with respect to $\hat{\omega}_0$ and $\hat{\Omega}_0$ [2]. This family of special Lagrangians presents one nodal singular fiber – the fiber which corresponds to $(t_1, t_2) = (0, \epsilon)$ and passes through the isolated S^1-fixed point. Moreover, if ϵ is small then away from a neighborhood of the exceptional divisor this family is close to the initial family of special Lagrangians in $\mathbb{C} \times \mathbb{C}^*$.

Even though the local model is only an asymptotic description of the geometry of the special Lagrangian fibration on $\mathbb{CP}^2 \setminus D$ near a point of D, it should be possible to glue this local construction into the fibration of Conjecture 2.9, and thereby construct a special Lagrangian fibration on the rational elliptic surface X obtained by blowing up $\mathbb{CP}^2$ at 9 points on the elliptic curve D. Each blow-up operation inserts a nodal singular fiber into the fibration; thus the base $\hat{B}$ of the special Lagrangian fibration on X presents 12 singular points. From the point of view of the symplectic affine structure, an easy calculation on the local model shows that each new singular point lies at a distance from the boundary of $\hat{B}$ equal to the symplectic area of the exceptional curve of the corresponding blowup; in fact the exceptional curve can be seen as a complex ray that runs from the singular point to the boundary of $\hat{B}$. Moreover, the monodromy of the fibration along the boundary of $\hat{B}$ is trivial, reflecting the fact that the anticanonical divisor $\hat{D} \subset X$ has trivial normal bundle.

The general case, where the exceptional divisors of the blowups are not assumed to have small symplectic areas, should again follow from a careful analysis of deformations of two-dimensional special Lagrangian torus fibrations with nodal singularities (with the same caveats as in the case of $\mathbb{CP}^2$).

Remark 2.11. Assume D is smooth. Then the holomorphic $(n, 0)$-form Ω on $X \setminus D$ induces a holomorphic volume form $\Omega_D = \mathrm{Res}_D(\Omega)$ on D: the *residue* of Ω along D. It is reasonable to expect that, as is the case in the various examples considered above, in a neighborhood of D the special Lagrangian fibration on $(X \setminus D, \omega, \Omega)$ consists of tori which are S^1-bundles over special Lagrangian submanifolds of $(D, \omega_{|D}, \Omega_D)$. As a toy example, consider $X = D \times \mathbb{C}$, $\omega = \omega_D + \frac{i}{2} dz \wedge d\bar{z}$, and $\Omega = \Omega_D \wedge dz/z$: then the product any special Lagrangian submanifold of D with a circle centered at the origin in $\mathbb{C}$ is easily seen to be special Lagrangian. We conjecture that the qualitative behavior is the same in the general case; see §7 of [3] for more details.

Assuming that this picture holds, the special Lagrangian fibration $f : X \setminus D \to B$ can be extended over the boundary of B by a special Lagrangian fibration on D. In particular, the boundary of B, with the induced affine structures, is the base B_D of an SYZ fibration on D. More precisely: with respect to the symplectic affine structure, the compactified base $\bar{B}$ is a singular affine manifold with boundary (and corners if D has normal crossings), and its boundary is B_D. With respect to the complex affine structure, $B^\vee$ (after adding in the interior singular fibers) is a complete singular affine manifold, isomorphic to $\mathbb{R}_+ \times B_D^\vee$ outside of a compact subset.

As already seen in Example 2.9, near ∂B the monodromy of the affine structures on B is determined explicitly by the affine structures on B_D and by the first Chern class of the normal bundle to D. Indeed, given a fiber of f near the boundary of B, i.e. an S^1-fibered special Lagrangian $L \subset X \setminus D$, the action of the monodromy on $H_1(L)$ can be determined by working in a basis consisting of a meridian loop linking D and $n-1$

longitudes running parallel to D; from this one deduces the corresponding actions on $H^1(L)$ (monodromy of B) and $H^{n-1}(L)$ (monodromy of $B^\vee$).

Next, we look at the mirror, and observe that near its boundary M consists of pairs (L, ∇) where L is an S^1-fibered special Lagrangian contained in a neighborhood of D. Denote by $\delta \in \pi_2(X, L)$ the homotopy class of a small meridian disc intersecting D transversely once (with boundary the meridian loop), and let $z_\delta(L, \nabla)$ be the corresponding weight as in equation (2.1). Then z_δ is a holomorphic function on M near its boundary. (In fact, z_δ is the dominant term in the expression of the superpotential W near ∂M, as the meridian discs have the smallest symplectic area among all Maslov index 2 holomorphic discs.) By construction, the boundary of M corresponds to the case where the area of the meridian disc reaches zero, i.e. $\partial M = \{|z_\delta| = 1\}$.

Consider the complex hypersurface $M_D = \{z_\delta = 1\}$ $(\subset \partial M)$. Geometrically, M_D corresponds to limits of sequences of pairs (L, ∇) where L collapses onto a special Lagrangian torus $\Lambda \subset D$ and the connection ∇ has trivial holonomy along the collapsed S^1-factor in L, i.e. is pulled back from a flat connection on the trivial bundle over Λ. Thus M_D is none other than the SYZ mirror to D. Moreover, the restriction of z_δ to ∂M induces a locally trivial fibration $z_\delta : \partial M \to S^1$ with fiber M_D. The monodromy of this fibration can be realized geometrically as follows. Start with a pair (L, ∇) where L is almost collapsed onto $\Lambda \subset D$ and ∇ has trivial holonomy along the meridian loop (so $z_\delta \in \mathbb{R}_+$): then we can change the holonomy of ∇ along the meridian loop by adding to it a multiple of $\sigma^{-1}\nabla\sigma$, where σ is the defining section of D and ∇ is a suitable connection on K_X^{-1}. From there it follows easily that the monodromy of the fibration $z_\delta : \partial M \to S^1$ is a symplectomorphism of M_D which geometrically realizes (as a fiberwise translation in the special Lagrangian fibration $M_D \to B_D$ dual to the SYZ fibration on D) the mirror to the autoequivalence $- \otimes K_{X|D}^{-1}$ of $D^b\mathrm{Coh}(D)$.

This rich geometric picture naturally leads to a formulation of mirror symmetry for the pairs (X, D) and (M, M_D); see §7 of [3] for details.

3. Special Lagrangian fibrations and double covers

3.1. **Special Lagrangians and Calabi-Yau double covers.** Let (X, ω, J) be a smooth compact Kähler manifold of complex dimension n, and let s be a nontrivial holomorphic section of K_X^{-2}. Unless otherwise specified we assume that the hypersurface $H = s^{-1}(0)$ is smooth. $\Theta = s^{-1}$ is a nonvanishing section of $K_X^{\otimes 2}$ over $X \setminus H$, with poles along H, and locally $\Omega = \Theta^{1/2}$ is a nonvanishing holomorphic n-form, defined up to sign. The restriction of Θ to a Lagrangian submanifold $L \subset X \setminus H$ does not vanish, and can be expressed in the form $\eta\,\mathrm{vol}_g^2$, where $\eta \in C^\infty(L, \mathbb{C}^*)$. By analogy with the situation considered previously, we make the following definition:

Definition 3.1. *A Lagrangian submanifold $L \subset X \setminus H$ is* special Lagrangian *if the argument of η is constant. (In fact Θ will usually be normalized so that η is real).*

It is easy to see that, if $L \subset X \setminus H$ is an orientable special Lagrangian submanifold, then over L the holomorphic quadratic differential Θ admits a globally defined square root Ω. Therefore Proposition 2.2 still applies in this setting; since $\Omega_{|L} = \eta^{1/2}\mathrm{vol}_g$, special Lagrangian deformations are now given by $\eta^{1/2}$-harmonic 1-forms on L.

As before, the base B of a special Lagrangian torus fibration carries two natural affine structures, one arising from the symplectic geometry of X and the other one arising from its complex geometry.

We now turn to the Calabi-Yau double cover of X branched along H, namely the unique double cover $\pi : Y \to X$ with the property that $\tilde{\Theta} = \pi^*\Theta$ admits a globally defined square root $\tilde{\Omega} \in \Omega^{n,0}(Y)$. More explicitly, the obstruction for Θ to admit a globally defined square root is given by an element of $H^1(X \setminus H, \mathbb{Z}/2) \simeq \mathrm{Hom}(\pi_1(X \setminus H), \mathbb{Z}/2)$, and we consider the branched cover with this monodromy.

The complex geometry of Y is fairly straightforward, as the complex structure $\tilde{J}$ and the holomorphic volume form $\tilde{\Omega}$ are simply lifted from those of X via π. In particular, it is easy to check that $\tilde{\Omega}$ is well-behaved along the ramification divisor. (To give the simplest example, consider the map $z \mapsto z^2$ from $\mathbb{C}$ to itself: the pullback of $\Theta = z^{-1}dz^{\otimes 2}$ is $\tilde{\Theta} = 4\,dz^{\otimes 2}$, which has a well-defined square root $\tilde{\Omega} = 2\,dz$.)

On the other hand, constructing a Kähler form on Y requires some choices, because the pullback form $\pi^*\omega$ is degenerate along the ramification locus $\tilde{H} = \pi^{-1}(H)$. One approach is to view Y as a complex hypersurface in the total space of the line bundle K_X^{-1} over X, equipped with a suitable Kähler metric. More directly, one can equip Y with a Kähler form $\tilde{\omega} = \pi^*\omega + \epsilon\,\lambda$, where $\epsilon > 0$ is a sufficiently small constant and λ is an exact real $(1,1)$-form whose restriction to the complex line $\mathrm{Ker}(d\pi)$ is positive at every point of the ramification locus. Any two forms obtained in this manner are symplectically isotopic; for example one can take $\lambda = -i\partial\bar{\partial}\phi$ where $\phi : Y \to [0,1]$ is supported in a neighborhood of $\tilde{H}$, equal to 1 on $\tilde{H}$, and strictly concave in the normal directions at every point of $\tilde{H}$.

Thus, given a compact special Lagrangian submanifold $L \subset X \setminus H$, the two lifts of L are in general not special Lagrangian submanifolds of Y, even though the restriction of $\tilde{\Omega}$ has constant phase, because they are not necessarily Lagrangian for $\tilde{\omega}$. In very specific cases (for instance in dimension 1 or in product situations) this is not an issue, but in general one needs to deform the lift of L to a nearby special Lagrangian submanifold $\tilde{L} \subset Y$, whose existence is guaranteed by the unobstructedness of deformations (Proposition 2.2) as long as $\tilde{\omega}$ is sufficiently close to $\pi^*\omega$.

When considering not just one submanifold but a whole special Lagrangian fibration on $X \setminus H$, it is natural to ask whether the lifts can be similarly deformed to a special Lagrangian fibration on Y. Away from H and from the singular fibers, we can rely on an implicit function theorem for special Lagrangian fibrations which again follows from unobstructedness. In spite of the wealth of results that have been obtained on singularities of special Lagrangians and their deformations (see e.g. [17]), to our knowledge there is no general result that would yield a special Lagrangian fibration on Y from one on $X \setminus H$. Nonetheless, it seems reasonable to expect that such a result might hold at least in low dimensions if the Kähler form on Y is chosen suitably and the family of special Lagrangians only presents generic singularities.

Thus, Conjecture 1.1 can be stated more precisely as follows:

Conjecture 3.2.

(1) *X carries a special Lagrangian fibration (or rather, foliation) $f : X \to \bar{B}$, where $\bar{B}$ is a singular affine manifold with boundary (with two affine structures), such that the generic fibers of f are special Lagrangian tori in $X \setminus H$, and the fibers of f above $\partial\bar{B}$ are special Lagrangians with boundary in H.*

(2) *Y carries a special Lagrangian torus fibration $\tilde{f} : Y \to \tilde{B}$, where $\tilde{B}$ is a singular affine manifold without boundary (with two affine structures), obtained by gluing together two copies of $\bar{B}$ along their boundary.*

Note that the boundaries of the two copies of $\bar{B}$ are identified using the identity map, whereas the normal direction is reflected; thus this is an orientation-reversing gluing, and the resulting singular affine manifold $\tilde{B}$ admits an orientation-reversing involution whose fixed point locus is the "seam" of the gluing.

3.2. **Example: $\mathbb{CP}^1$ and elliptic curves.** As our first example, we consider $X = \mathbb{CP}^1$ equipped with any Kähler form and a holomorphic quadratic differential Θ with poles at a subset $H \subset \mathbb{CP}^1$.

We first consider the special case $\Theta = dz^2/(z^2 - a^2)$, with simple poles at $\pm a$ and a double pole at infinity. Setting $a = 0$, we recover the classical situation discussed in Example 2.7, in which the circles centered at the origin are special Lagrangian. For arbitrary a, it follows from classical geometry that every ellipse with foci $\pm a$ is special Lagrangian with phase $\pi/2$ for $\Omega = \Theta^{1/2} = dz/\sqrt{z^2 - a^2}$. Thus we get a special Lagrangian foliation of $\mathbb{C} \setminus \{\pm a\}$ by this family of ellipses, the sole noncompact leaf being the real interval $(-a, a)$. The general case is less explicit but essentially amounts to modifying the special Lagrangian family in the same manner not only near zero but also near infinity.

More precisely, equip $\mathbb{CP}^1$ with a generic holomorphic quadratic differential $\Theta = z^2\,dz^2/(z-a)(z-b)(z-c)(z-d)$ with poles at $H = \{a, b, c, d\}$. Then, for a suitable choice of phase, $\mathbb{CP}^1 \setminus H$ admits a special Lagrangian foliation in which all the leaves are closed loops with the exception of two noncompact leaves, each connecting two of the points of H (say a and b on one hand, and c and d on the other hand). For instance, if $a < b < c < d$ are real, then we have such a foliation (with phase $\pi/2$) in which the two noncompact leaves are the real line segments (a, b) and (c, d). Indeed, after removing the two intervals $[a, b]$ and $[c, d]$, the quadratic differential Θ admits a well-defined square root Ω, which is a closed 1-form and hence has the same period (easily checked to be pure imaginary) on any homotopically nontrivial embedded curve. The general case follows from the same argument.

From a symplectic point of view, the base B of this foliation is again an interval of length equal to the symplectic area of $\mathbb{CP}^1$. However, unlike the situation of Example 2.7, the affine structure induced on B by the holomorphic volume form identifies it with a *finite* interval: if we normalize Ω so that the integral of $\operatorname{Im}\Omega$ over each special Lagrangian fiber is 1, then the length of this interval is equal to $\int_b^c \operatorname{Re}\Omega$.

The double cover of $\mathbb{CP}^1$ branched at H is an elliptic curve Y, and the family of special Lagrangians in $\mathbb{CP}^1 \setminus H$ lifts to a smooth special Lagrangian fibration on Y. The base $\tilde{B} \simeq S^1$ of this fibration, and its two affine structures, are obtained by doubling B along its boundary. For instance, the symplectic area of Y (which is the length of $\tilde{B}$ with respect to the symplectic affine structure, cf. Example 2.4) is twice that of $\mathbb{CP}^1$, whereas the integral of $\operatorname{Re}\tilde{\Omega}$ over a section of the special Lagrangian fibration (which is the length of $\tilde{B}$ with respect to the complex affine structure) is twice $\int_b^c \operatorname{Re}\Omega$.

Remark 3.3. With respect to the complex affine structure, the base B of the special Lagrangian foliation on $(\mathbb{CP}^1 \setminus H, \Omega)$ is a finite interval, whereas the base B_0 of the special Lagrangian fibration on $(\mathbb{CP}^1 \setminus \{0, \infty\}, \Omega_0 = dz/z)$ has infinite size. The reason is that, as $a, b \to 0$ and $c, d \to \infty$, the elliptic curve Y degenerates to a curve with two nodal singularities, and the base $\tilde{B}$ of its special Lagrangian fibration degenerates to a union of two infinite intervals. On the other hand, the symplectic structure on Y, which determines the length of the base with respect to the other affine structure, is unaffected by the degeneration.

3.3. **Example: Elliptic surfaces.** We revisit Example 2.10, and again denote by X a rational elliptic surface obtained by blowing up $\mathbb{CP}^2$ at the 9 base points of a pencil of cubics, equipped with a Kähler form $\hat{\omega}$. We previously considered a holomorphic volume form $\hat{\Omega}$ on X with poles along an elliptic fiber $\hat{D}$. Now we equip X with a section Θ of $K_X^{\otimes 2}$, with poles along the union $H = D_+ \cup D_-$ of two smooth fibers of the elliptic fibration; for simplicity we assume that $D_\pm$ lie close to a same smooth fiber $\hat{D}$, so that away from a neighborhood of $\hat{D}$ the quadratic volume element Θ is close to the square $\hat{\Omega}^{\otimes 2}$ of the volume form considered in Example 2.10.

Conjecture 3.4. *The special Lagrangian fibration on $X \setminus \hat{D}$ constructed in Conjecture 2.10 deforms to a special Lagrangian family on $X \setminus H$. The base B of this family is homeomorphic to a closed disc, and over its interior the fibers are special Lagrangian tori, with the exception of 12 nodal singular fibers. The fibers above ∂B are special Lagrangian annuli with one boundary component on D_+ and the other on D_-.*

We now explain the geometric intuition behind this conjecture by considering a simplified local model in which everything is explicit. The actual geometry of X near $\hat{D}$ differs from this local model by higher order terms; however the local model is expected to accurately describe all the qualitative features of the special Lagrangian families of Conjectures 2.10 and 3.4 in a small neighborhood of $\hat{D}$.

In a small neighborhood of the fiber $\hat{D}$, the elliptic fibration $X \to \mathbb{CP}^1$ is topologically trivial, and even though it is not holomorphically trivial, in first approximation we can consider a local model of the form $E \times U$, where E is an elliptic curve ($E \simeq \hat{D}$) and U is a neighborhood of the origin in $\mathbb{C}$ (with coordinate z). In this simplified local model, the holomorphic volume form $\hat{\Omega}$ can be written in the form $dw \wedge dz/z$, where dw is a holomorphic 1-form on E (in fact, the *residue* of $\hat{\Omega}$ along $\hat{D}$), the symplectic form $\hat{\omega}$ is a product form, and the special Lagrangian family of Conjecture 2.10 consists of product tori, where the first factor is a special Lagrangian circle in (E, dw) and the second factor is a circle centered at the origin.

We now equip $E \times U$ with the quadratic volume element $\Theta = (dw \wedge dz)^{\otimes 2}/(z^2 - \epsilon^2)$, with poles along $H = E \times \{\pm\epsilon\}$. Then the previous family of special Lagrangians deforms to one where each submanifold is again a product: the first factor is still a special Lagrangian circle in (E, dw), and the second factor is now an ellipse with foci at $\pm\epsilon$ (in the degenerate case, the line segment $[-\epsilon, \epsilon]$).

The bases of these two special Lagrangian fibrations on $E \times U$, equipped with their symplectic affine structures, are naturally isomorphic, as each ellipse with foci at $\pm\epsilon$ can be used interchangeably with the circle that encloses the same symplectic area (in fact, the corresponding product Lagrangian tori in $E \times U$ are Hamiltonian isotopic to each other). In this sense, passing from $X \setminus \hat{D}$ to $X \setminus H$ (i.e., from $\hat{B}$ to B) is expected to be a trivial operation from the symplectic point of view. However, the complex affine structures on $\hat{B}$ and B are very different: from that perspective $\hat{B}$ is "complete" (its boundary lies "at infinity", since the affine structure blows up near $\partial\hat{B}$ due to the singular behavior of $\hat{\Omega}$ along $\hat{D}$), whereas B has finite diameter. This is most easily seen in terms of the local model near $\hat{D}$, which allows us to reduce to the one-dimensional case (see Remark 3.3).

Finally, we consider the double cover Y of the rational elliptic surface X branched along H. It is easy to see that Y is an elliptically fibered K3 surface, carrying a holomorphic involution under which the holomorphic volume form $\tilde{\Omega} = (\pi^*\Theta)^{1/2}$ is anti-invariant. By Conjecture 1.1 we expect that Y, equipped with a suitably chosen Kähler form in the class $[\pi^*\hat{\omega}]$, carries a special Lagrangian fibration with 24 nodal singular fibers, whose base $\tilde{B} \simeq S^2$ is obtained by doubling B along its boundary.

In fact, it is well-known that such a fibration can be readily obtained using hyperkähler geometry as in Example 2.5. Indeed, consider an elliptically fibered K3 surface with a real structure for which the real part consists of two tori. For example, let Y' be the double cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along the zero set of a generic real homogeneous polynomial of bidegree $(4, 4)$ without any real roots. Composing the covering map with projection to the first $\mathbb{CP}^1$ factor, we obtain an elliptic fibration

$f : Y' \to \mathbb{CP}^1$ with 24 singular fibers. Complex conjugation lifts to an involution ι on Y' which is antiholomorphic with respect to the given complex structure J, and whose fixed point locus is the trivial (disconnected) double cover of $\mathbb{RP}^1 \times \mathbb{RP}^1$ (i.e., two tori). The involution ι maps each fiber of f to the fiber above the complex conjugate point of $\mathbb{CP}^1$, and in particular it interchanges pairs of complex conjugate singular fibers.

Equip Y' with a Calabi-Yau metric, such that the Kähler form ω_J is anti-invariant under ι (this is guaranteed by uniqueness of the Calabi-Yau metric if one imposes $[\omega_J]$ to be the pullback of a Kähler class on $\mathbb{CP}^1 \times \mathbb{CP}^1$ and hence anti-invariant). Denote by Ω_J a holomorphic $(2,0)$-form on Y': then $\iota^*\Omega_J$ is a scalar multiple of $\bar{\Omega}_J$, because $\dim H^{0,2}_J(Y) = 1$. So after normalization we can assume that $\iota^*\Omega_J = -\bar{\Omega}_J$, i.e. $\omega_K := \mathrm{Re}(\Omega_J)$ is anti-invariant and $\omega_I := \mathrm{Im}(\Omega_J)$ is invariant.

Now switch to the complex structure I determined by the Kähler form ω_I. Then ι becomes a holomorphic involution, and the holomorphic volume form $\Omega_I = \omega_J + i\omega_K$ is anti-invariant. Since the fibers of $f : Y' \to \mathbb{CP}^1$ are calibrated by ω_J, the map f is a special Lagrangian fibration on (Y', ω_I, Ω_I), compatible with the involution ι.

It seems likely that this construction can be used as an alternative approach to Conjecture 3.4, by considering the quotient of this special Lagrangian fibration by the involution ι.

Remark 3.5. The elliptic surface X contains nine exceptional spheres, arising from the nine blow-ups performed on $\mathbb{CP}^2$; these spheres intersect H in two points, so their preimages in the double cover Y are rational curves with normal bundle $\mathcal{O}(-2)$. These curves can be seen by looking at the complex affine structures on the bases B and $\tilde{B}$ of the special Lagrangian fibrations on X and Y, as discussed in Remark 2.6. Namely, the exceptional curves in X correspond to complex rays that run from singularities of the affine structure of B to its boundary (as in Example 2.10). Doubling B along its boundary to form $\tilde{B}$ creates alignments between pairs of singular points lying symmetrically across from each other. For at least 9 of the 12 pairs of points (those which correspond to the blowups) the corresponding complex rays match up to yield -2-curves in Y.

3.4. **Example: $\mathbb{CP}^2$ and K3.** We now revisit Example 2.9, and now equip $\mathbb{CP}^2$ with a section Θ of $K^{\otimes 2}$ with poles along a smooth curve H of degree 6. We assume that H lies in a small neighborhood of a cubic D, i.e. it is defined by a homogeneous polynomial of the form $p = \sigma^2 - \epsilon q$, where $\sigma \in H^0(\mathcal{O}(3))$ is the defining section of D and ϵ is a small constant. Thus, away from a neighborhood of D the quadratic volume element Θ is close to the square $\Omega^{\otimes 2}$ of the volume form considered in Example 2.9.

Conjecture 3.6. *The special Lagrangian fibration on $\mathbb{CP}^2 \setminus D$ constructed in Conjecture 2.9 deforms to a special Lagrangian family on $\mathbb{CP}^2 \setminus H$. The base B of this family is homeomorphic to a closed disc, and over its interior the fibers are special Lagrangian tori, with the exception of three nodal singular fibers. The fibers above ∂B*

are special Lagrangian annuli with boundary on H, with the exception of 18 pinched annuli (with one arc connecting the two boundaries collapsed to a point).

While we do not have a complete picture to propose, the rough idea is as follows. Looking at the defining section $p = \sigma^2 - \epsilon q$ of H, away from the zeroes of q we can think of H as two parallel copies of D, and special Lagrangians are expected to behave as in the previous example. Namely, near D a special Lagrangian in $\mathbb{CP}^2 \setminus D$ looks like the product of a special Lagrangian $\Lambda\,(\simeq S^1)$ in D with a small circle in the normal direction, and the corresponding special Lagrangian in $\mathbb{CP}^2 \setminus H$ should be obtained by replacing the circle factor by a family of ellipses whose foci lie on H. In the degenerate limit case, the ellipses become line segments joining the two foci, forming an annulus; when Λ passes through a zero of q, the corresponding line segment is collapsed to a point, giving a pinched annulus.

In fact, we are unable to provide an explicit local model for this behavior on $X \setminus H$. However, Conjecture 3.6 can be corroborated by calculations on a local model for the double cover Y of X branched along H.

Near a point of D, we can consider local coordinates (u, v) on a domain in $\mathbb{C}^2$ such that D is defined by the equation $u = 0$, and H is defined by the equation $u^2 - \epsilon q(v) = 0$ for some holomorphic function q. The corresponding section of $K_X^{\otimes 2}$ is given by $\Theta = (u^2 - \epsilon q(v))^{-1}\,(du \wedge dv)^{\otimes 2}$. As $\epsilon \to 0$, this converges to the square of the holomorphic volume form $u^{-1}\,du \wedge dv$, for which the cylinders $\{\mathrm{Re}\, v = a,\ |u|^2 = r\}$ are special Lagrangians (the circle factor corresponds to the direction normal to D, while the other factor corresponds to a local model for a special Lagrangian in D).

In this local model the double cover of $\mathbb{C}^2$ branched along H is the hypersurface $Y \subset \mathbb{C}^3$ defined by the equation $z^2 = u^2 - \epsilon q(v)$. The pullback of Θ under the projection map $(z, u, v) \mapsto (u, v)$ admits the square root

$$\tilde{\Omega} = z^{-1}\,du \wedge dv = u^{-1}\,dz \wedge dv.$$

It is worth noting that $\tilde{\Omega}$ is the natural holomorphic volume form induced on Y by the standard volume form of $\mathbb{C}^3$: denoting by $f = z^2 - u^2 + \epsilon q(v)$ the defining function of Y, we have $df \wedge \tilde{\Omega} = dz \wedge du \wedge dv$. We equip Y with the restriction of the standard Kähler form $\omega_0 = \frac{i}{2}\,dz \wedge d\bar{z} + \frac{i}{2}\,du \wedge d\bar{u} + \frac{i}{2}\,dv \wedge d\bar{v}$, which differs from the pullback of the standard Kähler form of $\mathbb{C}^2$ by the extra term $\frac{i}{2}\,dz \wedge d\bar{z} = \frac{i}{2}\partial\bar{\partial}|u^2 - \epsilon q(v)|^2$. We claim that the (possibly singular) submanifolds

$$\tilde{L}_{a,b} = \{(z, u, v) \in Y \mid \mathrm{Re}\,(v) = a,\ \mathrm{Re}\,(u\bar{z}) = b\} \qquad (a, b) \in \mathbb{R}^2$$

are special Lagrangian with respect to $\tilde{\Omega}$ and ω_0. Indeed, the vector field $\xi(z, u, v) = (iu, iz, 0)$ is tangent to the submanifolds $\tilde{L}_{a,b}$, and the 1-forms $\iota_\xi \mathrm{Im}\,\tilde{\Omega} = \mathrm{Re}\,dv$ and $\iota_\xi \omega_0 = -d\,\mathrm{Re}(u\bar{z}) + \frac{i}{2} dv \wedge d\bar{v}$ both vanish on $\tilde{L}_{a,b}$. Moreover, $\tilde{L}_{a,b}$ is singular if and only if it passes through a point $(0, 0, v_0)$ with v_0 a root of q.

The involution $(z, u, v) \mapsto (-z, u, v)$ maps $\tilde{L}_{a,b}$ to $\tilde{L}_{a,-b}$. Thus, the special Lagrangian fibration $(z, u, v) \mapsto (\mathrm{Re}\, v, \mathrm{Re}\,(u\bar{z}))$ descends to a family of submanifolds in $\mathbb{C}^2$, parameterized by the quotient of $\mathbb{R}^2$ by the reflection $(a, b) \mapsto (a, -b)$, i.e. the closed upper half-plane. The image of $\tilde{L}_{a,b}$ under this projection is

$$L_{a,b} = \{(u, v) \in \mathbb{C}^2 \,|\, \mathrm{Re}\,(v) = a,\ \mathrm{Re}\,(\bar{u}\sqrt{u^2 - \epsilon q(v)}) = \pm b\},$$

and behaves exactly as described above: fixing a value of v (i.e., a point of D), the intersection of $L_{a,b}$ with $\mathbb{C} \times \{v\}$ is an ellipse with foci the two square roots of $\epsilon q(v)$ (i.e. the two points where H intersects $\mathbb{C} \times \{v\}$). For $b = 0$ the ellipse degenerates to a line segment; when v is a root of q the ellipses become circles and the line segment collapses to a point. However, a quick calculation shows that $L_{a,b}$ is not Lagrangian with respect to the standard Kähler form on $\mathbb{C}^2$.

Thus, it may well be easier to construct a special Lagrangian fibration on the double cover of $\mathbb{CP}^2$ branched at H (namely, a K3 surface) than on $\mathbb{CP}^2 \setminus H$. In fact, as in the previous example, the easiest way to construct such a fibration is probably through hyperkähler geometry, starting from an elliptically fibered K3 surface with a real structure for which the real part is a smooth connected surface of genus 10. Let P be a real homogeneous polynomial of bidegree $(4, 4)$ whose zero set in $\mathbb{RP}^1 \times \mathbb{RP}^1$ consists of nine homotopically trivial circles $C_1, \dots, C_9$ bounding mutually disjoint discs D_i, and let Y' be the double cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along the zero set of P (over $\mathbb{C}$). Then complex conjugation lifts to a J-antiholomorphic involution ι of Y', whose fixed point locus is a connected surface of genus 10, namely the preimage of $\mathbb{RP}^1 \times \mathbb{RP}^1 \setminus (D_1 \cup \dots \cup D_9)$ (whereas the fixed point set of the composition of ι with the nontrivial deck transformation consists of 9 spheres, the preimages of $D_1, \dots, D_9$). After performing a hyperkähler rotation as in §3.3, we obtain a new complex structure I on Y' with respect to which ι is holomorphic and the elliptic fibration induced by projection to a $\mathbb{CP}^1$ factor is special Lagrangian.

Remark 3.7. The curve $H \subset \mathbb{CP}^2$ bounds a number of Lagrangian discs, arising as relative vanishing cycles for degenerations of H to a nodal curve. For instance, considering a degeneration of H to two intersecting cubics singles out 9 such discs. The preimages of these discs are Lagrangian spheres in the double cover Y, and can be seen by looking at the symplectic affine structure on the bases B and $\tilde{B}$ of the special Lagrangian fibrations on $\mathbb{CP}^2$ and Y. Namely, $\tilde{B}$ is obtained by doubling B along its boundary, and 18 of its singular points are aligned along the "seam" of this gluing. The rays emanating from these singular points run along the seam, and match with each other to give rise to Lagrangian spheres.

Remark 3.8. Consider a singular K3 surface Y_0 with 9 ordinary double point singularities, obtained as the double cover of $\mathbb{CP}^2$ branched along the union H_0 of two intersecting cubics. The singularities of Y_0 can be either smoothed, which amounts to smoothing H_0 to a smooth sextic curve, or blown up, which is equivalent to blowing up $\mathbb{CP}^2$ at the 9 intersection points between the two components of H_0. These two procedures yield respectively the K3 surface considered in the above discussion, and the K3 surface considered in §3.3. Y_0 admits a special Lagrangian fibration whose base $\tilde{B}_0$ presents 9 singularities with monodromy conjugate to $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$; viewing $\tilde{B}_0$ as two copies of a disc glued along the boundary, these 9 singularities all lie along the seam of the gluing. Smoothing Y_0 replaces each ordinary double point by a Lagrangian sphere, and resolves the corresponding singularity of $\tilde{B}_0$ into a pair of singular points aligned along the seam. Blowing up Y_0 replaces each ordinary double point by an exceptional curve, and resolves the corresponding singularity of $\tilde{B}_0$ into a pair of singular points lying symmetrically across from each other on either side of the seam.

3.5. **Towards mirror symmetry for double covers.** Conjecture 3.2 suggests that a mirror $Y^\vee$ of the Calabi-Yau double cover Y of X branched along H can be obtained by gluing two copies of the mirror of $X \setminus H$ along their boundary. From the point of view of affine geometry, we start with a special Lagrangian fibration $f^\vee : M \to B$ (T-dual to the special Lagrangian fibration on $X \setminus H$), and glue together two copies of M using an orientation-reversing diffeomorphism of ∂M which induces a reflection in each fiber of $f^\vee$ above ∂B.

Arguably the "usual" mirror of X arises from considering the complement of an anticanonical divisor D, rather than the hypersurface H. Consider a degeneration of H under which it collapses onto D (with multiplicity 2). At the level of double covers, this amounts to degenerating Y to the union of two copies of X glued together along D. By Moser's theorem, this deformation affects the complex geometry of Y but not its symplectic geometry. Hence, the special Lagrangian fibrations on $X \setminus H$ and $X \setminus D$ can reasonably be expected to have the same base B, as long as we only consider the symplectic affine structure. (The complex affine structures are very different: in the case of $X \setminus D$ the complex affine structure blows up near the boundary of B, while in the case of $X \setminus H$ it doesn't. See e.g. Remark 3.3.) So, as long as we only consider the complex geometry of the mirror and not its symplectic structure, it should be possible to construct the mirror of Y simply by gluing two copies of the mirror of $X \setminus D$ (which is also the mirror of X without its superpotential).

A complication arises when the normal bundle to D is not holomorphically trivial. In that case, the family of special Lagrangians in $X \setminus H$ presents additional singularities at the boundary of B; these singularities are not directly visible in the special Lagrangian fibration on $X \setminus D$. An example of this phenomenon is presented in §3.4 (compare Conjecture 3.6 with Conjecture 2.9). Thus, when building $\tilde{B}$ out of two copies of the base B of the special Lagrangian fibration on $X \setminus D$, we need to introduce extra singularities into the affine structure along the seam of the gluing. This is essentially the same phenomenon as in Gross and Siebert's program (where singularities of the affine structure also arise from the nontriviality of the normal bundles to the codimension 1 toric strata along which the smoothing takes place).

For simplicity, let us just consider the case where D has trivial normal bundle. In that case, the discussion in Remark 2.11 implies that the boundary of the (uncorrected) mirror M of $X \setminus D$ is the product of S^1 with a complex hypersurface $M_D \subset \partial M$ (the uncorrected SYZ mirror to D). In fact, we have a trivial fibration $z_\delta : \partial M \approx M_D \times S^1 \to S^1$, where z_δ is the weight associated to the homotopy class of a meridian disc (collapsing to a point as the special Lagrangian torus L collapses onto a special Lagrangian submanifold of D, whence $|z_\delta| = 1$ on ∂M). The orientation-reversing diffeomorphism $\varphi : \partial M \to \partial M$ used to glue the two copies of M together corresponds to a reversal of the coordinate dual to the class of the meridian loop. More precisely, view a point of ∂M as a pair (Λ, ∇) where Λ is a special Lagrangian torus in D and ∇ is a flat unitary connection on the trivial bundle over $\Lambda \times S^1$ (here we use the triviality of the normal bundle to D to view nearby special Lagrangians in $X \setminus D$ as products $\Lambda \times S^1$ rather than S^1-bundles over Λ). Then the gluing diffeomorphism φ is given by $(\Lambda, \nabla) \mapsto (\Lambda, \bar{\nabla})$, where $\bar{\nabla}$ is the pullback of ∇ by the diffeomorphism $(p, e^{i\theta}) \mapsto (p, e^{-i\theta})$ of $\Lambda \times S^1$. Thus, under the identification of ∂M with $M_D \times S^1$, the diffeomorphism φ is the product of the identity map in M_D and the complex conjugation map $z_\delta \mapsto \bar{z}_\delta = z_\delta^{-1}$ from S^1 to itself.

At this point it would be tempting to conclude that, if $K_{X|D}$ is holomorphically trivial, then a mirror of Y can be obtained (at least as a complex manifold) by gluing together two copies of the mirror of X along their boundary $S^1 \times M_D$, to obtain a Calabi-Yau variety with a holomorphic involution given near the "seam" of the gluing by $z_\delta \mapsto z_\delta^{-1}$. Unfortunately, in the presence of instanton corrections this seems to always fail; in particular, the fibers of $z_\delta : \partial X^\vee \to S^1$ above two complex conjugate points are not necessarily biholomorphic. The following example in complex dimension 2 (inspired by calculations in [2]) illustrates a fairly general phenomenon.

Example 3.9. We consider again the local model for blow-ups mentioned in Example 2.10, modified so the special Lagrangian fibers are tori rather than cylinders [2]. Start with $\mathbb{C}^* \times \mathbb{C}$ equipped the holomorphic volume form $d\log x \wedge d\log y$ with poles along $\mathbb{C}^* \times \{0\}$, and blow up the point $(1, 0)$ to obtain a complex manifold X equipped the holomorphic volume form $\Omega = \pi^*(d\log x \wedge d\log y)$, with poles along the proper transform D of $\mathbb{C}^* \times \{0\}$. Observe that the S^1-action $e^{i\theta} \cdot (x, y) = (x, e^{i\theta} y)$ lifts to X, and consider an S^1-invariant Kähler form ω for which the area of the exceptional divisor is ϵ. Denote by $\mu : X \to \mathbb{R}$ the moment map for the S^1-action, normalized to equal 0 on D and ϵ at the isolated fixed point. The S^1-invariant tori $L_{t_1,t_2} = \{\log |\pi^* x| = t_1,\ \mu = t_2\}$ define a special Lagrangian fibration on $X \setminus D$, with one nodal singularity at the isolated fixed point (for $(t_1, t_2) = (0, \epsilon)$) [2].

The base B of this special Lagrangian fibration is a half-plane, with a singular point at distance ϵ from the boundary (and nontrivial monodromy around the singularity), as pictured in Figure 1; we place the cut above the singular point in order to better visualize wall-crossing phenomena near the boundary of B. The complex rays emanating from the singular point (one of which corresponds precisely to the

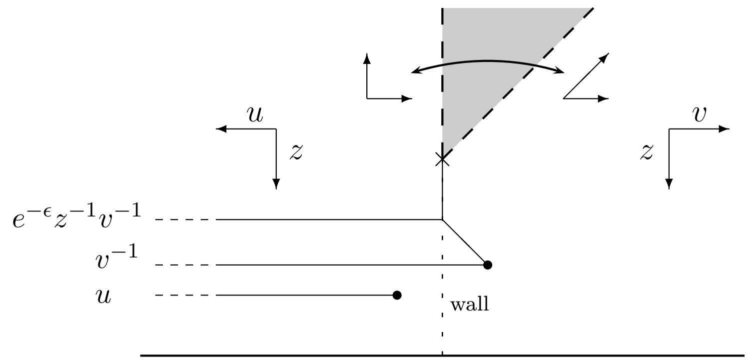

FIGURE 1. A special Lagrangian fibration on the blowup of $\mathbb{C}^* \times \mathbb{C}$

exceptional divisor of the blowup) are responsible for wall-crossing jumps in holomorphic disc counts, and split the mirror M into two chambers, which are essentially the preimages of the left and right halves of the figure.

Denote by z $(= z_\delta)$ the holomorphic coordinate on M which corresponds to the holomorphic disc $\{\pi^* x = e^{t_1},\ \mu < t_2\}$ in (X, L_{t_1,t_2}); it can be thought as a complexified and exponentiated version of the downward-pointing affine coordinate pictured on Figure 1. In one of the two chambers of M, denote by u the holomorphic coordinate that similarly corresponds to the leftward-pointing affine coordinate represented in the figure. For instance, if we partially compactify X to allow $\pi^* x$ to become zero (i.e., if we had blown up $\mathbb{C}^2$ at $(1, 0)$ rather than $\mathbb{C}^* \times \mathbb{C}$), then u becomes (up to a scaling factor) the weight associated to a disc that runs parallel to the x-axis. Similarly, denote by v the holomorphic coordinate in the other chamber of M corresponding to a rightward-pointing affine coordinate, normalized so that, if we ignore instanton corrections, the gluing across the wall is given by $u = v^{-1}$.

Imagine that L_{t_1,t_2} in the "left" chamber $(t_1 < 0)$ bounds a holomorphic disc with associated weight u (such a disc doesn't exist in X, but it exists in a suitable partial compactification), and increase the value of t_1 past zero, keeping t_2 less than ϵ: then this holomorphic disc deforms appropriately (and its weight is now called v^{-1}), but it also generates a new disc with weight $e^{-\epsilon} z^{-1} v^{-1}$, obtained by attaching an exceptional disc (the part of the exceptional divisor where $\mu > t_2$) as one crosses the wall. This phenomenon is pictured on Figure 1 (where the various discs are abusively represented as tropical curves, which actually should be drawn in the complex affine structure). Thus the instanton-corrected gluing is given by $u = v^{-1} + e^{-\epsilon} z^{-1} v^{-1}$, i.e.,

$$uv = 1 + e^{-\epsilon} z^{-1}. \tag{3.1}$$

Actually the portion of the wall where $t_2 > \epsilon$ also gives rise to the same instanton-corrected gluing, so that the corrected mirror is globally given by (3.1); see [2].

Now replace D by the union $H = D_+ \cup D_-$ of two disjoint complex curves, e.g. the proper transforms of two complex lines intersecting transversely at the blown up point $(1, 0)$, and consider the double cover Y of X branched along H. (We leave the details unspecified, as the construction should arguably be carried out in a global setting such as that of Conjecture 3.4 rather than in the local setting.)

Conjecture 1.1 suggests that Y should carry a special Lagrangian fibration whose base (considering only the symplectic affine structure) is obtained by doubling B along its boundary. Pictorially, this corresponds to flipping Figure 1 about the horizontal axis and gluing the two pictures together. On the mirror, before instanton corrections this amounts to reflecting the z variable via $z \mapsto z^{-1}$, and gluing M and its reflected copy along their common boundary $|z| = 1$. However, the gluing via $z \mapsto z^{-1}$ is not compatible with the instanton corrections discussed above; this is because when we cross the wall there are now two different exceptional discs to consider. Namely, Y contains a -2-curve C (the preimage of the exceptional curve in X), corresponding to the alignment between the walls that come out of the two singular fibers on either side of the seam. Special Lagrangian fibers which lie on the wall intersect C in a circle and split it into two Maslov index 0 discs, which both contribute to instanton corrections. A careful calculation shows that the instanton-corrected gluing is now

$$uv = (1 + e^{-\epsilon} z^{-1})(1 + e^{-\epsilon} z). \tag{3.2}$$

Thus the instanton-corrected mirror to Y does carry a holomorphic involution defined by $z \mapsto z^{-1}$, but restricting to the subset $|z| < 1$ does not yield the instanton-corrected mirror to X.

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