D. Auroux – Mirrors of curves and their Fukaya categories – Miami, Jan 27-28-29 2020 – Talk 1

Given a hypersurface H^{n-1} [over a nonarchimedean field, eg. Novikov field; not needed in simplest cases]
eg. degenerating family of complex hypersurfaces near tropical limit

embedded in alg. torus, toric variety, or abelian variety V^n

$\rightsquigarrow$ mirror LG-model (Y, W) where Y is a CY $(n+1)$-fold, $W \in \mathcal{O}(Y)$ superpotential.
(depends on choice of embedding)

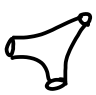
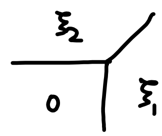
<u>Construction</u> (Abouzaid-A.-Katzarkov via SYZ; see also earlier Hori-Vafa, Clarke, ..., Chan-Lau-Leung, ...).

$H = f^{-1}(0)$, $f(x) = \sum_{\alpha \in A \subseteq \mathbb{Z}^n} c_\alpha t^{\rho(\alpha)} x^\alpha$ Laurent polynomial $(t \to 0)$

$\Rightarrow$ tropicalization $\varphi(\xi) = \max_{\alpha \in A} \left(\langle \alpha, \xi \rangle - \rho(\alpha) \right)$

★ Let (Y, ω) = toric CY with moment polytope $\Delta_Y = \{ (\xi, \eta) \,/\, \eta \geq \varphi(\xi) \} \subset \mathbb{R}^{n+1}$.
$W_0 = -z^{(0, \dots, 0, 1)}$ toric monomial which vanishes to order 1 on each toric divisor of Y

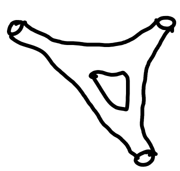
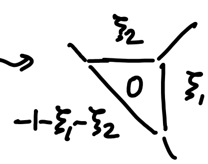
$\Rightarrow (Y, W_0)$ is mirror to $H \subset (\mathbb{C}^*)^n$. (eg: SYZ mirror to a LG-model w/ Morse-Bott sings along H).

Ex: (1)  $x_1 + x_2 + 1 = 0$ $\rightsquigarrow$ 

$\rightsquigarrow$ $Y \simeq \mathbb{C}^3$
$W_0 = -z_1 z_2 z_3$.

$\eta \geq \max(0, \xi_1, \xi_2)$

more generally, $\Pi_{n-1} = \{ x_1 + \dots + x_n + 1 = 0 \} \longleftrightarrow (\mathbb{C}^{n+1}, -\prod z_i)$.

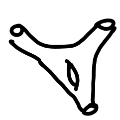
(2)  $x_1 + x_2 + \frac{t}{x_1 x_2} + 1 = 0$ $\rightsquigarrow$ 

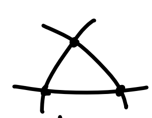
$\rightsquigarrow$ $Y \simeq \mathrm{Tot}(\overset{u}{\mathcal{O}(-3)} \to \overset{z_0 z_1 z_2}{\mathbb{P}^2})$
$W_0 = -u z_0 z_1 z_2$

<u>Remark 1</u>:
- the smooth fiber of W_0 is $\simeq (\mathbb{C}^*)^n$
- the sing. fiber $W_0^{-1}(0) = \bigcup$ toric strata in Y
- critical locus $= \bigcup$ codim 2 toric strata – for curves with max. degeneration ("tropically smooth") this is a trivalent configuration of $\mathbb{C}$'s and $\mathbb{P}^1$'s

<u>Remark 2</u>: In simple examples, Y = line bundle over some toric var., $W_0 = \overset{\text{fiber}}{u} \cdot \overset{\text{base}}{\sigma} \Rightarrow$ by Orlov's Knörrer periodicity $D^b_{sg}(Y, W_0) \simeq D^b\mathrm{Coh}(\sigma^{-1}(0))$ (usually singular) $(n-1)$-dim'l mirror

eg. (1) pants $\{x_1 + x_2 + 1 = 0\} \subset (\mathbb{C}^*)^2 \xleftrightarrow{\text{mirror}} \{ z_1 z_2 = 0 \} \subset \mathbb{C}^2$ (similarly $\Pi_{n-1} \longleftrightarrow \{ \prod z_i = 0 \} \subset \mathbb{C}^n$)

(2)  $\longleftrightarrow$ $\{ z_0 z_1 z_2 = 0 \} \subset \mathbb{P}^2$



For higher genus, such mirrors are stacky / non-reduced $\Rightarrow$ won't consider.

$\star$ For $H \subset V^n$ toric var. (Fano), def. by $f(x) = \sum_{\alpha \in A} c_\alpha t^{\rho(\alpha)} x^\alpha \in H^0(V, \mathcal{L})$ $(\mathrm{Conv}(A) = \mathrm{Newt}(\mathcal{L}))$

mirror is $(Y^{n+1}, w_0 + w_V)$

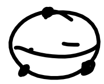
Y^{n+1}: same as before, CY toric var. determined by $\mathrm{Trop}(f)$

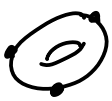
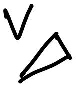
w_V: extra terms of superpotential, one per ray of Σ_V

$$w_V = \sum_{\nu \text{ ray of } \Sigma_V} T^{c(\nu)} z^{(\nu, \lambda(\nu))}$$

, $\lambda : \Sigma_V \to \mathbb{R}$ PL function describing $\mathcal{L} = \mathcal{O}(H)$

__Ex:__ (1) $\{x_0 + x_1 + x_2 = 0\} \subset \mathbb{P}^2 \longleftrightarrow (\mathbb{C}^3, \underbrace{-z_1 z_2 z_3}_{w_0} + \underbrace{T(z_1 + z_2 + z_3)}_{w_V})$



(2)  $\subset V$  $\longleftrightarrow \left(Y = \begin{matrix} \mathcal{O}(-3) \ni u \\ \downarrow \\ \mathbb{P}^2 \ni (z_0 : z_1 : z_2) \end{matrix}, \underbrace{-u z_0 z_1 z_2}_{w_0} + \underbrace{Tu(z_0^3 + z_1^3 + z_2^3)}_{w_V} \right)$.

(__NB:__ this is Morse-Bott along smooth cubic elliptic curve)

(3) genus g curve $\subset$ toric Del Pezzo ($g \geq 2$) $\longleftrightarrow$ $\mathrm{crit}(w_0 + w_V)$ = trivalent configuration of $(3g-3)$ $\mathbb{P}^1$'s meeting in $(2g-2)$ nodes

e.g. genus 2  $\longleftrightarrow$  $= \mathrm{crit}(w_0 + w_V)$.

$\star$ These constructions also work in abelian varieties (see eg. C. Cannizzo's thesis) & for complete intersections ($H^{n-k} \subset V^n \longleftrightarrow Y^{n+k}$, w_0 has k terms).

HMS relates:

(A) $\mathcal{W}(H)$ (wrapped Fukaya cat. of H) $\longleftrightarrow$ $D^b_{sg}(Y, W)$ (or $D^b\mathrm{Coh}(\sigma^{-1}(0))$ via Orlov)

well-studied: $\underbrace{\text{AAEKO, H. Lee}}_{\text{curves in } (\mathbb{C}^*)^2}$, $\underbrace{\text{Nadler}}_{\text{towards higher dim., but using microlocal sheaves. GPS} \rightsquigarrow \text{same thing!}}$, Gammage-Shende, Lekili-Polishchuk, ...]

(B). $D^b\mathrm{Coh}(H)$ to $\mathcal{F}(Y, W)$:

(0) via μlocal sheaves (Nadler) or sectors (GPS) – when Y is exact (rarely).

OUR FOCUS:

(1) via fiberwise wrapped Fukaya cat. (Abouzaid-A.).

(2) via "$\mathcal{F}(\sigma^{-1}(0))$" when expect a non-LG mirror via Knörrer periodicity (M. Jeffs)

(3) by directly working on $\mathrm{crit}(W)$ (or $(\mathrm{crit}(w_0), w_V)$) Bypass the need to embed H into V ??

$\rightarrow$ for mirrors of curves: Lagr. Floer theory in trivalent configurations of $\mathbb{C}$'s and $\mathbb{P}^1$'s?

(A-Efimov-Katzarkov in progress)

Today, focus on (1)

The fiberwise wrapped Fukaya cat. of (Y, w_0) or $(Y, w_0 + w_V)$ [Abouzaid - A.]

(ad hoc construction for toric LG models – can extend using language of sectors ... ⚠ non-exact Y!
↳ using monomial admissibility, see also Hanlon's thesis)

Objects: properly embedded Lagrangians $L \subset Y$ which are

- tautologically unobstructed (don't bound holom. discs)
- equipped with extra data (spin structure, grading, local system)
- monomially admissible:
  - → for $|w_0| > 1$, $\arg(w_0)_{|L}$ loc. constant (ie. $w_{0|L} \in$ union of radial arcs) $\in (-\frac{\pi}{2}, \frac{\pi}{2})$
  - → recalling that fibers of w_0 are $\simeq (\mathbb{C}^*)^n$, outside of a compact subset of these, $\exists$ finite open cover $\{U_\nu\}$ and a collection of monomials z^ν [exactly those in w_V if V compact, else choose a toric compactif.ⁿ $\overline{V}$ using $\mathrm{Newt}(f) = \mathrm{Conv}(A)$!]
  st. $\arg(z^\nu)_{|L \cap U_\nu}$ loc. constant (taking prescribed values, eg 0).

Ham. perturbations: L admissible $\rightsquigarrow$ flow L^t (Hamiltonian isotopic to L; admissible)
the flow increases the values of $\arg(w_0)$ and $\arg(z^\nu)$ at ∞

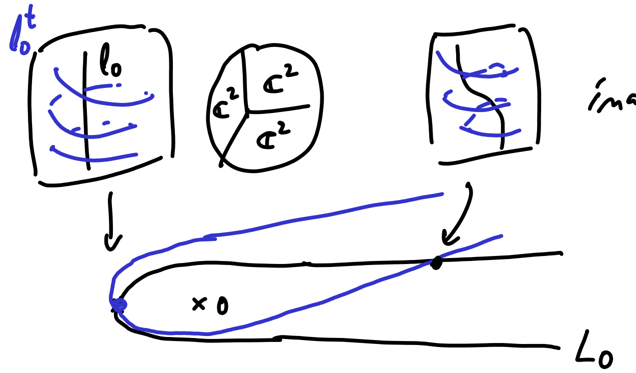
- for w_0 and for z^ν which appears in w_V, ↗ in <u>bounded</u> interval – NO WRAPPING
- for other monomials (in $w_{\overline{V}}$ from compactification but not in w_V), ↗ to ∞ (WRAP).

Then define $\hom(L_0, L_1) = \varinjlim_{t \to \infty} CF^*(L_0^t, L_1)$ under natural continuation maps.

★ The fiberwise partial wrapping Hamiltonians are essentially those which appear in Hanlon's thesis as monodromy for families of toric mirrors / induced functor on $\mathcal{F}((\mathbb{C}^*)^n, w_{\overline{V}}) \simeq D^b(\overline{V})$ is, under HMS, mirror to $- \otimes \mathcal{O}(D)$, $D = \overline{V} - V$ compactification divisors.
So the direct limit amounts, fiberwise, to the localization which restricts from $D^b(\overline{V})$ to $D^b(V)$.

★ **Ex:** $(\mathbb{C}^3, w_0 = -z_1 z_2 z_3)$ (mirror of △), L_0 = parallel-transport $\ell_0 = (\mathbb{R}_+)^2 \subset (\mathbb{C}^*)^2 \simeq w_0^{-1}(-1)$ along U-shaped arc.

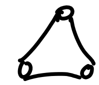
$w_0^{-1}(-1) \simeq (\mathbb{C}^*)^2$
$\cup$
$\ell_0 = (\mathbb{R}_+)^2$



images of ℓ_0 under monodromy + wrapping Ham.
$\arg(z_i) = f_i(\log|z|)$.
$\arg(z_i) = 0$ wherever $|z_i| \gg \min |z_j|$.
↓ wrapping
t (here $t \to +\infty$)

$$\hom(L_0,L_0) \simeq \underset{\mathbb{K}[x_1^{\pm 1},\, x_2^{\pm 1}]}{\underset{\wr\wr}{CW^*(\ell_0,\ell_0)}} \oplus CW^*(\ell_0,\ell_0)[-1] \qquad \text{(Abouzaid-A.)}$$

∂ = multiplication by $1+x_1+x_2$

so $$H^*\hom(L_0,L_0) \underset{\text{ring iso.}}{\simeq} \mathbb{K}[x_1^{\pm 1},x_2^{\pm 1}]/(1+x_1+x_2) \simeq \hom(\mathcal{O},\mathcal{O}) \text{ for}$$  ✓

Similarly for $H \subset (\mathbb{C}^*)^n$, find that L_0 & $\mathcal{O}_H$ match under HMS.
hypersurface

$\Rightarrow$ mod a generation statement (OK for pants), this yields HMS.

(same calculation!
∂ = mult. by f = def. eqn of H)