Mirror symmetry for open manifolds - 16/5/11 - (Abouzaid - Auroux - Katzarkov) ①

★ Mirror symmetry: for Calabi-Yau mflds $(X, \omega, J, \Omega) \longleftrightarrow (X^\vee, \omega^\vee, J^\vee, \Omega^\vee)$

kähler + $\Omega \in \Omega^{n,0} \simeq \mathcal{O}$

sympl. geometry of X $\longleftrightarrow$ cx. geometry of $X^\vee$

$DF(X,\omega) \simeq D^b Coh(X^\vee)$ (Kontsevich HMS)

(if X noncompact, this should be the wrapped Fukaya cat.)

Lagr. submflds + local systems $\longleftrightarrow$ coherent sheaves

intersection theory, holom. discs — Hom & Ext's

geometrically: SYZ conj: $X, X^\vee$ carry mutually dual fibrations by (real) (special) Lagr. tori (over B = sing real affine mfld aka "tropical mfld")

$$\begin{array}{ccc} T^n \hookrightarrow X & & \check{X} \hookleftarrow \check{T} \\ & \pi \searrow \ \swarrow \check{\pi} & \\ & B & \end{array}$$

where $\check{T} = \hom(\pi_1 T, U(1))$

up to corrections (dualization fails near singu. of fibration).

HMS motivation for SYZ: $p \in X^\vee \longleftrightarrow \mathcal{O}_p \in D^b Coh(X^\vee)$

$\longleftrightarrow \mathcal{L}_p \in DF(X,\omega)$

$HF^*(\mathcal{L}_p, \mathcal{L}_p) \simeq Ext^*(\mathcal{O}_p, \mathcal{O}_p) \simeq H^*(T^n; \mathbb{C})$

$\Rightarrow$ guess $\mathcal{L}_p \simeq T^n$ (+ rank 1 local system)

So: build $X^\vee$ as moduli space of $\{(L, \nabla) \mid L \subset X$ Lagr. torus fiber of π, ∇ $U(1)$ local system $\in \hom(\pi_1 L, \nabla)\}$.

(carries natural $\left\{\begin{matrix}\text{cx. structure} \\ \text{kähler}\end{matrix}\right\}$ if $L \left\{\begin{matrix}\text{Lagr.} \\ \text{SLag}\end{matrix}\right\}$)

- Can't do this for all CYs, need a reasonable Lagr. torus fibration!

★ Non CY case: (X, ω, J) kähler, $D \in |-K_X|$ hypersurf. w/ normal crossings, $\sigma_D^{-1} = \Omega \in \Omega^{n,0}(X-D)$ with simple poles along D

Then $X - D$ open CY; if carries Lagr. T^n-fibration, get a mirror $X^\vee$ by SYZ.

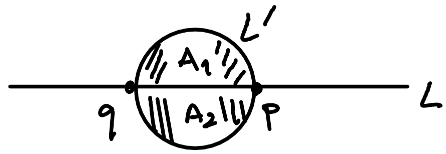
$DF(X)$ is a deformation of $DF(X-D)$ - by holom. discs intersecting D.

On mirror, deform $X^\vee$ by a superpotential $W: X^\vee \to \mathbb{C}$.

<u>Point</u>: $\mathcal{F}(X)$ is a <u>curved</u> A_∞-category, i.e. has m_0 term.

so on $CF(L,L')$, $\partial^2_{Floer}(x) = m_0^{L'} \cdot x - x \cdot m_0^{L}$

<u>Ex</u>: in $\mathbb{C}$



$CF(L,L') = \mathrm{span}(p,q)$

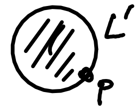
$\partial p = t^{A_1} q$

$\partial q = t^{A_2} p$

$\partial^2 = \underbrace{t^{A_1+A_2} id}_{m_0^{L'}\,!!}$

on the other hand, $HF(L',L')$ ok (m_0 obstructions cancel).

$$m_0^{L,\nabla} = \sum_{\beta \in \pi_2(X,L)} \underbrace{t^{\omega(\beta)} \nabla(\partial\beta)}_{\substack{\text{weight by} \\ \text{area + holonomy} \\ =: z_\beta}} ev_* [\mathcal{M}_1(L')] \in C_*(L) = CF(L,L).$$



$\rightsquigarrow$ replace $D^b Coh(X^\vee)$ by $\underline{\text{matrix factorizations}}$ on $(X^\vee, W)$ $\quad \mathcal{E}_0 \underset{d_1}{\overset{d_0}{\leftrightarrows}} \mathcal{E}_1 \quad d_0 d_1 = (W-\lambda)\, id$

(trivial unless $\lambda \in crit\, W$).

(obstructed for $\lambda \neq \lambda'$, but curvature terms cancel if $\lambda = \lambda'$)

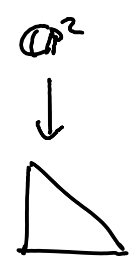
<u>Classical example: toric Fano varieties</u>.

X toric (possibly noncompact) Kähler ω

$\pi \downarrow$ moment map

Δ convex polytope (poss. noncompact) $\subset \mathbb{R}^n$

ex.



$D = \pi^{-1}(\partial\Delta)$ toric divisor, $[D] = c_1(X)$; assume $D > 0$. (X Fano)

$X - D = \pi^{-1}(\mathrm{int}\,\Delta) \cong (\mathbb{C}^*)^n$ CY (carries $\Omega = \prod d\log z_i$)

$\rightsquigarrow$ fibers of π are SLag. product tori $S^1(r_1) \times \dots \times S^1(r_n) \subset (\mathbb{C}^*)^n$

Mirror of $(\mathbb{C}^*)^n = (\mathbb{C}^*)^n$ (but "exchanging symplectic moment map with complex Log map")

$X \leftrightarrow (X^\vee = (\mathbb{C}^*)^n, W)$

$\Delta = \{ x \in \mathbb{R}^n \,/\, \langle \nu_i, x \rangle + \alpha_i \geq 0 \}_{i = 1 \dots \# \text{facets}}$ $\quad \nu_i \in \mathbb{Z}^n$ primitive normal vector ($\Leftrightarrow$ ray of fan), $\alpha_i \in \mathbb{R}$

$\rightsquigarrow W = \sum_i t^{\alpha_i} z^{\nu_i} : (\mathbb{C}^*)^n \to \mathbb{C}$. $\quad$ Ex. $\mathbb{CP}^2$: $W = z_1 + z_2 + \dfrac{t^\lambda}{z_1 z_2}$

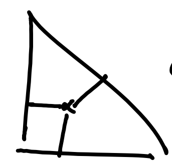
Namely: Points of $X^\vee$ $\longleftrightarrow$ Lagr. tori (L,∇) in $X \setminus D$
+ $U(1)$ loc. system

$$W(L,\nabla) = \sum_{\substack{\beta\in\pi_2(X,L)\\ \beta\cdot D=1}} n_\beta\, z_\beta(L,\nabla)$$

n_β = # holom. discs in class β through generic pt of L
$= \deg ev_*[\mathcal{M}_1(L,\beta)]$

$z_\beta(L,\nabla) = t^{\omega(\beta)}\nabla(\partial\beta) : X^\vee \to \mathbb{C}$
local holom. coords!

In toric case, 1 family of discs for each facet of Δ
symplectic area $\longleftrightarrow$ distance to facet



eg. $\mathbb{CP}^2$

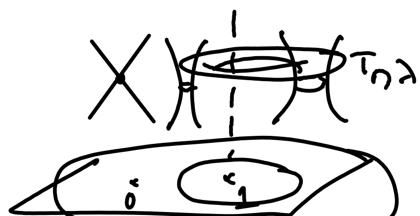
GOAL OF TALK: mirrors of A_n singularities: clarify statement "Milnor fiber $\longleftrightarrow$ resolution"

- Next example: $X^0 = \mathbb{C}^2 \setminus \{xy=1\}$ this is an open CY, $\Omega = \dfrac{dx\wedge dy}{xy-1}$.

S^1 acts by $(x,y)\mapsto(e^{i\theta}x, e^{-i\theta}y)$

$\exists$ S^1-inv't slag torus fibration with 1 sing. fiber: $T_{r,\lambda} = \left\{\begin{array}{l}|xy-1|=r\\ |x|^2-|y|^2=\lambda\end{array}\right\}$

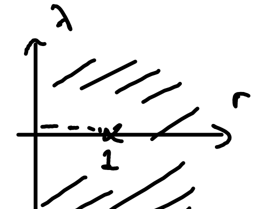
$(x,y) \mapsto xy$



(lift to level set of moment map $\mu_{S^1} = |x|^2-|y|^2=\lambda$ of slag torus fibration on reduced space $\cong \mathbb{C}^*$)

Sing fiber $= T_{1,0}$ through fixed point $(0,0)$.
(monodromy around it = shear!).

$T^2 \to X^0$
$\downarrow$
$B = \mathbb{R}_+\times\mathbb{R}$



Easier to understand mirror if we compactify partially.

Consider $\hat{X}$ = blowup of $\mathbb{P}^1\times\mathbb{C}$ at $(-1,0)$

$i \uparrow$

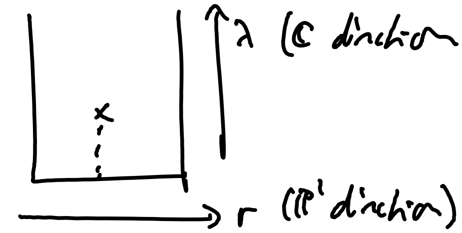
X^0 by $(x,y) \longmapsto (xy-1,\ x)$ in $\mathbb{P}^1\times\mathbb{C}$, lift to blowup

$\mathbb{C}^*\subset\mathbb{P}^1 \quad \mathbb{C}$ (y axis $\longmapsto$ exceptional $\mathbb{P}^1$)

$D = \hat{X}\setminus i(X^0)$ = proper transform of toric divisor of $\mathbb{P}^1\times\mathbb{C}$, $(\mathbb{P}^1\times 0)\cup(0\times\mathbb{C})\cup(\infty\times\mathbb{C})$.

Compared to above picture: $\begin{cases} \to \text{compactify base to } \mathbb{P}^1 \text{ (smooth fibers above 1 and } \infty) \\ \to \text{compactify 1 end of each conic fiber.} \end{cases}$ of conic fibration

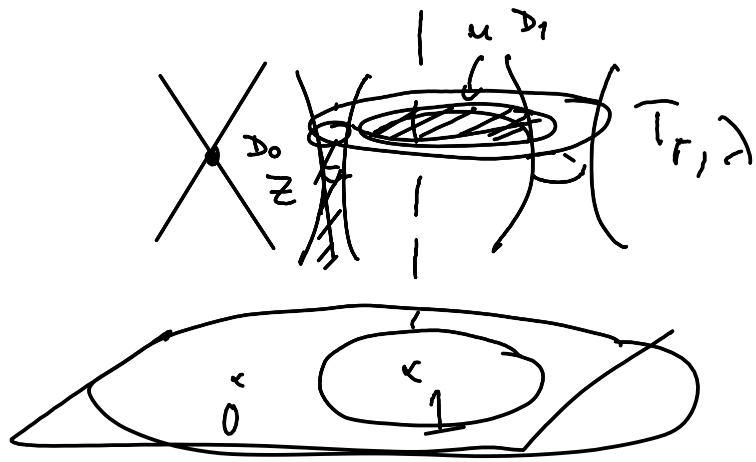
S^1 action extends & same story except now base =



Now easier to see complex charts of the mirror (coords z_β = weights of discs...)
(will have same mirror + superpotential) $= t^{\omega(\beta)}\,\mathrm{hol}(\partial\beta)$.

For $r<1$: $T_{r,\lambda}$ bounds
D_1 section over disc $\to$ weight $=: u$
D_0 piece of compactified fiber $=: z$

gives one chart.



For $r>1$: more naturally,
D_2 section over $\mathbb{P}^1$-disc $\to$ weight $= v$
D_0 ——— fiber $= z$

gluing b/w the charts: uv = weight of global section over $\mathbb{P}^1 = t^A$ ($A = \mathrm{area}(\mathbb{P}^1)$ if $\lambda>0$)
($D_1 \cup D_2 \sim \mathbb{P}^1\times\{pt\}$)

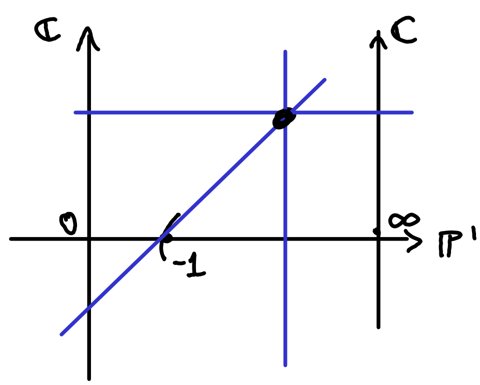
but monodromy $\Rightarrow$ mismatch b/w ∂D_1 & ∂D_2, instead $D_1 \cup D_2 \sim D_0 \cup$ compactif-section ($\widetilde{\mathbb{P}^1\times 0}$)

so $uv = t^{A-\varepsilon} z$ (ε = area exc. $\mathbb{P}^1$)

These gluings are inconsistent; but in fact so is disc counting for superpotential.

Namely, far from $\widetilde{0\times\mathbb{C}}$, the Slag fibration is $\simeq$ product tori, and can see disks:

for $r>1$



4 families of discs w/ $\beta\cdot D = 1$: approx.
- disc in pt $\times\,\mathbb{C}$
- two halves of $\mathbb{P}^1 \times$ pt
- proper transform of disc in affine line through $(-1,0)$

$$\Rightarrow W = \underbrace{z}_{\text{hits } \widetilde{\mathbb{C}\times 0}} + \underbrace{v}_{\infty\times\mathbb{C}} + \underbrace{\frac{t^A}{v} + \frac{z\,t^{A-\varepsilon}}{v}}_{\text{hit } 0\times\mathbb{C}}$$

Similarly for $r<1$, $W = z + \underbrace{\frac{t^A}{u} + \frac{z\,t^{A-\varepsilon}}{u}}_{\text{hit } \infty\times\mathbb{C}} + \underbrace{u}_{\text{hit } 0\times\mathbb{C}}$

match terms for each component of D

$\Rightarrow$ Corrected mirror: $X^\vee = \{(u,v,z)\in\mathbb{C}^2\times\mathbb{C}^* \,/\, uv = t^A(1+t^{-\varepsilon}z)\}$

($\cong$ complement of conic $uv = t^A$ in $\mathbb{C}^2$)

$$X = \mathbb{C}^2 \setminus \{xy=1\} \longleftrightarrow X^\vee \qquad \text{open CYs.}$$

$$\hat{X} = Bl_{(-1,0)}(\mathbb{P}^1\times\mathbb{C}) \longleftrightarrow (X^\vee,\ W = u+v+z)$$

or add only some divisors to X $\leftrightarrow$ some terms in W

especially: keeping only $0\times\mathbb{C}$, $\mathbb{C}^2 \longleftrightarrow (X^\vee,\ W=u)$

$\uparrow$ as conic bundle over $\mathbb{C}$ w/ one sing. fiber, via $(x,y)\mapsto xy-1$.

(see James Pascaleff: HMS for these examples).

$\star$ Generalization to A_n spaces:

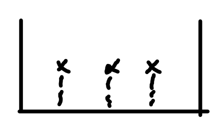
$\hat{X}$ = blowup of $\mathbb{P}^1\times\mathbb{C}$ at $n+1$ points $(z_i,0)$ in general position on $\mathbb{P}^1\times 0$ (w/ equal areas ε.)

$i \uparrow$

$X^0 = \{(x,y,z)\in\mathbb{C}^2\times\mathbb{C}^* \,/\, xy = p(z)\}$, $\quad p(z) = \prod_{i=1}^{n+1}(z-z_i)$

open CY.

$i(X^0)$ = complement of proper transform of toric divisor.

Same construction (using: S^1 still acts!) $\Rightarrow$ SLag fibration w/ base 

$\Rightarrow$ mirror Y^0 to X^0 in SYZ sense (after corrections) is covered by $(n+2)$ charts

$$\left((u_i,v_i,z)\in(\mathbb{C}^*)^3,\ u_iv_i = t^A\right) \quad (\simeq(\mathbb{C}^*)^2) \qquad i=0\ldots n+1$$

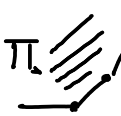
glued to each other via: $u_i = (1+t^{-\varepsilon}z)u_{i+1}$ or $v_{i+1} = (1+t^{-\varepsilon}z)v_i$

$\Leftrightarrow$ Y^0 = resolution of $\{(\underset{u_0}{\overset{\shortparallel}{u}}, \underset{v_{n+1}}{\overset{\shortparallel}{v}}, z)\in\mathbb{C}^2\times\mathbb{C}^* \,/\, uv = t^A(1+t^{-\varepsilon}z)^{n+1}\}$

resolution replaces $(u,v)=(0,0)$ by A_n-chain of (-2)-spheres

In fact, if forget condition $z\neq 0$, get $Y' = Y^0 \cup \{z=0\}$ = toric var.

$$\downarrow \text{resolution}$$

Π  slopes $0,\dots,n+1$

$$\mathbb{C}^2/\mathbb{Z}_{n+1} \cong \{uv = \zeta^{n+1}\} \subset \mathbb{C}^3$$

n (-2-curves)

(acts w/ weights $(1,-1)$) $[x,y] \longmapsto (x^{n+1}, y^{n+1}, xy)$

fan =  $(0,1) \dots (n+1,1)$

Y^0 = complement of the divisor $(z=0)$ in Y' (disjoint from A_n-chain, which is @ $z=-t^{\varepsilon}$).

- Now:

$$X^0 = \{xy = p_{n+1}(z)\} \subset \mathbb{C}^2\times\mathbb{C}^* \quad \xleftrightarrow{\text{mirror}} \quad Y^0 = \left(\begin{array}{l}\text{open dense subset in}\\ \text{toric var} = A_n\text{-resolution}\end{array}\right)$$

$$\cap$$

$$X = \{xy = p_{n+1}(z)\} \subset \mathbb{C}^3 \qquad Y^0,\ W = u$$

(NB: vanishes to order k on k^{th} $\mathbb{P}^1$ of the A_n-chain !)

$$\cap$$

$$\hat{X} = Bl_{(n+1)}(\mathbb{P}^1\times\mathbb{C}) \qquad Y^0,\ W = u+v+z+(\text{add}^l\text{ terms}).$$

(not explicit)

- Q: what about mirror to Y' ? ie. add missing $z=0$ to Y^0 ?

  Ans: X^0 with a superpotential on it ! namely, $W = x : X^0 \to \mathbb{C}$.

★ The story generalizes to $\left\{\begin{array}{l}\text{"split" conic bundles}\\ \text{\& their compactifications}\end{array}\right\}$ over toric varieties. [AAK]

Namely: V toric variety, $H = \{f=0\} \subset V$ smooth hypersurface

(suff. generic, "close to tropical limit", nef)

tropicalization $\mathrm{Trop}\, f = \varphi : \mathbb{R}^n \to \mathbb{R}$ PL function

$\Rightarrow$ mirror to $\hat{X} = Bl_{H\times 0}(V\times\mathbb{C})$ is the complement Y^0 of a hypersurface in the toric var. Y' assoc^d to polytope $\Pi = \{x_{n+1} \geq \varphi(x_1 \dots x_n)\} \subset \mathbb{R}^{n+1}$,

equipped w/ superpotential $W = z^{(0,\dots,0,1)} + \left(\begin{array}{l}1\text{ monomial for each}\\ \text{toric divisor of } V\end{array}\right) + (\text{extra terms})$

namely, $\mathcal{L}\to V$ line bundle defining H given by PL function λ on fan of V; for each ray of fan, w/ primitive generator $\nu\in\mathbb{Z}^n$, get $\lambda(\nu)\in\mathbb{Z}$, then get term $z^{(\nu,\lambda(\nu))}$

& restricting to affine conic bundle $X := \{(x,y,z)\in\mathbb{C}\times\mathcal{L}\times V \,/\, xy = f(z)\}$

(affine conic bundle over V, singular over H)

$$X \xleftrightarrow{\text{mirror}} \left(Y^0,\ W = \text{monomials for divisors of } V = \sum_{\text{rays of } V} z^{(\nu,\lambda(\nu))}\right).$$